Derivatives and the Shape of a Graph

Calculus 1 — Section 4.5

The first derivative test · concavity and points of inflection · the second derivative test

The first derivative test

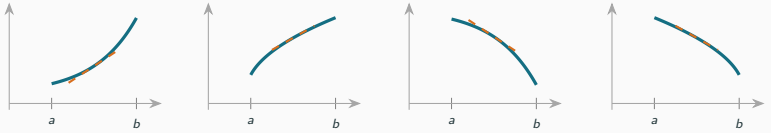
The sign of f'

A critical point is only a *candidate* for a local extremum: for $f(x) = x^3$, $f'(0) = 0$ but f has no local extremum at 0. We now decide which critical points give local extrema.

The sign of f'

A critical point is only a *candidate* for a local extremum: for $f(x) = x^3$, $f'(0) = 0$ but f has no local extremum at 0. We now decide which critical points give local extrema.

By Corollary 3 of the Mean Value Theorem: if $f' > 0$ over an interval, f is increasing there; if $f' < 0$, f is decreasing.



(a) $f' > 0$: increasing (b) $f' > 0$: increasing (c) $f' < 0$: decreasing (d) $f' < 0$: decreasing

Figure 4.30

Local extrema and sign changes of f'

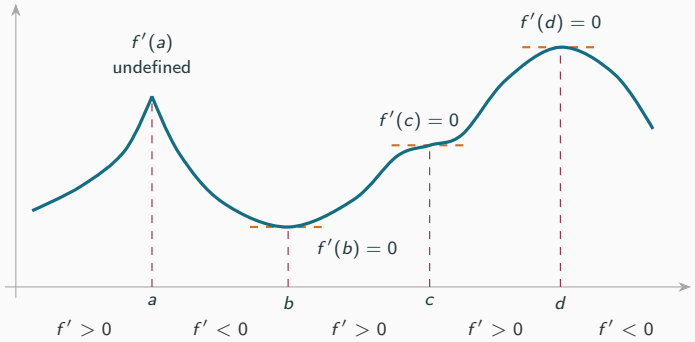


Figure 4.31: local maxima at a and d, a local minimum at b, no local extremum at c.

Local extrema and sign changes of f'

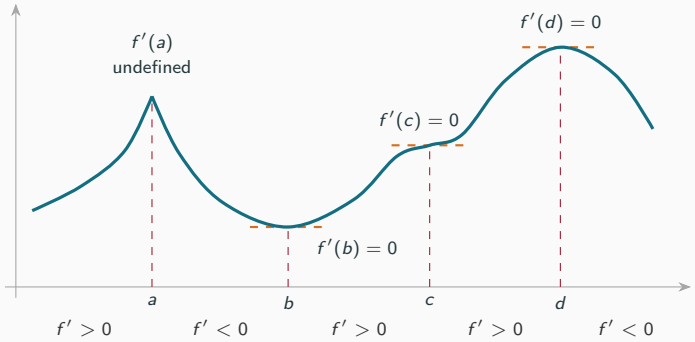


Figure 4.31: local maxima at a and d, a local minimum at b, no local extremum at c.

- If a continuous function f has a local extremum, it must occur at a critical point c.
- f has a local extremum at the critical point c if and only if f' changes sign as x increases through c.

The first derivative test

Theorem 4.9 (First Derivative Test)

Suppose that f is continuous over an interval I containing a critical point c, and differentiable over I, except possibly at c.

i. If f' changes sign from positive for $x < c$ to negative for $x > c$, then $f(c)$ is a local maximum of f.

ii. If f' changes sign from negative for $x < c$ to positive for $x > c$, then $f(c)$ is a local minimum of f.

iii. If f' has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of f.

Using the first derivative test

Problem-solving strategy: using the first derivative test

Consider a function f that is continuous over an interval I.

1. Find all critical points of f and divide the interval into smaller intervals, using the critical points as endpoints.
2. On each subinterval, choose a test point and determine the sign of f' there; f is increasing or decreasing over that subinterval accordingly.
3. Use the first derivative test to decide whether f has a local maximum, a local minimum, or neither at each critical point.

