Differentiation Rules and Techniques

Calculus 1 — Sections 3.3, 3.5–3.9

Differentiation rules · trigonometric functions · chain rule · inverse functions · implicit differentiation · exponentials and logarithms

Outline

§3.3 Differentiation Rules

Learning objectives

By the end of this section you should be able to:

- State the constant, constant multiple, and power rules.
- Apply the sum and difference rules to combine derivatives.
- Use the product rule and the quotient rule.
- Extend the power rule to negative exponents.
- Combine the rules to differentiate any polynomial or rational function.

The idea

Computing every derivative from the limit definition is tedious. A few rules, each proved once from the definition, let us differentiate whole families of functions directly.

The constant rule

Theorem 3.2 — Constant rule

Let c be a constant. If $f(x) = c$, then $f'(x) = 0$. That is,

$$\frac{d}{dx}(c) = 0.$$

Why? From the definition,

$$f'(x) = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

The graph of $f(x) = c$ is a horizontal line, so every tangent line has slope 0.

Example 3.17

If $f(x) = 8$, then $f'(x) = 0$.

Example 3.18 — differentiating x^3 from the definition

We know $\frac{d}{dx}(x^2) = 2x$ and $\frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}$. What about x^3?

$$\begin{aligned}
\frac{d}{dx}(x^3) &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} && \text{expand } (x+h)^3 \\
&= \lim_{h\to 0} \frac{h\left(3x^2 + 3xh + h^2\right)}{h} && \text{cancel } x^3\text{, factor out } h \\
&= \lim_{h\to 0} \left(3x^2 + 3xh + h^2\right) = 3x^2. && \text{let } h \to 0
\end{aligned}$$

The exponent comes down as a coefficient, and the power drops by 1.

The power rule

Theorem 3.3 — Power rule

Let n be a positive integer. If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}, \qquad \text{i.e.} \qquad \frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof. By the binomial theorem, $(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + h^n$, so

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + h^{n-1} \xrightarrow[h\to 0]{} nx^{n-1}.$$

Every term except the first contains a factor of h. □

Example 3.19

$\frac{d}{dx}(x^{10}) = 10x^9$. *Caution:* the rule is for x^n, not for a constant raised to a variable power such as 3^x.

Sum, difference, and constant multiple rules

Theorem 3.4

Let f and g be differentiable and k a constant. Then

$$(f+g)' = f' + g', \qquad (f-g)' = f' - g', \qquad (kf)' = k\,f'.$$

Proof of the sum rule. With $j = f + g$, regroup the difference quotient and use the sum law for limits:

$$j'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right) = f'(x) + g'(x).$$

The difference and constant multiple rules are proved in the same way. □

Examples 3.20 & 3.21 — using the basic rules

3.20 — constant multiple

For $g(x) = 3x^2$,

$$g'(x) = 3\,\frac{d}{dx}(x^2) = 3(2x) = 6x,$$

three times the derivative of $f(x) = x^2$.

3.21 — step by step

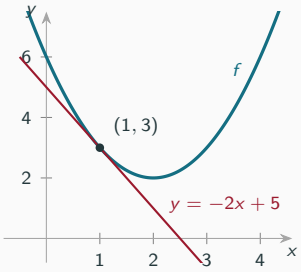
For $f(x) = 2x^5 + 7$:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(2x^5) + \frac{d}{dx}(7) && \text{sum rule} \\ &= 2\,\frac{d}{dx}(x^5) + \frac{d}{dx}(7) && \text{constant multiple} \\ &= 2(5x^4) + 0 && \text{power and constant rules} \\ &= 10x^4. \end{aligned}$$

Example 3.22 — finding a tangent line

Find the line tangent to $f(x) = x^2 - 4x + 6$ at $x = 1$.

- **Point:** $f(1) = 1 - 4 + 6 = 3$, so $(1, 3)$.
- **Slope:** by the rules, $f'(x) = 2x - 4$, so $f'(1) = -2$.
- **Line:** $y - 3 = -2(x - 1)$, that is, $\boxed{y = -2x + 5}$.



Linear approximation

This line is the linearization of f at 1:
$L(x) = f(1) + f'(1)(x - 1) = -2x + 5$. So $f(x) \approx -2x + 5$ for x near 1. For instance, $f(1.1) \approx L(1.1) = 2.8$; the exact value is $f(1.1) = 2.81$.

The product rule

The derivative of a product is not the product of the derivatives

Write $x^2 = x \cdot x$. Then $\frac{d}{dx}(x^2) = 2x$, but $\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$.

