L'Hôpital's Rule

Calculus 1 — Section 4.8

Applying L'Hôpital's rule · other indeterminate forms · growth rates of functions

Applying L'Hôpital's rule

The indeterminate form $0/0$

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2 \neq 0$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L_1}{L_2}$.

The indeterminate form $0/0$

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2 \neq 0$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L_1}{L_2}$.

But what if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$? This is an indeterminate form of type $\frac{0}{0}$: we cannot determine the behavior of $\frac{f(x)}{g(x)}$ without further analysis.

The indeterminate form $0/0$

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2 \neq 0$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L_1}{L_2}$.

But what if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$? This is an indeterminate form of type $\frac{0}{0}$: we cannot determine the behavior of $\frac{f(x)}{g(x)}$ without further analysis.

We have met such limits before:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 4 \qquad \text{(factoring)},$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{(a geometric argument)}.$$

The indeterminate form $0/0$

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2 \neq 0$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L_1}{L_2}$.

But what if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$? This is an indeterminate form of type $\frac{0}{0}$: we cannot determine the behavior of $\frac{f(x)}{g(x)}$ without further analysis.

We have met such limits before:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 4 \qquad \text{(factoring)},$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{(a geometric argument)}.$$

L'Hôpital's rule uses derivatives to evaluate such limits, and many more that we could not compute before.

The idea: linear approximations

Suppose f and g are differentiable with $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, so that $f(a) = 0 = g(a)$, and $g'(a) \neq 0$. For x near a,

$$f(x) \approx f(a) + f'(a)(x-a) = f'(a)(x-a),$$

$$g(x) \approx g(a) + g'(a)(x-a) = g'(a)(x-a).$$

The idea: linear approximations

Suppose f and g are differentiable with $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, so that $f(a) = 0 = g(a)$, and $g'(a) \neq 0$. For x near a,

$$f(x) \approx f(a) + f'(a)(x-a) = f'(a)(x-a),$$

$$g(x) \approx g(a) + g'(a)(x-a) = g'(a)(x-a).$$

Therefore

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}.$$

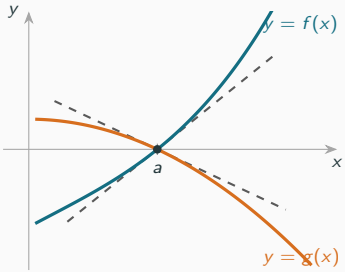


Figure 4.71: near a, $\frac{f(x)}{g(x)}$ is close to the ratio of the linear approximations.

L'Hôpital's rule: the case $0/0$

Theorem 4.12 (L'Hôpital's Rule, $0/0$ case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This also holds for one-sided limits, and for $a = \infty$ or $a = -\infty$.

L'Hôpital's rule: the case $0/0$

Theorem 4.12 (L'Hôpital's Rule, $0/0$ case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This also holds for one-sided limits, and for $a = \infty$ or $a = -\infty$.

Careful

We take the limit of the quotient of the derivatives $\frac{f'}{g'}$; we do not compute the derivative of the quotient $\frac{f}{g}$.

Proof in a special case

Assume that f, g, f' and g' are continuous over an open interval containing a, and that $g'(a) \neq 0$. Since $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$, continuity gives $f(a) = 0 = g(a)$. Therefore

Proof in a special case

Assume that f, g, f' and g' are continuous over an open interval containing a, and that $g'(a) \neq 0$. Since $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, continuity gives $f(a) = 0 = g(a)$. Therefore

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} \qquad \text{since } f(a) = 0 = g(a)$$

Proof in a special case

Assume that f, g, f' and g' are continuous over an open interval containing a, and that $g'(a) \neq 0$. Since $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, continuity gives $f(a) = 0 = g(a)$. Therefore

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} \qquad \text{since } f(a) = 0 = g(a)$$

$$= \lim_{x\to a} \frac{\big(f(x) - f(a)\big)/(x-a)}{\big(g(x) - g(a)\big)/(x-a)} \qquad \text{algebra}$$