Note. If f' is not continuous throughout I, include its points of discontinuity, along with the critical points, as endpoints of the subintervals.

Example 4.17 — using the first derivative test

Use the first derivative test to find all local extrema of $f(x) = x^3 - 3x^2 - 9x - 1$.

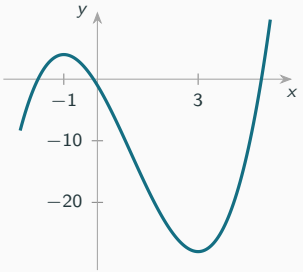


Figure 4.32: $f(x) = x^3 - 3x^2 - 9x - 1$

Example 4.17 — using the first derivative test

Use the first derivative test to find all local extrema of $f(x) = x^3 - 3x^2 - 9x - 1$.

Step 1. $f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$, so the critical points are $x = -1$ and $x = 3$.

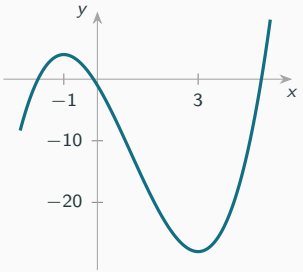


Figure 4.32: $f(x) = x^3 - 3x^2 - 9x - 1$

Example 4.17 — using the first derivative test

Use the first derivative test to find all local extrema of $f(x) = x^3 - 3x^2 - 9x - 1$.

Step 1. $f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1)$, so the critical points are $x = -1$ and $x = 3$.

Step 2.

Interval	Test point	Sign of f'	Conclusion
$(-\infty, -1)$	-2	$(+)(-)(-) = +$	increasing
$(-1, 3)$	0	$(+)(-)(+) = -$	decreasing
$(3, \infty)$	4	$(+)(+)(+) = +$	increasing

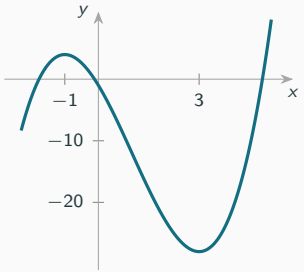


Figure 4.32: $f(x) = x^3 - 3x^2 - 9x - 1$

Example 4.17 — using the first derivative test

Use the first derivative test to find all local extrema of $f(x) = x^3 - 3x^2 - 9x - 1$.

Step 1. $f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1)$, so the critical points are $x = -1$ and $x = 3$.

Step 2.

Interval	Test point	Sign of f'	Conclusion
$(-\infty, -1)$	-2	$(+)(-)(-) = +$	increasing
$(-1, 3)$	0	$(+)(-)(+) = -$	decreasing
$(3, \infty)$	4	$(+)(+)(+) = +$	increasing

Step 3. f' changes from $+$ to $-$ at -1: a local maximum $f(-1) = 4$. It changes from $-$ to $+$ at 3: a local minimum $f(3) = -28$.

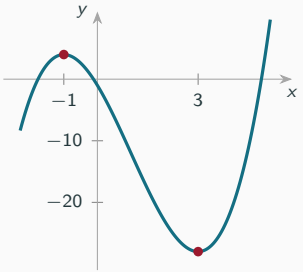


Figure 4.32: $f(x) = x^3 - 3x^2 - 9x - 1$

Example 4.18 — using the first derivative test

Use the first derivative test to find all local extrema of $f(x) = 5x^{1/3} - x^{5/3}$.

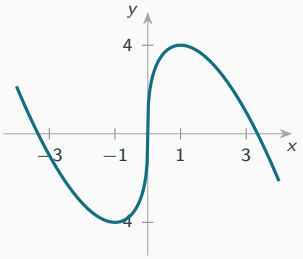


Figure 4.33: $f(x) = 5x^{1/3} - x^{5/3}$

Example 4.18 — using the first derivative test

Use the first derivative test to find all local extrema of $f(x) = 5x^{1/3} - x^{5/3}$.

