Limits at Infinity and Asymptotes

Calculus 1 — Section 4.6

Limits at infinity · end behavior · guidelines for drawing the graph of a function

Limits at infinity

Limits at infinity and horizontal asymptotes

Consider $f(x) = 2 + \dfrac{1}{x}$. As x gets larger, $f(x)$ approaches 2:

x	10	100	1000	10,000
$2 + \frac{1}{x}$	2.1	2.01	2.001	2.0001

x	-10	-100	-1000	$-10{,}000$
$2 + \frac{1}{x}$	1.9	1.99	1.999	1.9999

Table 4.2

We write

$$\lim_{x \to \infty} f(x) = 2 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2.$$

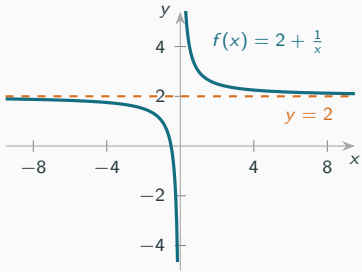


Figure 4.40: f approaches $y = 2$ as $x \to \pm\infty$.

Limits at infinity — definition

Definition

If the values of $f(x)$ become arbitrarily close to L as x becomes sufficiently large, we say that f has a limit at infinity and write $\lim_{x \to \infty} f(x) = L$.

If the values of $f(x)$ become arbitrarily close to L for $x < 0$ as $|x|$ becomes sufficiently large, we say that f has a limit at negative infinity and write $\lim_{x \to -\infty} f(x) = L$.

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Definition

If $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, we say the line $y = L$ is a horizontal asymptote of f.

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Definition

If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, we say the line $y = L$ is a horizontal asymptote of f.

For example, since $\lim_{x\to\infty} \frac{1}{x} = 0$, the line $y = 0$ is a horizontal asymptote of $f(x) = \frac{1}{x}$.

Crossing a horizontal asymptote

A graph cannot cross a vertical asymptote. However, it may cross a horizontal asymptote, even an unlimited number of times.

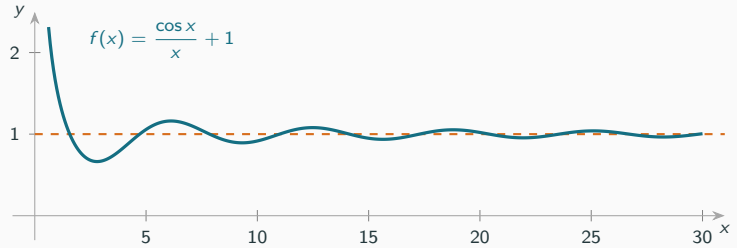


Figure 4.42: f crosses its horizontal asymptote $y = 1$ an infinite number of times.

The algebraic limit laws and the squeeze theorem also apply to limits at infinity.

Example 4.21(a) — computing limits at infinity

Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = 5 - \dfrac{2}{x^2}$, and determine its horizontal asymptotes.

Example 4.21(a) — computing limits at infinity

Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = 5 - \dfrac{2}{x^2}$, and determine its horizontal asymptotes.

By the algebraic limit laws,

$$\lim_{x\to\infty}\left(5 - \frac{2}{x^2}\right) = 5 - 2\Big(\lim_{x\to\infty}\frac{1}{x}\Big)^2 = 5 - 2\cdot 0 = 5.$$

Example 4.21(a) — computing limits at infinity

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Similarly, $\lim_{x\to-\infty} f(x) = 5$. Therefore $y = 5$ is a horizontal asymptote, and f approaches it as $x \to \pm\infty$.

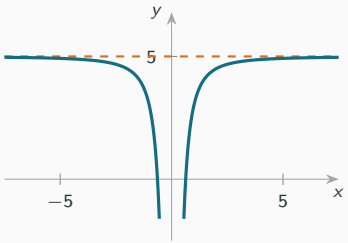


Figure 4.43: $f(x) = 5 - \frac{2}{x^2}$

Example 4.21(b) — computing limits at infinity

Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = \dfrac{\sin x}{x}$, and determine its horizontal asymptotes.