Proof in a special case

Assume that f, g, f' and g' are continuous over an open interval containing a, and that $g'(a) \neq 0$. Since $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, continuity gives $f(a) = 0 = g(a)$. Therefore

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} && \text{since } f(a) = 0 = g(a) \\
&= \lim_{x\to a} \frac{\big(f(x) - f(a)\big)/(x-a)}{\big(g(x) - g(a)\big)/(x-a)} && \text{algebra} \\
&= \frac{f'(a)}{g'(a)} && \text{definition of the derivative}
\end{aligned}$$

Proof in a special case

Assume that f, g, f' and g' are continuous over an open interval containing a, and that $g'(a) \neq 0$. Since $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, continuity gives $f(a) = 0 = g(a)$. Therefore

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} && \text{since } f(a) = 0 = g(a) \\
&= \lim_{x\to a} \frac{\big(f(x) - f(a)\big)/(x-a)}{\big(g(x) - g(a)\big)/(x-a)} && \text{algebra} \\
&= \frac{f'(a)}{g'(a)} && \text{definition of the derivative} \\
&= \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} && \text{continuity of } f' \text{ and } g'
\end{aligned}$$

□

Example 4.38(a),(b) — applying L'Hôpital's rule $(0/0)$

(a) $\lim_{x \to 0} \dfrac{1 - \cos x}{x}$.

(b) $\lim_{x \to 1} \dfrac{\sin(\pi x)}{\ln x}$.

Example 4.38(a),(b) — applying L'Hôpital's rule $(0/0)$

(a) $\lim_{x\to 0} \dfrac{1-\cos x}{x}$. Since $1-\cos x \to 0$ and $x \to 0$, we apply L'Hôpital's rule:

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{\sin x}{1} = \frac{0}{1} = 0.$$

(b) $\lim_{x\to 1} \dfrac{\sin(\pi x)}{\ln x}$.

Example 4.38(a),(b) — applying L'Hôpital's rule $(0/0)$

(a) $\lim_{x\to 0} \dfrac{1-\cos x}{x}$. Since $1-\cos x \to 0$ and $x \to 0$, we apply L'Hôpital's rule:

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{\sin x}{1} = \frac{0}{1} = 0.$$

(b) $\lim_{x\to 1} \dfrac{\sin(\pi x)}{\ln x}$. As $x \to 1$, $\sin(\pi x) \to 0$ and $\ln x \to 0$, so

$$\lim_{x\to 1} \frac{\sin(\pi x)}{\ln x} = \lim_{x\to 1} \frac{\pi\cos(\pi x)}{1/x} = \lim_{x\to 1} (\pi x)\cos(\pi x) = (\pi\cdot 1)(-1) = -\pi.$$

Example 4.38(c),(d) — applying L'Hôpital's rule $(0/0)$

(c) $\lim_{x\to\infty} \dfrac{e^{1/x}-1}{1/x}$.

(d) $\lim_{x\to 0} \dfrac{\sin x - x}{x^2}$.

Example 4.38(c),(d) — applying L'Hôpital's rule (0/0)

(c) $\lim_{x\to\infty} \dfrac{e^{1/x}-1}{1/x}$. As $x\to\infty$, $e^{1/x}-1\to 0$ and $\frac{1}{x}\to 0$, so

$$\lim_{x\to\infty} \frac{e^{1/x}-1}{1/x} = \lim_{x\to\infty} \frac{e^{1/x}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} e^{1/x} = e^0 = 1.$$

(d) $\lim_{x\to 0} \dfrac{\sin x - x}{x^2}$.