Step 1. $f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \dfrac{5(1 - x^{4/3})}{3x^{2/3}}$. It is zero at $x = \pm 1$ and undefined at $x = 0$: the critical points are $x = -1, 0, 1$.

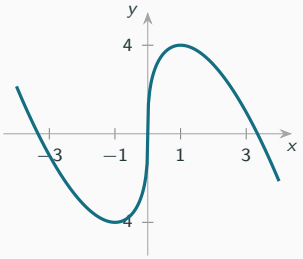


Figure 4.33: $f(x) = 5x^{1/3} - x^{5/3}$

Example 4.18 — using the first derivative test

Use the first derivative test to find all local extrema of $f(x) = 5x^{1/3} - x^{5/3}$.

Step 1. $f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \dfrac{5(1 - x^{4/3})}{3x^{2/3}}$. It is zero at $x = \pm 1$ and undefined at $x = 0$: the critical points are $x = -1, 0, 1$.

Step 2.

Interval	Test point	Sign of f'	Conclusion
$(-\infty, -1)$	-2	$-$	decreasing
$(-1, 0)$	$-\frac{1}{2}$	$+$	increasing
$(0, 1)$	$\frac{1}{2}$	$+$	increasing
$(1, \infty)$	2	$-$	decreasing

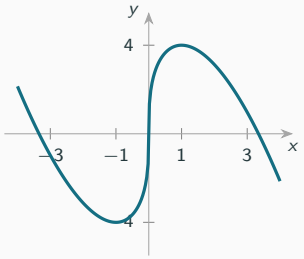


Figure 4.33: $f(x) = 5x^{1/3} - x^{5/3}$

Example 4.18 — using the first derivative test

Use the first derivative test to find all local extrema of $f(x) = 5x^{1/3} - x^{5/3}$.

Step 1. $f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \dfrac{5(1 - x^{4/3})}{3x^{2/3}}$. It is zero at $x = \pm 1$ and undefined at $x = 0$: the critical points are $x = -1, 0, 1$.

Step 2.

Interval	Test point	Sign of f'	Conclusion
$(-\infty, -1)$	-2	$-$	decreasing
$(-1, 0)$	$-\frac{1}{2}$	$+$	increasing
$(0, 1)$	$\frac{1}{2}$	$+$	increasing
$(1, \infty)$	2	$-$	decreasing

Step 3. A local minimum $f(-1) = -4$, a local maximum $f(1) = 4$, and no local extremum at $x = 0$.

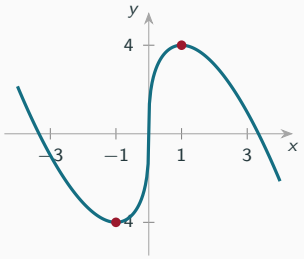


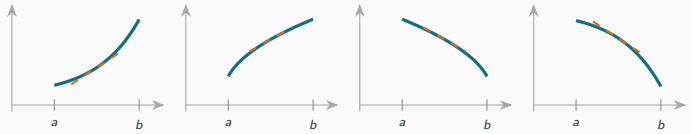
Figure 4.33: $f(x) = 5x^{1/3} - x^{5/3}$

Concavity and points of inflection

Concavity

Definition

Let f be differentiable over an open interval I. If f' is increasing over I, we say f is concave up over I. If f' is decreasing over I, we say f is concave down over I.



(a) f' increasing: concave up (b) f' decreasing: concave down (c) f' increasing: concave up (d) f' decreasing: concave down

Figure 4.34: concave up — the graph lies above its tangent lines; concave down — below.

The test for concavity and inflection points

If $f'' > 0$, then f' is increasing (Corollary 3); if $f'' < 0$, then f' is decreasing.

Theorem 4.10 (Test for Concavity)

Let f be twice differentiable over an interval I.

i. If $f''(x) > 0$ for all $x \in I$, then f is concave up over I.

ii. If $f''(x) < 0$ for all $x \in I$, then f is concave down over I.