Example 4.21(b) — computing limits at infinity

Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = \dfrac{\sin x}{x}$, and determine its horizontal asymptotes.

Since $-1 \le \sin x \le 1$, we have

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x} \qquad \text{for } x > 0,$$

and $\lim_{x\to\infty}\left(-\frac{1}{x}\right) = 0 = \lim_{x\to\infty}\frac{1}{x}$. By the squeeze theorem, $\lim_{x\to\infty} \dfrac{\sin x}{x} = 0$.

Example 4.21(b) — computing limits at infinity

Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = \dfrac{\sin x}{x}$, and determine its horizontal asymptotes.

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Similarly, $\lim_{x\to-\infty} \dfrac{\sin x}{x} = 0$. So $y = 0$ is a horizontal asymptote, which the graph crosses infinitely many times.

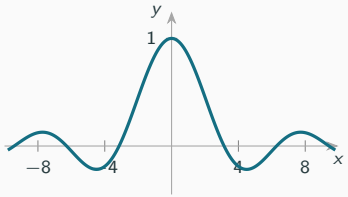


Figure 4.44: $f(x) = \frac{\sin x}{x}$

Example 4.21(c) — computing limits at infinity

Evaluate $\lim_{x\to\infty} \tan^{-1} x$ and $\lim_{x\to-\infty} \tan^{-1} x$.

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$\tan^{-1}$ is the inverse of tan restricted to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and

$$\lim_{x\to(\pi/2)^-} \tan x = \infty, \qquad \lim_{x\to(-\pi/2)^+} \tan x = -\infty.$$

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It follows that

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}, \qquad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}.$$

So $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes of $\tan^{-1} x$.

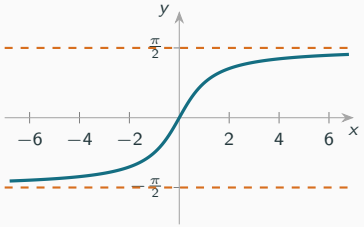


Figure 4.46: $f(x) = \tan^{-1} x$ has two horizontal asymptotes.

Infinite limits at infinity

For $f(x) = x^3$, the values become arbitrarily large as $x \to \infty$, and negative but arbitrarily large in magnitude as $x \to -\infty$:

$$\lim_{x \to \infty} x^3 = \infty, \qquad \lim_{x \to -\infty} x^3 = -\infty.$$

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Definition

f has an infinite limit at infinity, $\lim_{x\to\infty} f(x) = \infty$, if $f(x)$ becomes arbitrarily large for x sufficiently large; and $\lim_{x\to\infty} f(x) = -\infty$ if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large for x sufficiently large. Similarly as $x \to -\infty$.

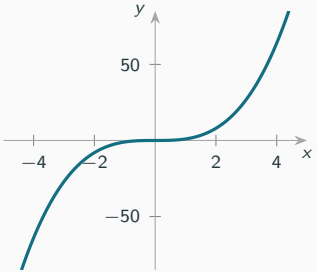


Figure 4.47: $f(x) = x^3$

End behavior

End behavior and power functions

The behavior of f as $x \to \pm\infty$ is its end behavior. At each end, $f(x)$ may approach a horizontal asymptote $y = L$, tend to ∞ or $-\infty$, or do neither (e.g. oscillate).

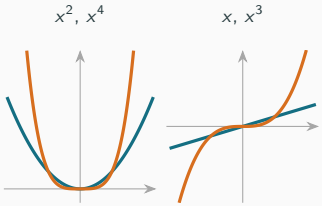
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For $n = 1, 2, 3, \ldots$ (Figures 4.50 and 4.51):

$$\lim_{x\to\infty} x^n = \infty, \qquad \lim_{x\to-\infty} x^n = \begin{cases} \infty & n \text{ even}, \\ -\infty & n \text{ odd}. \end{cases}$$

For a constant $c \neq 0$, $\lim cx^n = \lim x^n$ if $c > 0$ and $\lim cx^n = -\lim x^n$ if $c < 0$ (as $x \to \infty$ or $x \to -\infty$).