Example 4.38(c),(d) — applying L'Hôpital's rule (0/0)

(c) $\lim_{x\to\infty} \dfrac{e^{1/x}-1}{1/x}$. As $x\to\infty$, $e^{1/x}-1\to 0$ and $\frac{1}{x}\to 0$, so

$$\lim_{x\to\infty} \frac{e^{1/x}-1}{1/x} = \lim_{x\to\infty} \frac{e^{1/x}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} e^{1/x} = e^0 = 1.$$

(d) $\lim_{x\to 0} \dfrac{\sin x - x}{x^2}$. Both numerator and denominator approach 0:

$$\lim_{x\to 0} \frac{\sin x - x}{x^2} = \lim_{x\to 0} \frac{\cos x - 1}{2x}.$$

Example 4.38(c),(d) — applying L'Hôpital's rule (0/0)

(c) $\lim_{x\to\infty} \dfrac{e^{1/x}-1}{1/x}$. As $x\to\infty$, $e^{1/x}-1\to 0$ and $\frac{1}{x}\to 0$, so

$$\lim_{x\to\infty} \frac{e^{1/x}-1}{1/x} = \lim_{x\to\infty} \frac{e^{1/x}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} e^{1/x} = e^0 = 1.$$

(d) $\lim_{x\to 0} \dfrac{\sin x - x}{x^2}$. Both numerator and denominator approach 0:

$$\lim_{x\to 0} \frac{\sin x - x}{x^2} = \lim_{x\to 0} \frac{\cos x - 1}{2x}.$$

This is again of the form $\frac{0}{0}$, so we apply L'Hôpital's rule again:

$$\lim_{x\to 0} \frac{\cos x - 1}{2x} = \lim_{x\to 0} \frac{-\sin x}{2} = 0. \qquad \text{Hence} \quad \lim_{x\to 0} \frac{\sin x - x}{x^2} = 0.$$

L'Hôpital's rule: the case ∞/∞

Theorem 4.13 (L'Hôpital's Rule, ∞/∞ case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim_{x\to a} f(x) = \infty$ (or $-\infty$) and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$). Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This also holds if $a = \infty$ or $-\infty$, or the limit is one-sided.

L'Hôpital's rule: the case ∞/∞

Theorem 4.13 (L'Hôpital's Rule, ∞/∞ case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim_{x\to a} f(x) = \infty$ (or $-\infty$) and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$). Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This also holds if $a = \infty$ or $-\infty$, or the limit is one-sided.

Example 4.39(a). $3x + 5 \to \infty$ and $2x + 1 \to \infty$ as $x \to \infty$, so

$$\lim_{x\to\infty} \frac{3x+5}{2x+1} = \lim_{x\to\infty} \frac{3}{2} = \frac{3}{2}.$$

(Dividing numerator and denominator by x, as in Section 4.6, gives the same answer.)

Example 4.39(b) — applying L'Hôpital's rule (∞/∞)

Evaluate $\displaystyle\lim_{x\to 0^+} \frac{\ln x}{\cot x}$.

Example 4.39(b) — applying L'Hôpital's rule (∞/∞)

Evaluate $\lim_{x\to 0^+} \dfrac{\ln x}{\cot x}$.

As $x \to 0^+$, $\ln x \to -\infty$ and $\cot x \to \infty$. By L'Hôpital's rule,

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} = \lim_{x\to 0^+} \frac{1/x}{-\csc^2 x} = \lim_{x\to 0^+} \frac{1}{-x\csc^2 x}.$$

Example 4.39(b) — applying L'Hôpital's rule (∞/∞)

Evaluate $\lim_{x\to 0^+} \dfrac{\ln x}{\cot x}$.

As $x \to 0^+$, $\ln x \to -\infty$ and $\cot x \to \infty$. By L'Hôpital's rule,

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} = \lim_{x\to 0^+} \frac{1/x}{-\csc^2 x} = \lim_{x\to 0^+} \frac{1}{-x\csc^2 x}.$$

Now $x \to 0$ but $\csc^2 x \to \infty$: we cannot conclude yet. Using $\csc x = \frac{1}{\sin x}$,

$$\lim_{x\to 0^+} \frac{1}{-x\csc^2 x} = \lim_{x\to 0^+} \frac{\sin^2 x}{-x}.$$

Example 4.39(b) — applying L'Hôpital's rule (∞/∞)

Evaluate $\lim_{x\to 0^+} \dfrac{\ln x}{\cot x}$.

