Related Rates

Calculus 1 — Section 4.1

Setting up related-rates problems · examples of the process

Setting up related-rates problems

Example 4.1 — inflating a balloon

A spherical balloon is being filled with air at the constant rate of $2\,\mathrm{cm}^3/\mathrm{sec}$. How fast is the radius increasing when the radius is $3\,\mathrm{cm}$?

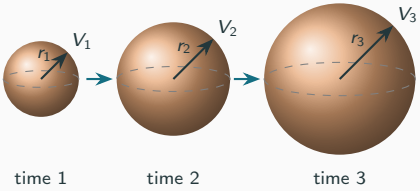


Figure 4.2: both the radius and the volume increase with time.

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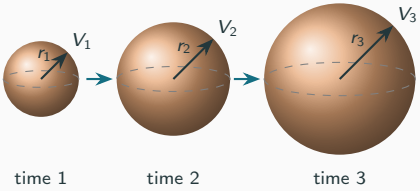


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Differentiating with respect to t (chain rule):

$$V'(t) = 4\pi\,[r(t)]^2\,r'(t).$$

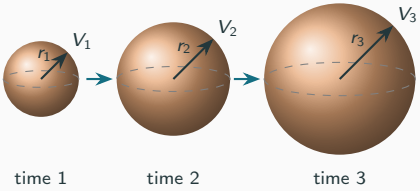


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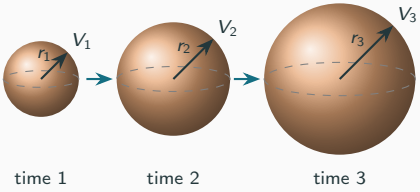


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When $r = 3\,\mathrm{cm}$:
$r'(t) = \dfrac{1}{18\pi}\,\mathrm{cm/sec} \approx 0.018\,\mathrm{cm/sec}$.

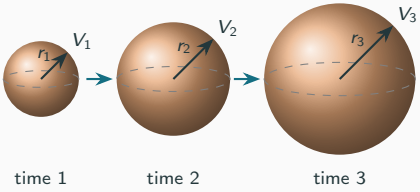


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Problem-solving strategy

Solving a related-rates problem

1. Assign symbols to all variables involved in the problem. Draw a figure if applicable.
2. State, in terms of the variables, the information that is given and the rate to be determined.
3. Find an equation relating the variables introduced in step 1.
4. Using the chain rule, differentiate both sides of the equation found in step 3 with respect to the independent variable. This new equation will relate the derivatives.
5. Substitute all known values into the equation from step 4, then solve for the unknown rate of change.

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Warning

Do not substitute the value of a changing quantity before differentiating: it would behave as a constant and its derivative would disappear.

Examples of the process

Example 4.2 — an airplane flying at a constant elevation

An airplane is flying overhead at a constant elevation of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man at 600 ft/sec. At what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?

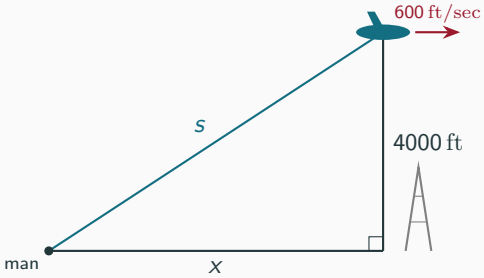


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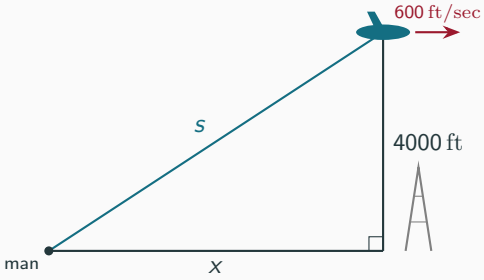


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Step 1. x = horizontal distance from the man to the point on the ground below the plane; s = distance from the man to the plane. Both are functions of time.