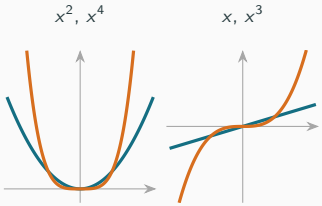


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Example 4.24. $\lim_{x\to\infty}(-5x^3) = -\infty$, $\lim_{x\to-\infty}(-5x^3) = \infty$; $\lim_{x\to\infty} 2x^4 = \infty$, $\lim_{x\to-\infty} 2x^4 = \infty$.

End behavior of polynomials

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ with $n \geq 1$ and $a_n \neq 0$. Factoring,

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n} \frac{1}{x} + \cdots + \frac{a_0}{a_n} \frac{1}{x^n}\right).$$

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As $x \to \pm\infty$, all the terms inside the parentheses except the first approach 0. Hence

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} a_n x^n.$$

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As $x \to \pm\infty$, all the terms inside the parentheses except the first approach 0. Hence

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} a_n x^n.$$

For example, $f(x) = 5x^3 - 3x^2 + 4$ behaves like $g(x) = 5x^3$ as $x \to \pm\infty$.

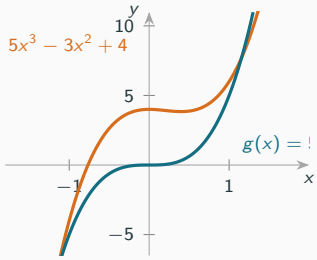


Figure 4.52: the end behavior is determined by the term with the largest exponent.

Example 4.25(a) — end behavior of a rational function

Determine the limits as $x \to \pm\infty$ of $f(x) = \dfrac{3x - 1}{2x + 5}$ (numerator and denominator have the same degree).

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Divide numerator and denominator by x, the highest power of x in the denominator:

$$\lim_{x\to\pm\infty} \frac{3x-1}{2x+5} = \lim_{x\to\pm\infty} \frac{3-1/x}{2+5/x} = \frac{3-0}{2+0} = \frac{3}{2}.$$

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So $y = \frac{3}{2}$ is a horizontal asymptote of f.

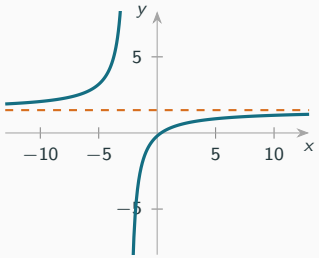


Figure 4.53: f approaches $y = \frac{3}{2}$ as $x \to \pm\infty$.

Example 4.25(b) — end behavior of a rational function

Determine the limits as $x \to \pm\infty$ of $f(x) = \dfrac{3x^2 + 2x}{4x^3 - 5x + 7}$ (the degree of the numerator is less than the degree of the denominator).

Example 4.25(b) — end behavior of a rational function

Determine the limits as $x \to \pm\infty$ of $f(x) = \dfrac{3x^2 + 2x}{4x^3 - 5x + 7}$ (the degree of the numerator is less than the degree of the denominator).

Divide numerator and denominator by x^3:

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{3/x + 2/x^2}{4 - 5/x^2 + 7/x^3} = \frac{0}{4} = 0.$$

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So $y = 0$ is a horizontal asymptote of f.

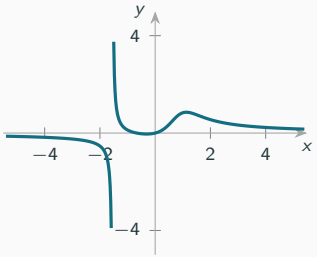


Figure 4.54: f approaches $y = 0$ as $x \to \pm\infty$.

