Derivatives of Exponential and Logarithmic Functions

Calculus 1 — Section 3.9

Derivatives of logarithmic and exponential functions · logarithmic differentiation · real exponents

Learning objectives

By the end of this section you should be able to:

- Find the derivative of exponential functions.
- Find the derivative of logarithmic functions.
- Use logarithmic differentiation to determine the derivative of a function.

Our route

We start from the natural logarithm, characterized by the way it turns products into sums. We deduce its derivative from its tangent line at $x = 1$, and then obtain e^x as its inverse function.

The natural logarithm

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Definition

The natural logarithm (or Napierian logarithm, after John Napier, 1614) is the unique differentiable function $\ln : (0, \infty) \to \mathbb{R}$ such that

$$\ln(xy) = \ln x + \ln y \quad \text{for all } x, y > 0, \qquad \ln 1 = 0, \qquad \ln'(1) = 1.$$

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- A logarithm turns products into sums: this is why logarithm tables were used to multiply large numbers by hand.

Properties of the natural logarithm

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From $\ln(xy) = \ln x + \ln y$ and $\ln 1 = 0$ we get, for all $x, y > 0$:

$$\ln\left(\frac{1}{x}\right) = -\ln x, \qquad \ln\left(\frac{x}{y}\right) = \ln x - \ln y, \qquad \ln\left(x^q\right) = q \ln x \text{ for every rational number } q.$$

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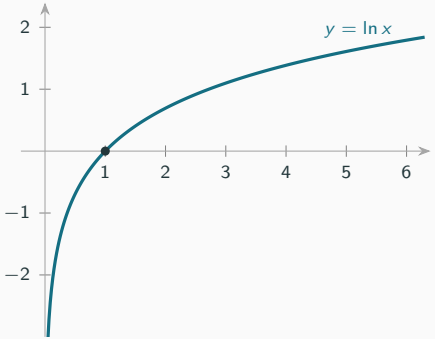
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- $\ln x + \ln\left(\frac{1}{x}\right) = \ln\left(x \cdot \frac{1}{x}\right) = \ln 1 = 0$.
- For a positive integer n: $\ln\left(x^n\right) = \ln(x \cdots x) = \ln x + \cdots + \ln x = n \ln x$.
- Then $n \ln\left(x^{1/n}\right) = \ln\left((x^{1/n})^n\right) = \ln x$, so $\ln\left(x^{1/n}\right) = \frac{1}{n} \ln x$.
- For positive integers m, n: $\ln\left(x^{m/n}\right) = \ln\left((x^{1/n})^m\right) = m \ln\left(x^{1/n}\right) = \frac{m}{n} \ln x$.
- Negative exponents follow from $\ln\left(\frac{1}{x}\right) = -\ln x$.

The graph of $\ln x$

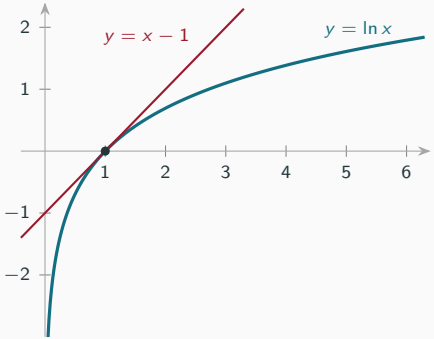
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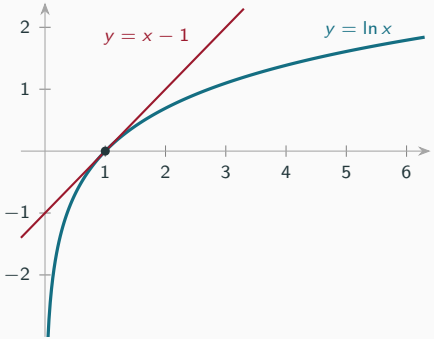


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- Near $x = 1$: $\ln x \approx x - 1$.



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$$= \lim_{k\to 0}\frac{\ln(1+k)}{kx} \qquad \text{substitute } k = \tfrac{h}{x} \quad (k \to 0 \text{ as } h \to 0)$$

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&= \frac{1}{x}\lim_{k\to 0}\frac{\ln(1+k)-\ln 1}{k} = \frac{1}{x}\ln'(1) = \frac{1}{x} && \text{the slope of the tangent at } x = 1 \text{ is } 1
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Theorem 3.15 (Derivative of the Natural Logarithm)

If $x > 0$, then

$$\frac{d}{dx}\ln x = \frac{1}{x}. \tag{3.30}$$

The derivative of $\ln\left(g(x)\right)$

Theorem 3.15 (continued)

Let $g(x)$ be a differentiable function. For all x with $g(x) > 0$,

$$\frac{d}{dx}\ln\left(g(x)\right) = \frac{g'(x)}{g(x)}. \tag{3.31}$$

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Since $f'(x) = \dfrac{1}{x}$, we have $f'\big(g(x)\big) = \dfrac{1}{g(x)}$.