The test for concavity and inflection points

If $f'' > 0$, then f' is increasing (Corollary 3); if $f'' < 0$, then f' is decreasing.

Theorem 4.10 (Test for Concavity)

Let f be twice differentiable over an interval I.

i. If $f''(x) > 0$ for all $x \in I$, then f is concave up over I.

ii. If $f''(x) < 0$ for all $x \in I$, then f is concave down over I.

Definition

If f is continuous at a and f changes concavity at a, the point $(a, f(a))$ is an inflection point of f.

The test for concavity and inflection points

If $f'' > 0$, then f' is increasing (Corollary 3); if $f'' < 0$, then f' is decreasing.

Theorem 4.10 (Test for Concavity)

Let f be twice differentiable over an interval I.

i. If $f''(x) > 0$ for all $x \in I$, then f is concave up over I.

ii. If $f''(x) < 0$ for all $x \in I$, then f is concave down over I.

Definition

If f is continuous at a and f changes concavity at a, the point $(a, f(a))$ is an inflection point of f.

f can change concavity only where $f''(x) = 0$ or $f''(x)$ is undefined; but it need not change concavity there.

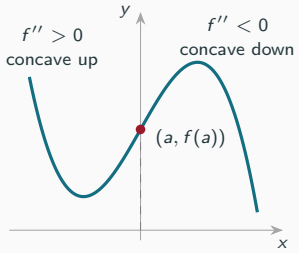


Figure 4.35: an inflection point.

Example 4.19 — testing for concavity

For $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up and where it is concave down. List all inflection points.

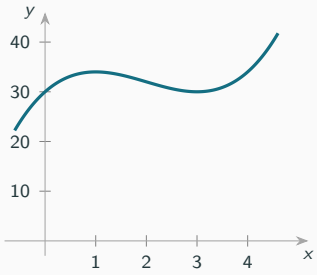


Figure 4.36: an inflection point at $(2, 32)$.

Example 4.19 — testing for concavity

For $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up and where it is concave down. List all inflection points.

$f'(x) = 3x^2 - 12x + 9$ and $f''(x) = 6x - 12$. Since f'' is defined everywhere, f can change concavity only where $f''(x) = 0$, that is, at $x = 2$.

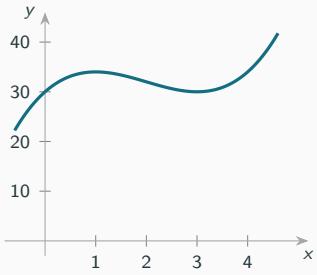


Figure 4.36: an inflection point at $(2, 32)$.

Example 4.19 — testing for concavity

For $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up and where it is concave down. List all inflection points.

$f'(x) = 3x^2 - 12x + 9$ and $f''(x) = 6x - 12$. Since f'' is defined everywhere, f can change concavity only where $f''(x) = 0$, that is, at $x = 2$.

Interval	Test point	Sign of f''	Conclusion
$(-\infty, 2)$	0	$-$	concave down
$(2, \infty)$	3	$+$	concave up

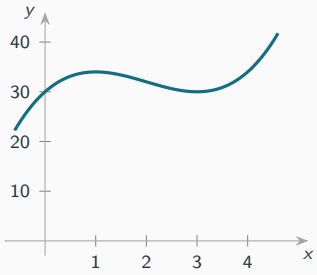


Figure 4.36: an inflection point at $(2, 32)$.

Example 4.19 — testing for concavity

For $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up and where it is concave down. List all inflection points.

$f'(x) = 3x^2 - 12x + 9$ and $f''(x) = 6x - 12$. Since f'' is defined everywhere, f can change concavity only where $f''(x) = 0$, that is, at $x = 2$.

Interval	Test point	Sign of f''	Conclusion
$(-\infty, 2)$	0	$-$	concave down
$(2, \infty)$	3	$+$	concave up

f is concave down over $(-\infty, 2)$ and concave up over $(2, \infty)$. It changes concavity at $x = 2$, so $(2, f(2)) = (2, 32)$ is an inflection point.