Example 4.25(c) — an oblique asymptote

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Dividing by x: $f(x) = \dfrac{3x + 4}{1 + 2/x}$. The denominator tends to 1, so $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

Example 4.25(c) — an oblique asymptote

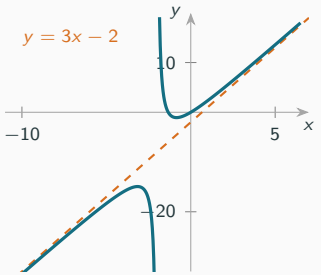
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By long division,

$$f(x) = 3x - 2 + \frac{4}{x + 2}.$$

Since $\lim_{x\to\pm\infty} \frac{4}{x+2} = 0$, we get $f(x) - (3x - 2) \to 0$ as $x \to \pm\infty$. The graph of f approaches the line $y = 3x - 2$ as $x \to \pm\infty$: an oblique asymptote.



Figures 4.55–4.56: $f(x) = \frac{3x^2+4x}{x+2}$

End behavior of rational functions

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$, with $a_n \neq 0$ and $b_m \neq 0$.

End behavior of rational functions

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1. If $n = m$, then f has the horizontal asymptote $y = \frac{a_n}{b_m}$ as $x \to \pm\infty$.

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2. If $n < m$, then f has the horizontal asymptote $y = 0$ as $x \to \pm\infty$.

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2. If $n < m$, then f has the horizontal asymptote $y = 0$ as $x \to \pm\infty$.
3. If $n > m$, then f has no horizontal asymptote: the limits at infinity are ∞ or $-\infty$. By long division,

$$f(x) = g(x) + \frac{r(x)}{q(x)}, \qquad \deg r < \deg q,$$

so $\lim_{x\to\pm\infty} \frac{r(x)}{q(x)} = 0$ and $f(x) - g(x) \to 0$. If $n = m + 1$, then g is linear and the line $y = g(x)$ is an oblique asymptote.

Example 4.26 — a function involving a radical

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$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3-2/x}{\sqrt{4+5/x^2}} = \frac{3}{\sqrt{4}} = \frac{3}{2},$$

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Two horizontal asymptotes: $y = \frac{3}{2}$ as $x \to \infty$ and $y = -\frac{3}{2}$ as $x \to -\infty$.

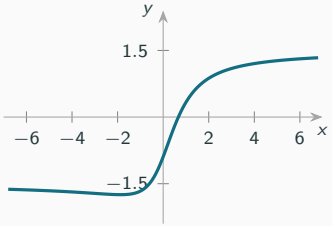


Figure 4.57: two horizontal asymptotes.

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For large $|x|$, $\sqrt{4x^2+5} \approx \sqrt{4x^2} = 2|x|$. We divide by $|x| = \sqrt{x^2}$, with $|x| = x$ for $x > 0$ and $|x| = -x$ for $x < 0$:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3-2/x}{\sqrt{4+5/x^2}} = \frac{3}{\sqrt{4}} = \frac{3}{2},$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{-3+2/x}{\sqrt{4+5/x^2}} = -\frac{3}{2}.$$

Two horizontal asymptotes: $y = \frac{3}{2}$ as $x \to \infty$ and $y = -\frac{3}{2}$ as $x \to -\infty$.

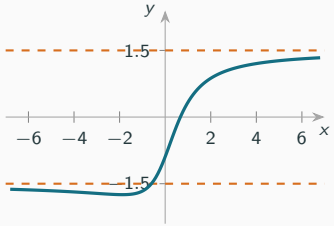


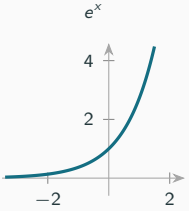
Figure 4.57: two horizontal asymptotes.

End behavior of transcendental functions

- $\sin x$ and $\cos x$ oscillate between -1 and 1; $\tan x$ has infinitely many vertical asymptotes. They have no limit (finite or infinite) as $x \to \pm\infty$.

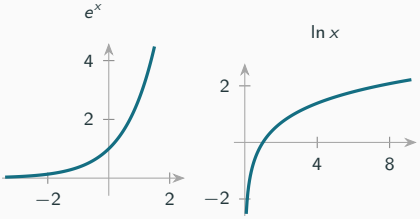
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End behavior of transcendental functions

- $\sin x$ and $\cos x$ oscillate between -1 and 1; $\tan x$ has infinitely many vertical asymptotes. They have no limit (finite or infinite) as $x \to \pm\infty$.
- $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$.
- $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to 0^+} \ln x = -\infty$.