By the chain rule,

$$h'(x) = f'\big(g(x)\big)\,g'(x) = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}. \qquad \square$$

Example 3.77 — taking a derivative of a natural logarithm

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Use Equation (3.31) with $g(x) = x^3 + 3x - 4$, so $g'(x) = 3x^2 + 3$:

$$f'(x) = \frac{1}{x^3 + 3x - 4} \cdot \left(3x^2 + 3\right) = \frac{3x^2 + 3}{x^3 + 3x - 4}.$$

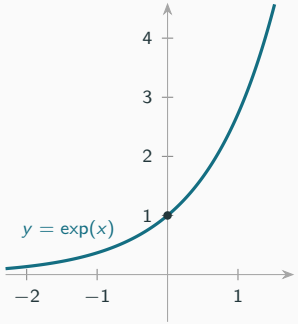
The natural exponential function

The exponential function

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Since ln is increasing ($\ln'(x) = \frac{1}{x} > 0$) and takes every real value, it has an inverse function. The exponential function $\exp : \mathbb{R} \to (0, \infty)$ is the inverse function of ln:

$$y = \exp(x) \iff \ln y = x.$$



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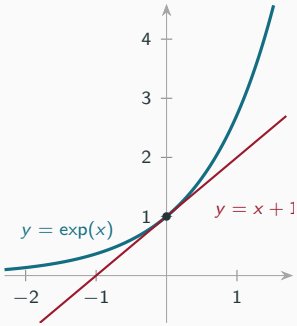
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Equivalently, exp is the unique differentiable function such that

$$\exp(x + y) = \exp(x)\exp(y), \quad \exp(0) = 1, \quad \exp'(0) = 1.$$



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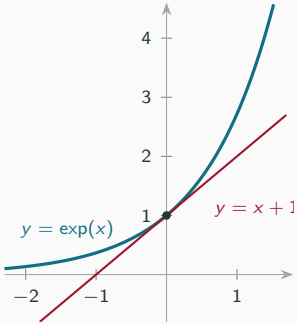
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Indeed, $\ln\big(\exp(x)\exp(y)\big) = \ln\exp(x) + \ln\exp(y) = x + y$; $\ln 1 = 0$; and by the inverse function theorem, $\exp'(0) = \frac{1}{\ln'(\exp(0))} = \frac{1}{\ln'(1)} = 1$.



Tangent line at $x = 0$: $y = x + 1$.

The number e and the notation e^x

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$$e = \exp(1), \qquad \text{that is,} \qquad \ln e = 1. \qquad \text{Numerically, } e \approx 2.71828.$$

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In this notation

$$e^{x+y} = e^x e^y, \qquad e^0 = 1, \qquad \ln\left(e^x\right) = x \ \ (x \in \mathbb{R}), \qquad e^{\ln x} = x \ \ (x > 0).$$

The derivative of e^x

By the inverse function theorem (§3.7), with $g(x) = e^x$ the inverse of $f(x) = \ln x$ and $f'(x) = \frac{1}{x}$:

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Theorem 3.14 (Derivative of the Natural Exponential Function)

For all real x,

$$\frac{d}{dx}e^x = e^x.$$

The derivative of $e^{g(x)}$

Chain rule

If $g(x)$ is differentiable, then

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Example 3.75 — find the derivative of $y = \frac{e^{x^2}}{x}$

Use the derivative of e^x, the quotient rule, and the chain rule:

$$y' = \frac{\left(e^{x^2} \cdot 2x\right) x - 1 \cdot e^{x^2}}{x^2} = \frac{e^{x^2}\left(2x^2 - 1\right)}{x^2}.$$

Example 3.76 — applying the natural exponential function

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'(t)$, to the population, $A(t)$, is constant.

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First find $A'(t)$. By the chain rule,

$$A'(t) = 1000e^{0.3t} \cdot 0.3 = 300e^{0.3t}.$$

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Thus the ratio of the rate of change of the population to the population is

$$\frac{A'(t)}{A(t)} = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3.$$

General exponential and logarithmic functions

Other exponential functions

Let $b > 0$.

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Proof: by the chain rule, $\frac{d}{dx} e^{x \ln b} = e^{x \ln b} \cdot \ln b = b^x \ln b$.

Example 3.79 — applying the derivative of b^x

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Find the derivative of $h(x) = \dfrac{3^x}{3^x + 2}$.