As $x \to 0^+$, $\ln x \to -\infty$ and $\cot x \to \infty$. By L'Hôpital's rule,

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} = \lim_{x\to 0^+} \frac{1/x}{-\csc^2 x} = \lim_{x\to 0^+} \frac{1}{-x\csc^2 x}.$$

Now $x \to 0$ but $\csc^2 x \to \infty$: we cannot conclude yet. Using $\csc x = \frac{1}{\sin x}$,

$$\lim_{x\to 0^+} \frac{1}{-x\csc^2 x} = \lim_{x\to 0^+} \frac{\sin^2 x}{-x}.$$

This is of the form $\frac{0}{0}$; we apply L'Hôpital's rule again:

$$\lim_{x\to 0^+} \frac{\sin^2 x}{-x} = \lim_{x\to 0^+} \frac{2\sin x\cos x}{-1} = \frac{0}{-1} = 0. \qquad \text{So} \quad \lim_{x\to 0^+} \frac{\ln x}{\cot x} = 0.$$

Example 4.40 — when L'Hôpital's rule does not apply

Consider $\lim_{x \to 1} \dfrac{x^2 + 5}{3x + 4}$.

Example 4.40 — when L'Hôpital's rule does not apply

Consider $\lim_{x \to 1} \dfrac{x^2 + 5}{3x + 4}$.

The limits of the numerator and denominator are not both zero, nor both infinite: L'Hôpital's rule does not apply. If we tried to use it anyway, we would get

$$\lim_{x \to 1} \frac{2x}{3} = \frac{2}{3} \qquad \text{(wrong!)}.$$

Example 4.40 — when L'Hôpital's rule does not apply

Consider $\lim_{x\to 1} \dfrac{x^2+5}{3x+4}$.

The limits of the numerator and denominator are not both zero, nor both infinite: L'Hôpital's rule does not apply. If we tried to use it anyway, we would get

$$\lim_{x\to 1} \frac{2x}{3} = \frac{2}{3} \qquad \text{(wrong!)}.$$

In fact, $\lim_{x\to 1}(x^2+5) = 6$ and $\lim_{x\to 1}(3x+4) = 7$, so

$$\lim_{x\to 1} \frac{x^2+5}{3x+4} = \frac{6}{7} \neq \lim_{x\to 1} \frac{\frac{d}{dx}(x^2+5)}{\frac{d}{dx}(3x+4)}.$$

Example 4.40 — when L'Hôpital's rule does not apply

Consider $\lim_{x\to 1} \dfrac{x^2+5}{3x+4}$.

The limits of the numerator and denominator are not both zero, nor both infinite: L'Hôpital's rule does not apply. If we tried to use it anyway, we would get

$$\lim_{x\to 1} \frac{2x}{3} = \frac{2}{3} \qquad \text{(wrong!)}.$$

In fact, $\lim_{x\to 1}(x^2+5) = 6$ and $\lim_{x\to 1}(3x+4) = 7$, so

$$\lim_{x\to 1} \frac{x^2+5}{3x+4} = \frac{6}{7} \neq \lim_{x\to 1} \frac{\frac{d}{dx}(x^2+5)}{\frac{d}{dx}(3x+4)}.$$

Check first

Before applying L'Hôpital's rule, check that the limit is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Other indeterminate forms

Other indeterminate forms

The expressions

$$0 \cdot \infty, \qquad \infty - \infty, \qquad 1^{\infty}, \qquad \infty^{0}, \qquad 0^{0}$$

are also indeterminate forms. They are not real numbers: they describe forms that arise when evaluating certain limits.

Other indeterminate forms

The expressions

$$0 \cdot \infty, \qquad \infty - \infty, \qquad 1^{\infty}, \qquad \infty^{0}, \qquad 0^{0}$$

are also indeterminate forms. They are not real numbers: they describe forms that arise when evaluating certain limits.

Key idea: rewrite the expression so that we arrive at the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and then apply L'Hôpital's rule.

Indeterminate form of type $0 \cdot \infty$

Suppose $f(x) \to 0$ and $g(x) \to \infty$ (or $-\infty$) as $x \to a$. One term in the product $f(x)g(x)$ approaches zero, the other becomes arbitrarily large in magnitude: anything can happen to the product.