Theorem 3.5 — Product rule

Let f and g be differentiable. If $j(x) = f(x)\,g(x)$, then

$$j'(x) = f'(x)\,g(x) + g'(x)\,f(x).$$

In words: the derivative of the first times the second, plus the derivative of the second times the first.

Proof of the product rule

Let $j(x) = f(x)g(x)$. Add and subtract $f(x)\,g(x+h)$ in the numerator:

$$\begin{aligned} j'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h) \right) + \lim_{h\to 0} \left(\frac{g(x+h) - g(x)}{h} \cdot f(x) \right) \\ &= f'(x)\,g(x) + g'(x)\,f(x). \end{aligned}$$

The key step

g is differentiable, hence continuous (Theorem 3.1), so $\lim_{h\to 0} g(x+h) = g(x)$. □

Examples 3.23 & 3.24 — applying the product rule

3.23 — at a point

For $j = fg$ with $f(2) = 3$, $f'(2) = -4$, $g(2) = 1$, $g'(2) = 6$:

$$j'(2) = f'(2)g(2) + g'(2)f(2) = (-4)(1) + (6)(3) = 14.$$

3.24 — binomials

For $j(x) = (x^2 + 2)(3x^3 - 5x)$, take $f = x^2 + 2$, $g = 3x^3 - 5x$, so $f' = 2x$, $g' = 9x^2 - 5$:

$$j'(x) = (2x)(3x^3 - 5x) + (9x^2 - 5)(x^2 + 2) = 15x^4 + 3x^2 - 10.$$

Check: $j(x) = 3x^5 + x^3 - 10x$, so $j'(x) = 15x^4 + 3x^2 - 10$.

Not the quotient of the derivatives either

Write $x^2 = \dfrac{x^3}{x}$. Then $\frac{d}{dx}(x^2) = 2x$, but $\dfrac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(x)} = \dfrac{3x^2}{1} = 3x^2$.

Theorem 3.6 — Quotient rule

Let f and g be differentiable. If $j(x) = \dfrac{f(x)}{g(x)}$, then

$$j'(x) = \frac{f'(x)\,g(x) - g'(x)\,f(x)}{\big(g(x)\big)^2}.$$

Example 3.25 — applying the quotient rule

Find the derivative of $k(x) = \dfrac{5x^2}{4x+3}$.

Take $f(x) = 5x^2$ and $g(x) = 4x + 3$, so $f'(x) = 10x$ and $g'(x) = 4$. Then

$$k'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{\big(g(x)\big)^2} = \frac{10x(4x+3) - 4(5x^2)}{(4x+3)^2}.$$

Expanding the numerator, $40x^2 + 30x - 20x^2 = 20x^2 + 30x$, so

$$k'(x) = \frac{20x^2 + 30x}{(4x+3)^2}.$$

The extended power rule

Theorem 3.7 — Extended power rule

If k is a negative integer, then

$$\frac{d}{dx}\left(x^k\right) = kx^{k-1}.$$

Proof. Set $n = -k$, a positive integer, so $x^k = x^{-n} = \dfrac{1}{x^n}$. Apply the quotient rule with $f(x) = 1$ and $g(x) = x^n$, so $f' = 0$ and $g' = nx^{n-1}$:

$$\frac{d}{dx}\left(x^{-n}\right) = \frac{0 \cdot x^n - nx^{n-1} \cdot 1}{(x^n)^2} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1} = kx^{k-1}. \qquad \square$$

So $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ holds for every integer n (for $n = 0$, it is the constant rule).

Examples 3.26 & 3.27 — negative exponents

3.26

With $k = -4$: $\frac{d}{dx}\left(x^{-4}\right) = -4x^{-4-1} = -4x^{-5}$.

3.27 — rewrite first

For $f(x) = \dfrac{6}{x^2}$, the quotient rule works, but rewriting as $f(x) = 6x^{-2}$ is quicker:

$$\begin{aligned} f'(x) &= 6\,\frac{d}{dx}\left(x^{-2}\right) && \text{constant multiple rule} \\ &= 6\left(-2x^{-3}\right) && \text{extended power rule} \\ &= -12x^{-3}. \end{aligned}$$

Combining differentiation rules

Rule of thumb

Apply the rules in the reverse of the order in which you would evaluate the function.