Example 4.27 — a transcendental function

Find the limits as $x \to \pm\infty$ of $f(x) = \dfrac{2 + 3e^x}{7 - 5e^x}$.

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Find the limits as $x \to \pm\infty$ of $f(x) = \dfrac{2 + 3e^x}{7 - 5e^x}$.

As $x \to \infty$, divide by e^x:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2/e^x + 3}{7/e^x - 5} = \frac{0+3}{0-5} = -\frac{3}{5},$$

since $\lim_{x\to\infty} \frac{2}{e^x} = 0 = \lim_{x\to\infty} \frac{7}{e^x}$.

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As $x \to -\infty$, $e^x \to 0$, so $\lim_{x\to-\infty} f(x) = \dfrac{2}{7}$.

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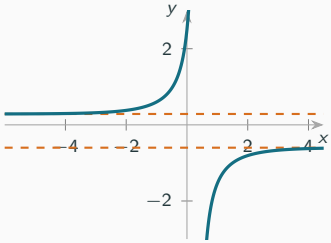
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As $x \to -\infty$, $e^x \to 0$, so $\lim_{x\to-\infty} f(x) = \dfrac{2}{7}$.

Horizontal asymptotes: $y = -\frac{3}{5}$ as $x \to \infty$ and $y = \frac{2}{7}$ as $x \to -\infty$.



$f(x) = \frac{2+3e^x}{7-5e^x}$ (vertical asymptote at $x = \ln\frac{7}{5}$).

Guidelines for drawing the graph of a function

Drawing the graph of a function

Problem-solving strategy: drawing the graph of a function

1. Determine the domain of the function.
2. Locate the x- and y-intercepts.
3. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ to determine the end behavior. If either is a finite number L, then $y = L$ is a horizontal asymptote. If either is $\pm\infty$, determine whether f has an oblique asymptote.
4. Determine whether f has any vertical asymptotes.
5. Calculate f'. Find all critical points, the intervals where f is increasing and decreasing, and the local extrema.
6. Calculate f''. Find the intervals where f is concave up and concave down, and the inflection points.

Example 4.28 — sketching a polynomial

Sketch a graph of $f(x) = (x-1)^2(x+2)$.

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Step 2. $f(0) = 2$: y-intercept $(0, 2)$; x-intercepts $(1, 0)$ and $(-2, 0)$.

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$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$.

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Step 5. $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$: f is increasing on $(-\infty, -1)$ and $(1, \infty)$, decreasing on $(-1, 1)$. Local maximum $f(-1) = 4$, local minimum $f(1) = 0$.

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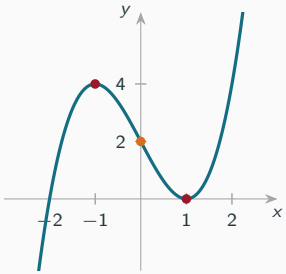
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$f(x) = (x-1)^2(x+2)$

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Step 4. The denominator is zero at $x = \pm 1$, and

$$\lim_{x\to 1^-} f(x) = \infty, \quad \lim_{x\to 1^+} f(x) = -\infty, \quad \lim_{x\to -1^-} f(x) = -\infty, \quad \lim_{x\to -1^+} f(x) = \infty :$$

$x = 1$ and $x = -1$ are vertical asymptotes.

Example 4.29 — derivatives and graph

Step 5. $f'(x) = \dfrac{2x}{(1-x^2)^2}$: the only critical point is $x = 0$. $f' < 0$ on $(-\infty, -1)$ and $(-1, 0)$; $f' > 0$ on $(0, 1)$ and $(1, \infty)$. So f has a local minimum at $x = 0$ and no local maximum.