Indeterminate form of type $0 \cdot \infty$

Suppose $f(x) \to 0$ and $g(x) \to \infty$ (or $-\infty$) as $x \to a$. One term in the product $f(x)g(x)$ approaches zero, the other becomes arbitrarily large in magnitude: anything can happen to the product.

For example, let $f(x) = \dfrac{1}{x^n + 1} \to 0$ and $g(x) = 3x^2 \to \infty$ as $x \to \infty$. Then $f(x)g(x) = \dfrac{3x^2}{x^n + 1}$, and

$$\lim_{x \to \infty} f(x)g(x) = \begin{cases} 3 & \text{if } n = 2, \\ \infty & \text{if } n = 1, \\ 0 & \text{if } n = 3. \end{cases}$$

Example 4.41 — indeterminate form of type $0 \cdot \infty$

Evaluate $\lim_{x \to 0^+} x \ln x$.

Example 4.41 — indeterminate form of type $0 \cdot \infty$

Evaluate $\lim_{x \to 0^+} x \ln x$.

Rewrite the product as a quotient:

$$x \ln x = \frac{\ln x}{1/x}.$$

As $x \to 0^+$, $\ln x \to -\infty$ and $\frac{1}{x} \to \infty$. By L'Hôpital's rule,

Example 4.41 — indeterminate form of type $0 \cdot \infty$

Evaluate $\lim_{x \to 0^+} x \ln x$.

Rewrite the product as a quotient:

$$x \ln x = \frac{\ln x}{1/x}.$$

As $x \to 0^+$, $\ln x \to -\infty$ and $\frac{1}{x} \to \infty$. By L'Hôpital's rule,

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.$$

Example 4.41 — indeterminate form of type $0 \cdot \infty$

Evaluate $\lim_{x \to 0^+} x \ln x$.

Rewrite the product as a quotient:

$$x \ln x = \frac{\ln x}{1/x}.$$

As $x \to 0^+$, $\ln x \to -\infty$ and $\frac{1}{x} \to \infty$. By L'Hôpital's rule,

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.$$

We conclude that $\lim_{x \to 0^+} x \ln x = 0$.

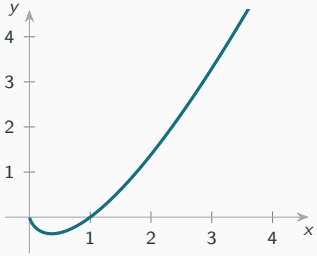


Figure 4.72: $f(x) = x \ln x$

Indeterminate form of type $\infty - \infty$

Let $f(x) = 3x^n$ and $g(x) = 3x^2 + 5$. As $x \to \infty$, $f(x) \to \infty$ and $g(x) \to \infty$, but:

$$\lim_{x\to\infty} \big(f(x) - g(x)\big) = \begin{cases} \lim_{x\to\infty} \left(3x^3 - 3x^2 - 5\right) = \infty & \text{if } n = 3, \\ \lim_{x\to\infty} \left(3x^2 - 3x^2 - 5\right) = -5 & \text{if } n = 2, \\ \lim_{x\to\infty} \left(3x - 3x^2 - 5\right) = -\infty & \text{if } n = 1. \end{cases}$$

The limit cannot be determined from "$\infty - \infty$" alone. We rewrite the expression as a fraction to apply L'Hôpital's rule.

Example 4.42 — indeterminate form of type $\infty - \infty$

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right)$.

Example 4.42 — indeterminate form of type $\infty - \infty$

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right)$.

Combining the fractions over the common denominator $x^2 \tan x$,

$$\frac{1}{x^2} - \frac{1}{\tan x} = \frac{\tan x - x^2}{x^2 \tan x} \qquad \left(\text{form } \frac{0}{0} \right).$$

Example 4.42 — indeterminate form of type $\infty - \infty$

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right)$.