Example 3.28

For $k(x) = 3h(x) + x^2 g(x)$ (sum, constant multiple, product rules):

$$k'(x) = 3h'(x) + \frac{d}{dx}(x^2)g(x) + g'(x)\,x^2 = 3h'(x) + 2x\,g(x) + x^2 g'(x).$$

Example 3.29 — a product of three functions

For $k(x) = f(x)g(x)h(x) = \big(f(x)g(x)\big) \cdot h(x)$, apply the product rule twice:

$$k'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

Example 3.30 — quotient rule and product rule together

For $h(x) = \dfrac{2x^3 k(x)}{3x+2}$, find $h'(x)$.

$$h'(x) = \frac{\frac{d}{dx}\left(2x^3 k(x)\right) \cdot (3x+2) - \frac{d}{dx}(3x+2) \cdot 2x^3 k(x)}{(3x+2)^2}$$ quotient rule

$$= \frac{\left(6x^2 k(x) + 2x^3 k'(x)\right)(3x+2) - 3 \cdot 2x^3 k(x)}{(3x+2)^2}$$ product rule; $\frac{d}{dx}(3x+2) = 3$

$$= \frac{-6x^3 k + 18x^3 k + 12x^2 k + 6x^4 k' + 4x^3 k'}{(3x+2)^2}$$ expand

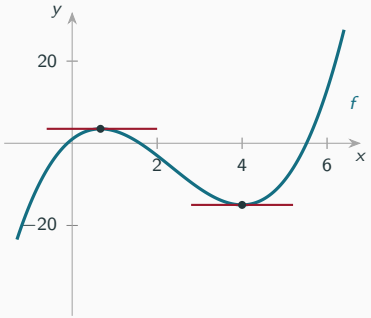
$$= \frac{12k(x)\left(x^3 + x^2\right) + 2k'(x)\left(3x^4 + 2x^3\right)}{(3x+2)^2}.$$ simplify

Example 3.31 — where is the tangent line horizontal?

For $f(x) = x^3 - 7x^2 + 8x + 1$, a horizontal tangent means $f'(x) = 0$.

- $f'(x) = 3x^2 - 14x + 8 = (3x - 2)(x - 4)$.
- $f'(x) = 0$ when $x = \frac{2}{3}$ or $x = 4$.

At these points the graph levels off: a local high point at $x = \frac{2}{3}$ and a local low point at $x = 4$.



Example 3.32 — finding a velocity

An object's position at time t is $s(t) = \dfrac{t}{t^2+1}$. What is its initial velocity?

The initial velocity is $v(0) = s'(0)$. By the quotient rule,

$$s'(t) = \frac{1 \cdot (t^2+1) - 2t \cdot t}{(t^2+1)^2} = \frac{1-t^2}{(t^2+1)^2}.$$

Evaluating at $t = 0$:

$$v(0) = s'(0) = \frac{1-0}{(0+1)^2} = 1.$$

Summary

Constant rule	$\frac{d}{dx}(c) = 0$
Power rule (any integer n)	$\frac{d}{dx}(x^n) = nx^{n-1}$
Sum, difference, constant multiple	$(f \pm g)' = f' \pm g', \quad (kf)' = kf'$
Product rule	$(fg)' = f'g + g'f$
Quotient rule	$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$

Combine the rules in the reverse of the order of evaluation, and remember: the derivative of a product (or quotient) is not the product (or quotient) of the derivatives.

§3.5 Derivatives of Trigonometric Functions

Learning objectives

By the end of this section you should be able to:

- Find the derivatives of the sine and cosine functions.
- Find the derivatives of the other standard trigonometric functions.
- Calculate higher-order derivatives of sine and cosine.

Why it matters

Simple harmonic motion, such as a mass oscillating on a spring, is described by sines and cosines. Their derivatives give the velocity and acceleration of that motion.

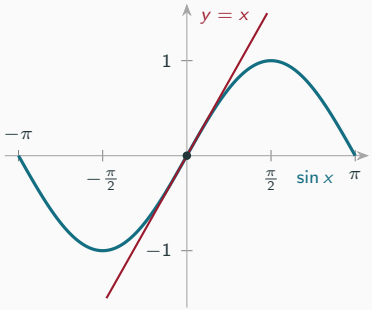
The derivative of $\sin x$ at $x = 0$

From the graph. Near the origin, the graph of $\sin x$ hugs the line $y = x$: the tangent line at 0 is $y = x$, of slope 1.

Proof. By the definition of the derivative,

$$\left.\frac{d}{dx}(\sin x)\right|_{x=0} = \lim_{h\to 0} \frac{\sin(0+h) - \sin 0}{h} = \lim_{h\to 0} \frac{\sin h}{h} = 1,$$

the limit proved in §2.3 with the Squeeze Theorem.