Use the quotient rule and $\frac{d}{dx} 3^x = 3^x \ln 3$:

$$h'(x) = \frac{3^x \ln 3 \left(3^x + 2\right) - 3^x \ln 3 \left(3^x\right)}{\left(3^x + 2\right)^2} = \frac{2 \cdot 3^x \ln 3}{\left(3^x + 2\right)^2}.$$

Logarithms in other bases

Definition

Let $b > 0$, $b \neq 1$. The function b^x is one-to-one with range $(0, \infty)$. The logarithm in base b, $\log_b$, is its inverse function: for $x > 0$,

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Since $b^y = x \iff \ln\left(b^y\right) = \ln x \iff y \ln b = \ln x$, we get $\log_b x = \dfrac{\ln x}{\ln b}$.

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Derivative of $\log_b x$

$$\frac{d}{dx} \log_b x = \frac{1}{x \ln b} \qquad (x > 0).$$

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Derivative of $\log_b x$

$$\frac{d}{dx} \log_b x = \frac{1}{x \ln b} \qquad (x > 0).$$

Proof: $\ln b$ is a constant, so $\frac{d}{dx} \frac{\ln x}{\ln b} = \frac{1}{\ln b} \cdot \frac{1}{x}$.

Example 3.80 — finding the slope of a tangent line

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By the derivative of $\log_2$ and the chain rule,

$$\frac{dy}{dx} = \frac{1}{(3x + 1)\ln 2} \cdot 3 = \frac{3}{(3x + 1)\ln 2}.$$

Example 3.80 — finding the slope of a tangent line

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By the derivative of $\log_2$ and the chain rule,

$$\frac{dy}{dx} = \frac{1}{(3x+1)\ln 2} \cdot 3 = \frac{3}{(3x+1)\ln 2}.$$

So the tangent line at $x = 1$ has slope

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{3}{4\ln 2} = \frac{3}{\ln 16}.$$

Logarithmic differentiation

Logarithmic differentiation

We still cannot differentiate functions such as $y = x^x$ or $y = x^\pi$. Logarithmic differentiation handles them.

Problem-solving strategy: using logarithmic differentiation

1. To differentiate $y = h(x)$, take the natural logarithm of both sides: $\ln y = \ln\big(h(x)\big)$.
2. Use properties of logarithms to expand $\ln\big(h(x)\big)$ as much as possible.
3. Differentiate both sides. On the left we have $\frac{1}{y}\,\frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h(x)$.

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$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln \left(2x^4 + 1\right) + \frac{8x^3}{2x^4 + 1} \cdot \tan x$$ Step 3: differentiate (product rule)

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$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln \left(2x^4 + 1\right) + \frac{8x^3}{2x^4 + 1} \cdot \tan x$$ Step 3: differentiate (product rule)

$$\frac{dy}{dx} = y \cdot \left(\sec^2 x \ln \left(2x^4 + 1\right) + \frac{8x^3}{2x^4 + 1} \cdot \tan x\right)$$ Step 4: multiply by y

Example 3.81 — using logarithmic differentiation

Find the derivative of $y = \left(2x^4 + 1\right)^{\tan x}$.

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The power rule for real exponents

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- So far: $\frac{d}{dx}x^n = nx^{n-1}$ for integers n (§3.3), and $\frac{d}{dx}x^{m/n} = \frac{m}{n}x^{(m/n)-1}$ for rational exponents (§3.7).

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For example, $\frac{d}{dx}x^\pi = \pi x^{\pi-1}$ and $\frac{d}{dx}x^{\sqrt{2}} = \sqrt{2}\,x^{\sqrt{2}-1}$.

Proof of the power rule (Example 3.83)

Let $y = x^r$, where r is an arbitrary real number and $x > 0$. We use logarithmic differentiation.

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Alternatively, $x^r = e^{r \ln x}$, so by the chain rule $\frac{d}{dx}x^r = e^{r \ln x} \cdot \frac{r}{x} = x^r \cdot \frac{r}{x} = rx^{r-1}$.

Summary

Key results

- $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$, $\dfrac{d}{dx}\ln\big(g(x)\big) = \dfrac{g'(x)}{g(x)}$.
- $\dfrac{d}{dx}e^x = e^x$, $\dfrac{d}{dx}e^{g(x)} = e^{g(x)}g'(x)$.
- Other bases: $\dfrac{d}{dx}b^x = b^x \ln b$, $\dfrac{d}{dx}\log_b x = \dfrac{1}{x\ln b}$.
- Logarithmic differentiation: take ln of both sides, expand, differentiate, multiply by y.
- Power rule for real exponents: $\dfrac{d}{dx}x^r = rx^{r-1}$ (every real r, $x > 0$).