Combining the fractions over the common denominator $x^2 \tan x$,

$$\frac{1}{x^2} - \frac{1}{\tan x} = \frac{\tan x - x^2}{x^2 \tan x} \qquad \left(\text{form } \frac{0}{0} \right).$$

By L'Hôpital's rule,

$$\lim_{x \to 0^+} \frac{\tan x - x^2}{x^2 \tan x} = \lim_{x \to 0^+} \frac{\sec^2 x - 2x}{x^2 \sec^2 x + 2x \tan x} = \infty,$$

since the numerator $\to 1$ and the denominator $\to 0$ through positive values.

Indeterminate forms involving exponents

The expressions 0^0, ∞^0 and 1^∞ are indeterminate forms. To evaluate $\lim_{x\to a} f(x)^{g(x)}$, we use the natural logarithm. Let

$$y = f(x)^{g(x)}. \qquad \text{Then} \qquad \ln y = \ln\left(f(x)^{g(x)}\right) = g(x)\ln\left(f(x)\right).$$

Indeterminate forms involving exponents

The expressions 0^0, ∞^0 and 1^∞ are indeterminate forms. To evaluate $\lim_{x\to a} f(x)^{g(x)}$, we use the natural logarithm. Let

$$y = f(x)^{g(x)}. \qquad \text{Then} \qquad \ln y = \ln\big(f(x)^{g(x)}\big) = g(x)\ln\big(f(x)\big).$$

The limit $\lim_{x\to a} g(x)\ln\big(f(x)\big)$ is typically of the form $0\cdot\infty$, which we can rewrite as a quotient and evaluate with L'Hôpital's rule.

Indeterminate forms involving exponents

The expressions 0^0, ∞^0 and 1^∞ are indeterminate forms. To evaluate $\lim_{x\to a} f(x)^{g(x)}$, we use the natural logarithm. Let

$$y = f(x)^{g(x)}. \qquad \text{Then} \qquad \ln y = \ln\big(f(x)^{g(x)}\big) = g(x)\ln\big(f(x)\big).$$

The limit $\lim_{x\to a} g(x)\ln\big(f(x)\big)$ is typically of the form $0\cdot\infty$, which we can rewrite as a quotient and evaluate with L'Hôpital's rule.

Suppose $\lim_{x\to a} \ln y = L$ (where L may be ∞ or $-\infty$). Since the natural logarithm is continuous, $\ln\big(\lim_{x\to a} y\big) = L$, and so

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} y = e^L.$$

Example 4.43 — indeterminate form of type ∞^0

Evaluate $\lim_{x \to \infty} x^{1/x}$.

Example 4.43 — indeterminate form of type ∞^0

Evaluate $\lim_{x\to\infty} x^{1/x}$.

Let $y = x^{1/x}$. Then

$$\ln y = \ln\left(x^{1/x}\right) = \frac{1}{x}\ln x = \frac{\ln x}{x}.$$

Example 4.43 — indeterminate form of type ∞^0

Evaluate $\lim_{x\to\infty} x^{1/x}$.

Let $y = x^{1/x}$. Then

$$\ln y = \ln\left(x^{1/x}\right) = \frac{1}{x}\ln x = \frac{\ln x}{x}.$$

By L'Hôpital's rule $\left(\frac{\infty}{\infty}\right)$,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Example 4.43 — indeterminate form of type ∞^0

Evaluate $\lim_{x\to\infty} x^{1/x}$.

Let $y = x^{1/x}$. Then

$$\ln y = \ln\left(x^{1/x}\right) = \frac{1}{x}\ln x = \frac{\ln x}{x}.$$

By L'Hôpital's rule $\left(\frac{\infty}{\infty}\right)$,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Since the natural logarithm is continuous, $\ln\left(\lim_{x\to\infty} y\right) = 0$, and so

$$\lim_{x\to\infty} x^{1/x} = e^0 = 1.$$

Example 4.44 — indeterminate form of type 0^0

Evaluate $\lim_{x \to 0^+} x^{\sin x}$.

Example 4.44 — indeterminate form of type 0^0

Evaluate $\lim_{x \to 0^+} x^{\sin x}$.

Let $y = x^{\sin x}$, so $\ln y = \sin x \ln x$, of the form $0 \cdot \infty$ as $x \to 0^+$.

Example 4.44 — indeterminate form of type 0^0

Evaluate $\lim_{x \to 0^+} x^{\sin x}$.