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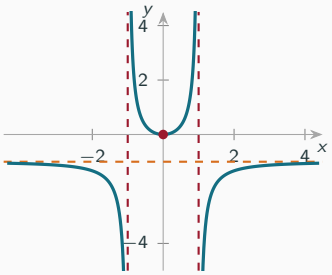
Step 6. $f''(x) = \dfrac{6x^2 + 2}{(1-x^2)^3}$, never zero. $f'' < 0$ on $(-\infty, -1)$ and $(1, \infty)$: concave down; $f'' > 0$ on $(-1, 1)$: concave up.

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f changes concavity at $x = \pm 1$, but there are no inflection points there, since f is not continuous at $x = \pm 1$.



$f(x) = \frac{x^2}{1-x^2}$

Example 4.30 — a rational function with an oblique asymptote

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Sketch the graph of $f(x) = \dfrac{x^2}{x-1}$.

Step 1. Domain: all real numbers except $x = 1$.

Step 2. The only intercept is $(0, 0)$.

Step 3. The degree of the numerator is one more than the degree of the denominator. By long division,

$$f(x) = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1},$$

and $\frac{1}{x-1} \to 0$ as $x \to \pm\infty$: $y = x + 1$ is an oblique asymptote.

Example 4.30 — a rational function with an oblique asymptote

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Step 4. $\lim_{x\to 1^+} f(x) = \infty$ and $\lim_{x\to 1^-} f(x) = -\infty$: $x = 1$ is a vertical asymptote.

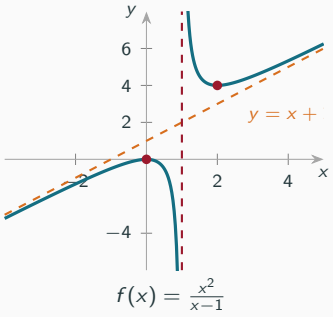
Example 4.30 — derivatives and graph

Step 5. $f'(x) = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$: critical points $x = 0$ and $x = 2$. $f' > 0$ on $(-\infty, 0)$ and $(2, \infty)$, $f' < 0$ on $(0, 1)$ and $(1, 2)$. Local maximum $f(0) = 0$, local minimum $f(2) = 4$.

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Example 4.31 — a function with a cusp

Sketch a graph of $f(x) = (x-1)^{2/3} = \left(\sqrt[3]{x-1}\right)^2$.

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Step 5. $f'(x) = \dfrac{2}{3(x-1)^{1/3}}$ is never zero and is undefined at $x = 1$. $f' < 0$ on $(-\infty, 1)$ and $f' > 0$ on $(1, \infty)$: local minimum $f(1) = 0$. Since

$$\lim_{x\to 1^+} f'(x) = \infty, \qquad \lim_{x\to 1^-} f'(x) = -\infty,$$

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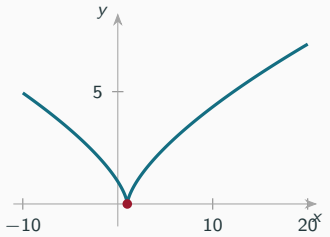
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Step 6. $f''(x) = -\dfrac{2}{9(x-1)^{4/3}} < 0$ for $x \neq 1$: f is concave down on $(-\infty, 1)$ and $(1, \infty)$.



$f(x) = (x-1)^{2/3}$: a cusp at $(1,0)$.

Summary

Key points

- If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then $y = L$ is a horizontal asymptote; a graph may cross it.
- Polynomials behave like their leading term $a_n x^n$ as $x \to \pm\infty$.
- Rational functions: degree of the numerator $=$, $<$, $>$ degree of the denominator gives the horizontal asymptote $y = \frac{a_n}{b_m}$, $y = 0$, or no horizontal asymptote (an oblique asymptote when it exceeds it by one).
- For radicals, divide by the appropriate power of $|x|$; for exponentials, use $e^x \to \infty$ as $x \to \infty$ and $e^x \to 0$ as $x \to -\infty$.
- To graph f: domain, intercepts, end behavior, vertical asymptotes, then f' (increasing/decreasing, extrema) and f'' (concavity, inflection points).