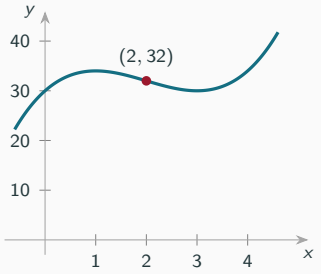


Figure 4.36: an inflection point at $(2, 32)$.

What derivatives tell us about graphs

f'	f''	f is ...	Concavity
$+$	$+$	increasing	concave up
$+$	$-$	increasing	concave down
$-$	$+$	decreasing	concave up
$-$	$-$	decreasing	concave down

Table 4.1: what derivatives tell us about graphs.

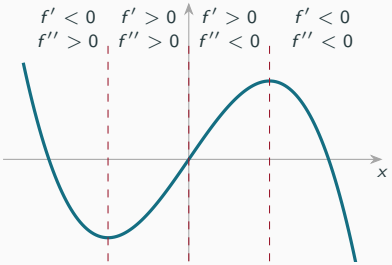


Figure 4.37

The second derivative test

The second derivative test

Suppose $f'(a) = 0$ and f'' is continuous with $f''(a) < 0$. Then $f'' < 0$ near a, so f' is decreasing near a: $f' > 0$ to the left of a and $f' < 0$ to the right. By the first derivative test, f has a local maximum at a.

The second derivative test

Suppose $f'(a) = 0$ and f'' is continuous with $f''(a) < 0$. Then $f'' < 0$ near a, so f' is decreasing near a: $f' > 0$ to the left of a and $f' < 0$ to the right. By the first derivative test, f has a local maximum at a.

Theorem 4.11 (Second Derivative Test)

Suppose $f'(c) = 0$ and f'' is continuous over an interval containing c.

i. If $f''(c) > 0$, then f has a local minimum at c.

ii. If $f''(c) < 0$, then f has a local maximum at c.

iii. If $f''(c) = 0$, then the test is inconclusive.

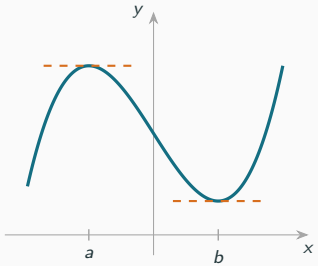
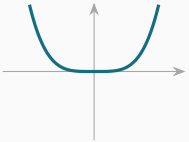


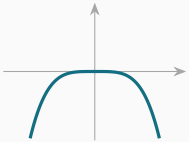
Figure 4.38: $f'(a) = 0$, $f''(a) < 0$: local maximum at a; $f'(b) = 0$, $f''(b) > 0$: local minimum at b.

When $f''(c) = 0$: the test is inconclusive

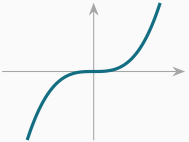
The functions $f(x) = x^4$, $f(x) = -x^4$ and $f(x) = x^3$ all have $f'(0) = 0$ and $f''(0) = 0$, but:



x^4: local minimum at 0



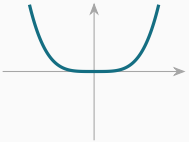
$-x^4$: local maximum at 0



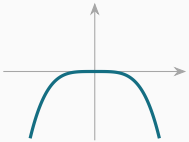
x^3: no local extremum at 0

When $f''(c) = 0$: the test is inconclusive

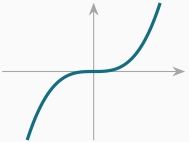
The functions $f(x) = x^4$, $f(x) = -x^4$ and $f(x) = x^3$ all have $f'(0) = 0$ and $f''(0) = 0$, but:



x^4: local minimum at 0



$-x^4$: local maximum at 0



x^3: no local extremum at 0

When $f''(c) = 0$, use the first derivative test instead.

Example 4.20 — using the second derivative test

Use the second derivative test to find the location of all local extrema of $f(x) = x^5 - 5x^3$.