Let $y = x^{\sin x}$, so $\ln y = \sin x \ln x$, of the form $0 \cdot \infty$ as $x \to 0^+$.

Writing $\sin x \ln x = \frac{\sin x}{1/\ln x}$ leads to a more complicated limit. Instead, we write $\sin x \ln x = \dfrac{\ln x}{\csc x}$ $\left(\frac{\infty}{\infty}\right)$ and apply L'Hôpital's rule:

Example 4.44 — indeterminate form of type 0^0

Evaluate $\lim_{x\to 0^+} x^{\sin x}$.

Let $y = x^{\sin x}$, so $\ln y = \sin x \ln x$, of the form $0 \cdot \infty$ as $x \to 0^+$.

Writing $\sin x \ln x = \frac{\sin x}{1/\ln x}$ leads to a more complicated limit. Instead, we write $\sin x \ln x = \dfrac{\ln x}{\csc x}$ $\left(\frac{\infty}{\infty}\right)$ and apply L'Hôpital's rule:

$$\lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x \cos x} = \left(\lim_{x\to 0^+} \frac{\sin x}{x}\right)\left(\lim_{x\to 0^+} (-\tan x)\right) = 1 \cdot 0 = 0.$$

Example 4.44 — indeterminate form of type 0^0

Evaluate $\lim_{x\to 0^+} x^{\sin x}$.

Let $y = x^{\sin x}$, so $\ln y = \sin x \ln x$, of the form $0 \cdot \infty$ as $x \to 0^+$.

Writing $\sin x \ln x = \frac{\sin x}{1/\ln x}$ leads to a more complicated limit. Instead, we write $\sin x \ln x = \dfrac{\ln x}{\csc x}$ $\left(\frac{\infty}{\infty}\right)$ and apply L'Hôpital's rule:

$$\lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x \cos x} = \left(\lim_{x\to 0^+} \frac{\sin x}{x}\right)\left(\lim_{x\to 0^+} (-\tan x)\right) = 1 \cdot 0 = 0.$$

So $\lim_{x\to 0^+} \ln y = 0$, and therefore

$$\lim_{x\to 0^+} x^{\sin x} = e^0 = 1.$$

Growth rates of functions

Comparing growth rates

Both x^2 and x^3 approach ∞ as $x \to \infty$, but x^3 grows much faster:

x	10	100	1000	10,000
x^2	100	10,000	1,000,000	100,000,000
x^3	1000	1,000,000	1,000,000,000	1,000,000,000,000

Table 4.7

In fact, $\lim_{x\to\infty} \frac{x^3}{x^2} = \lim_{x\to\infty} x = \infty$. On the other hand, $\lim_{x\to\infty} \frac{x^2}{3x^2+4x+1} = \frac{1}{3}$: x^2 and $3x^2 + 4x + 1$ grow at the same rate.

Comparing growth rates

Both x^2 and x^3 approach ∞ as $x \to \infty$, but x^3 grows much faster:

x	10	100	1000	10,000
x^2	100	10,000	1,000,000	100,000,000
x^3	1000	1,000,000	1,000,000,000	1,000,000,000,000

Table 4.7

In fact, $\lim_{x\to\infty} \frac{x^3}{x^2} = \lim_{x\to\infty} x = \infty$. On the other hand, $\lim_{x\to\infty} \frac{x^2}{3x^2+4x+1} = \frac{1}{3}$: x^2 and $3x^2 + 4x + 1$ grow at the same rate.

Definition

Suppose $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$.

- g grows more rapidly than f if $\lim_{x\to\infty} \frac{g(x)}{f(x)} = \infty$, or equivalently, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.
- f and g grow at the same rate if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$ for some constant $M \neq 0$.

Example 4.45(a) — comparing x^2 and e^x

Evaluate $\lim_{x \to \infty} \frac{x^2}{e^x}$.

Example 4.45(a) — comparing x^2 and e^x

Evaluate $\lim_{x\to\infty} \frac{x^2}{e^x}$.