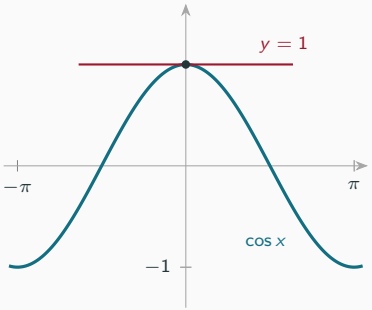
The derivative of $\cos x$ at $x = 0$

From the graph. The graph of $\cos x$ is at its highest point at $x = 0$: the tangent line there is the horizontal line $y = 1$, of slope 0.

Proof. By the definition of the derivative,

$$\left.\frac{d}{dx}(\cos x)\right|_{x=0} = \lim_{h\to 0}\frac{\cos h - \cos 0}{h} = \lim_{h\to 0}\frac{\cos h - 1}{h} = 0,$$

by §2.3 (Example 2.25).



The derivatives of $\sin x$ and $\cos x$

Theorem 3.8

The derivative of the sine is the cosine, and the derivative of the cosine is the negative sine:

$$\frac{d}{dx}(\sin x) = \cos x, \qquad \frac{d}{dx}(\cos x) = -\sin x.$$

Ingredients for the proof.

- The derivatives at 0 we just computed, i.e. the two trigonometric limits from §2.3:

$$\lim_{h\to 0} \frac{\sin h}{h} = 1, \qquad \lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$$

- The sum identity $\sin(x + h) = \sin x \cos h + \cos x \sin h$.

Proof that $\frac{d}{dx}(\sin x) = \cos x$

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} && \text{definition of the derivative}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} && \text{sum identity}\\
&= \lim_{h\to 0}\left(\sin x\cdot\frac{\cos h-1}{h}+\cos x\cdot\frac{\sin h}{h}\right) && \text{regroup}\\
&= \sin x\cdot 0+\cos x\cdot 1 && \text{the two trig limits}\\
&= \cos x. && \square
\end{aligned}$$

The proof for $\cos x$ is similar, using $\cos(x+h) = \cos x\cos h - \sin x\sin h$.

Examples 3.39 & 3.40 — with the product and quotient rules

3.39 — product rule

For $f(x) = 5x^3 \sin x$:

$$f'(x) = \frac{d}{dx}(5x^3) \cdot \sin x + \frac{d}{dx}(\sin x) \cdot 5x^3 = 15x^2 \sin x + 5x^3 \cos x.$$

3.40 — quotient rule

For $g(x) = \dfrac{\cos x}{4x^2}$:

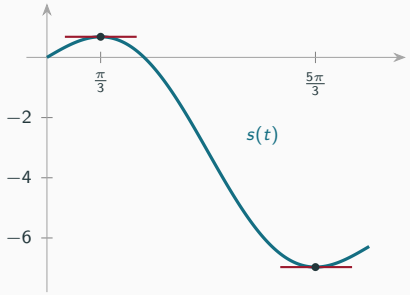
$$g'(x) = \frac{(-\sin x)\, 4x^2 - 8x \cos x}{(4x^2)^2} = \frac{-4x^2 \sin x - 8x \cos x}{16x^4} = \frac{-x \sin x - 2 \cos x}{4x^3}.$$

Example 3.41 — when is the particle at rest?

A particle's position is $s(t) = 2\sin t - t$ for $0 \le t \le 2\pi$.

- At rest means $v(t) = s'(t) = 0$.
- $s'(t) = 2\cos t - 1$, so we solve $2\cos t - 1 = 0$, i.e. $\cos t = \frac{1}{2}$.
- On $[0, 2\pi]$: $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$.

At these times the graph of s has horizontal tangents.



Example 3.42 — the derivative of $\tan x$

Write $\tan x$ as a quotient and use the quotient rule:

$$\begin{aligned}
\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) &= \frac{\cos x \cdot \cos x - (-\sin x)\cdot \sin x}{(\cos x)^2} && \text{quotient rule} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} && \text{simplify} \\
&= \frac{1}{\cos^2 x} && \cos^2 x + \sin^2 x = 1 \\
&= \sec^2 x. && \sec x = \frac{1}{\cos x}
\end{aligned}$$

The other three are found the same way, since $\cot x$, $\sec x$, $\csc x$ are all quotients of $\sin x$ and $\cos x$.