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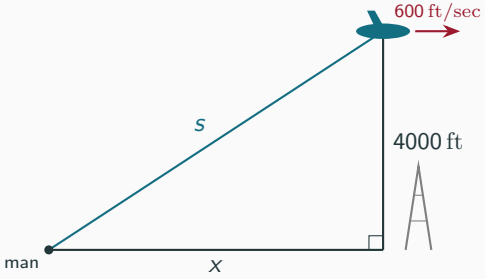


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Step 1. x = horizontal distance from the man to the point on the ground below the plane; s = distance from the man to the plane. Both are functions of time.

Step 2. Given: $\dfrac{dx}{dt} = 600\,\text{ft/sec}$. Find: $\dfrac{ds}{dt}$ when $x = 3000\,\text{ft}$.

Example 4.2 — continued

Step 3. By the Pythagorean theorem,

$$[x(t)]^2 + 4000^2 = [s(t)]^2.$$

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Step 4. Differentiating with respect to t (the derivative of the constant 4000^2 is 0):

$$2x\,\frac{dx}{dt} = 2s\,\frac{ds}{dt}, \qquad \text{that is,} \qquad x\,\frac{dx}{dt} = s\,\frac{ds}{dt}.$$

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Step 5. When $x = 3000$: $3000^2 + 4000^2 = s^2$, so $s = 5000\,\text{ft}$. Then

$$(3000)(600) = (5000)\,\frac{ds}{dt}, \qquad \text{so} \qquad \frac{ds}{dt} = \frac{3000 \cdot 600}{5000} = 360\,\text{ft/sec}.$$

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Note. Substituting $x = 3000$ before differentiating would give $3000^2 + 4000^2 = [s(t)]^2$, hence $0 = s\,\frac{ds}{dt}$ and the wrong answer $\frac{ds}{dt} = 0$.

Example 4.3 — a rocket launch

A rocket is launched so that it rises vertically. A camera is positioned 5000 ft from the launch pad. When the rocket is 1000 ft above the launch pad, its velocity is 600 ft/sec. Find the necessary rate of change of the camera's angle as a function of time so that it stays focused on the rocket.

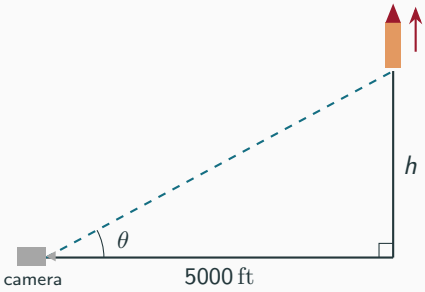


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Step 1. h = height of the rocket above the launch pad; θ = angle between the camera lens and the ground.

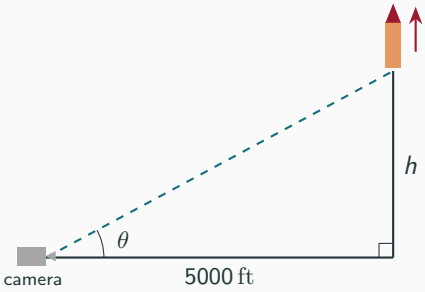


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Step 1. h = height of the rocket above the launch pad; θ = angle between the camera lens and the ground.

Step 2. Given: $\dfrac{dh}{dt} = 600\,\text{ft/sec}$. Find: $\dfrac{d\theta}{dt}$ when $h = 1000\,\text{ft}$.

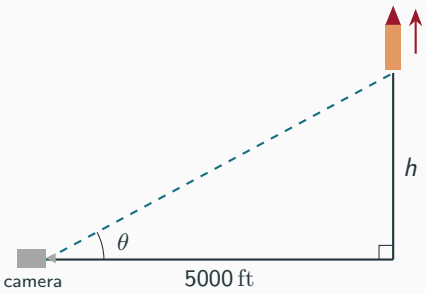


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Step 3. In the right triangle, $\tan\theta = \dfrac{h}{5000}$, that is, $h = 5000\tan\theta$.