Since $x^2 \to \infty$ and $e^x \to \infty$, by L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}.$$

Example 4.45(a) — comparing x^2 and e^x

Evaluate $\lim_{x\to\infty} \frac{x^2}{e^x}$.

Since $x^2 \to \infty$ and $e^x \to \infty$, by L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}.$$

Again $2x \to \infty$ and $e^x \to \infty$, so we apply L'Hôpital's rule again:

$$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

Example 4.45(a) — comparing x^2 and e^x

Evaluate $\lim_{x\to\infty} \frac{x^2}{e^x}$.

Since $x^2 \to \infty$ and $e^x \to \infty$, by L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}.$$

Again $2x \to \infty$ and $e^x \to \infty$, so we apply L'Hôpital's rule again:

$$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

Hence $\lim_{x\to\infty} \frac{x^2}{e^x} = 0$: e^x grows more rapidly than x^2 as $x \to \infty$.

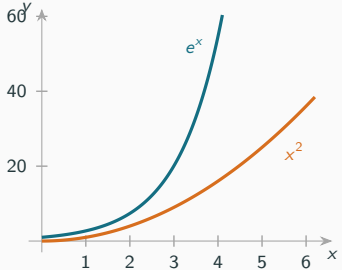


Figure 4.73: an exponential function grows faster than a power function.

Example 4.45(b) — comparing $\ln x$ and x^2

Evaluate $\lim_{x \to \infty} \frac{\ln x}{x^2}$.

Example 4.45(b) — comparing $\ln x$ and x^2

Evaluate $\lim_{x\to\infty} \dfrac{\ln x}{x^2}$.

Since $\ln x \to \infty$ and $x^2 \to \infty$, by L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

Example 4.45(b) — comparing $\ln x$ and x^2

Evaluate $\lim_{x\to\infty} \dfrac{\ln x}{x^2}$.

Since $\ln x \to \infty$ and $x^2 \to \infty$, by L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

Thus x^2 grows more rapidly than $\ln x$ as $x \to \infty$.

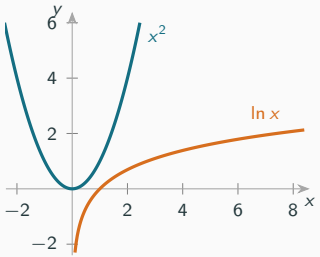


Figure 4.74: a power function grows faster than a logarithmic function.

Exponentials, powers and logarithms

Using the same ideas as in Example 4.45:

- e^x grows more rapidly than x^p for any $p > 0$;
- x^p grows more rapidly than $\ln x$ for any $p > 0$.

Exponentials, powers and logarithms

Using the same ideas as in Example 4.45:

- e^x grows more rapidly than x^p for any $p > 0$;
- x^p grows more rapidly than $\ln x$ for any $p > 0$.

For instance, $\ln x$ grows more slowly than $\sqrt[3]{x}$ and $\sqrt{x}$:

x	10	100	1000	10,000
$\ln x$	2.303	4.605	6.908	9.210
$\sqrt[3]{x}$	2.154	4.642	10	21.544
$\sqrt{x}$	3.162	10	31.623	100

Table 4.12

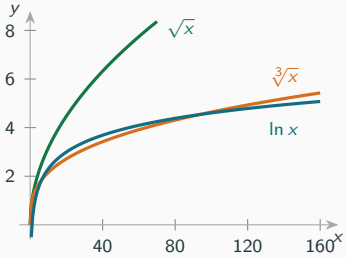


Figure 4.76: $\ln x$ grows more slowly than x^p for any $p > 0$.

Summary

Key points

- **L'Hôpital's rule:** if $\lim \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$ (if the latter exists or is $\pm\infty$). Check the form first!
- $0 \cdot \infty$ **and** $\infty - \infty$**:** rewrite as a quotient of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
- 0^0**,** ∞^0**,** 1^∞**:** set $y = f(x)^{g(x)}$, compute $\lim \ln y = L$; then $\lim y = e^L$.
- **Growth rates:** $\ln x \ll x^p \ll e^x$ as $x \to \infty$, for any $p > 0$.