Example 4.20 — using the second derivative test

Use the second derivative test to find the location of all local extrema of $f(x) = x^5 - 5x^3$.

The critical points are given by

$$f'(x) = 5x^4 - 15x^2 = 5x^2\left(x^2 - 3\right) = 0 \iff x = 0,\ \pm\sqrt{3}.$$

Example 4.20 — using the second derivative test

Use the second derivative test to find the location of all local extrema of $f(x) = x^5 - 5x^3$.

The critical points are given by

$$f'(x) = 5x^4 - 15x^2 = 5x^2\big(x^2 - 3\big) = 0 \iff x = 0,\ \pm\sqrt{3}.$$

The second derivative is $f''(x) = 20x^3 - 30x = 10x\big(2x^2 - 3\big)$.

Example 4.20 — using the second derivative test

Use the second derivative test to find the location of all local extrema of $f(x) = x^5 - 5x^3$.

The critical points are given by

$$f'(x) = 5x^4 - 15x^2 = 5x^2\left(x^2 - 3\right) = 0 \iff x = 0,\ \pm\sqrt{3}.$$

The second derivative is $f''(x) = 20x^3 - 30x = 10x\left(2x^2 - 3\right)$.

x	$f''(x)$	Conclusion
$-\sqrt{3}$	$-30\sqrt{3}$	local maximum
0	0	second derivative test is inconclusive
$\sqrt{3}$	$30\sqrt{3}$	local minimum

Example 4.20 — the critical point $x = 0$

At $x = 0$ the second derivative test is inconclusive, so we apply the first derivative test.

Example 4.20 — the critical point $x = 0$

At $x = 0$ the second derivative test is inconclusive, so we apply the first derivative test.

With the test points $x = -1 \in (-\sqrt{3}, 0)$ and $x = 1 \in (0, \sqrt{3})$:

$$f'(-1) = -10 < 0, \qquad f'(1) = -10 < 0.$$

Example 4.20 — the critical point $x = 0$

At $x = 0$ the second derivative test is inconclusive, so we apply the first derivative test.

With the test points $x = -1 \in (-\sqrt{3}, 0)$ and $x = 1 \in (0, \sqrt{3})$:

$$f'(-1) = -10 < 0, \qquad f'(1) = -10 < 0.$$

So f is decreasing on both sides of 0: no local extremum at $x = 0$.

Example 4.20 — the critical point $x = 0$

At $x = 0$ the second derivative test is inconclusive, so we apply the first derivative test.

With the test points $x = -1 \in (-\sqrt{3}, 0)$ and $x = 1 \in (0, \sqrt{3})$:

$$f'(-1) = -10 < 0, \qquad f'(1) = -10 < 0.$$

So f is decreasing on both sides of 0: no local extremum at $x = 0$.

Conclusion: a local maximum $f(-\sqrt{3}) = 6\sqrt{3}$ and a local minimum $f(\sqrt{3}) = -6\sqrt{3}$.

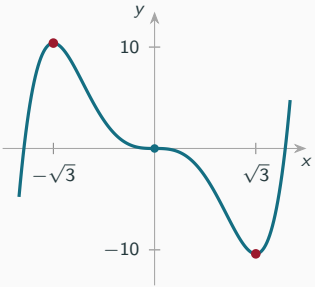


Figure 4.39: $f(x) = x^5 - 5x^3$

Summary

Key points

- **First derivative test:** at a critical point c, f has a local maximum if f' changes from $+$ to $-$, a local minimum if f' changes from $-$ to $+$, and no local extremum if f' does not change sign.
- **Concavity:** f is concave up where f' is increasing (e.g. $f'' > 0$) and concave down where f' is decreasing (e.g. $f'' < 0$). An inflection point is where f changes concavity.
- **Second derivative test:** if $f'(c) = 0$, then $f''(c) > 0$ gives a local minimum and $f''(c) < 0$ a local maximum; if $f''(c) = 0$, the test is inconclusive.