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Step 5. When $h = 1000\,\text{ft}$, the hypotenuse is $c = \sqrt{1000^2 + 5000^2} = 1000\sqrt{26}\,\text{ft}$, so

$$\sec^2\theta = \left(\frac{c}{5000}\right)^2 = \left(\frac{1000\sqrt{26}}{5000}\right)^2 = \frac{26}{25}.$$

Example 4.3 — continued

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$$\sec^2\theta = \left(\frac{c}{5000}\right)^2 = \left(\frac{1000\sqrt{26}}{5000}\right)^2 = \frac{26}{25}.$$

With $\dfrac{dh}{dt} = 600$:

$$600 = 5000\cdot\frac{26}{25}\cdot\frac{d\theta}{dt}, \qquad \text{so} \qquad \frac{d\theta}{dt} = \frac{3}{26}\,\text{rad/sec}.$$

Example 4.4 — water draining from a funnel

Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03\,\mathrm{ft}^3/\mathrm{sec}$. The height of the funnel is $2\,\mathrm{ft}$ and the radius at the top of the funnel is $1\,\mathrm{ft}$. At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}\,\mathrm{ft}$?

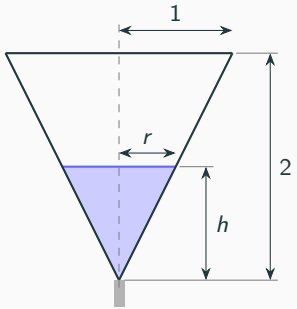


Figure 4.6

Example 4.4 — water draining from a funnel

Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03\,\mathrm{ft}^3/\mathrm{sec}$. The height of the funnel is $2\,\mathrm{ft}$ and the radius at the top of the funnel is $1\,\mathrm{ft}$. At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}\,\mathrm{ft}$?

Step 1. h = height of the water; r = radius of the water at its surface; V = volume of the water.

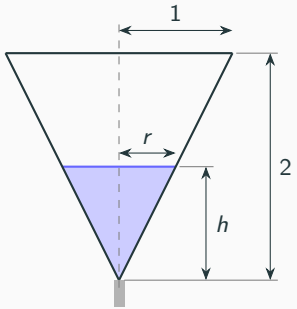


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Step 1. h = height of the water; r = radius of the water at its surface; V = volume of the water.

Step 2. Given: $\dfrac{dV}{dt} = -0.03\,\mathrm{ft}^3/\mathrm{sec}$ (the volume decreases). Find: $\dfrac{dh}{dt}$ when $h = \frac{1}{2}\,\mathrm{ft}$.

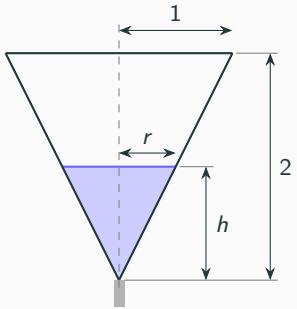


Figure 4.6

Example 4.4 — continued

Step 3. The volume of water in the cone is $V = \frac{1}{3}\pi r^2 h$. By similar triangles, $\frac{r}{h} = \frac{1}{2}$, so $r = \frac{h}{2}$ and

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}\,h^3.$$

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Step 5. With $\dfrac{dV}{dt} = -0.03$ and $h = \frac{1}{2}$:

$$-0.03 = \frac{\pi}{4}\left(\frac{1}{2}\right)^2 \frac{dh}{dt} = \frac{\pi}{16}\,\frac{dh}{dt}, \qquad \text{so} \qquad \frac{dh}{dt} = -\frac{0.48}{\pi} \approx -0.153\,\text{ft/sec}.$$

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The negative sign means that the water level is going down.

Summary

Key points

- When quantities depending on time are related by an equation, differentiating that equation with respect to t (chain rule) relates their rates of change.
- Strategy: variables and figure $\rightarrow$ given and wanted rates $\rightarrow$ equation $\rightarrow$ differentiate $\rightarrow$ substitute and solve.
- Substitute the values of changing quantities only after differentiating.
- A negative rate means that the quantity is decreasing.