The remaining four derivatives

Theorem 3.9

$$\frac{d}{dx}(\tan x) = \sec^2 x, \qquad \frac{d}{dx}(\cot x) = -\csc^2 x,$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x, \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x.$$

Example 3.44 — sum and product rules together

For $f(x) = \csc x + x \tan x$:

$$f'(x) = \frac{d}{dx}(\csc x) + \frac{d}{dx}(x \tan x) = -\csc x \cot x + \left(1 \cdot \tan x + \sec^2 x \cdot x\right)$$

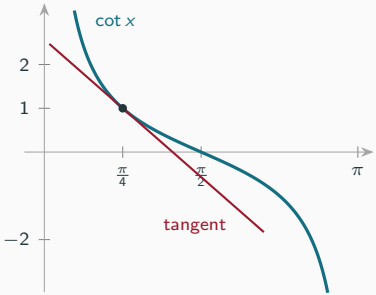
$$= -\csc x \cot x + \tan x + x \sec^2 x.$$

Example 3.43 — a tangent line to $\cot x$

Find the line tangent to $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

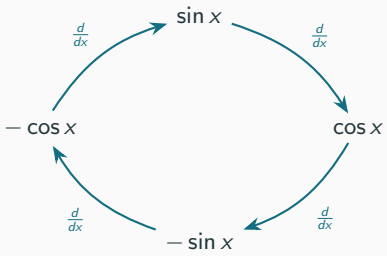
- **Point:** $f\left(\frac{\pi}{4}\right) = \cot \frac{\pi}{4} = 1$, so $\left(\frac{\pi}{4}, 1\right)$.
- **Slope:** $f'(x) = -\csc^2 x$, so $f'\left(\frac{\pi}{4}\right) = -\csc^2 \frac{\pi}{4} = -2$.
- **Line:** $y - 1 = -2\left(x - \frac{\pi}{4}\right)$, that is,

$$\boxed{y = -2x + 1 + \frac{\pi}{2}}.$$



Example 3.45 — the derivatives of $\sin x$ repeat

$$
\begin{aligned}
y &= \sin x \\
\frac{dy}{dx} &= \cos x \\
\frac{d^2y}{dx^2} &= -\sin x \\
\frac{d^3y}{dx^3} &= -\cos x \\
\frac{d^4y}{dx^4} &= \sin x
\end{aligned}
$$



The pattern repeats every four steps: $\dfrac{d^{4n}}{dx^{4n}}(\sin x) = \sin x$ and $\dfrac{d^{4n+1}}{dx^{4n+1}}(\sin x) = \cos x$.

Derivatives of all orders of $\sin x$ and $\cos x$

Write $k = 4n + r$ with $r \in \{0, 1, 2, 3\}$ the remainder of k divided by 4. Only r matters:

k	$4n$	$4n+1$	$4n+2$	$4n+3$
$\frac{d^k}{dx^k}(\sin x)$	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$
$\frac{d^k}{dx^k}(\cos x)$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$

Example 3.46 — using the pattern

Find $\dfrac{d^{74}}{dx^{74}}(\sin x)$.

Divide by 4: $74 = 4(18) + 2$. Every block of four derivatives brings us back to $\sin x$, so only the remainder 2 matters:

$$\frac{d^{74}}{dx^{74}}(\sin x) = \frac{d^{72+2}}{dx^{72+2}}(\sin x) = \frac{d^2}{dx^2}(\sin x) = -\sin x.$$

Summary

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\cos x) = -\sin x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$

- sin and cos come from the definition, using $\lim_{h \to 0} \frac{\sin h}{h} = 1$ and $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$.
- The other four follow from the quotient rule.
- Derivatives of $\sin x$ and $\cos x$ repeat every four steps.

§3.6 The Chain Rule

Learning objectives

By the end of this section you should be able to:

- State the chain rule for the composition of two functions.
- Apply the chain rule together with the power rule.
- Apply the chain rule and the product/quotient rules in combination when both are necessary.
- Describe the proof of the chain rule.

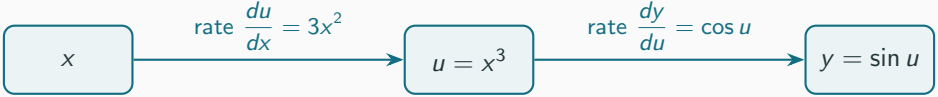
Why it matters

We can differentiate x^n, $\sin x$, $\cos x$, ..., and their sums, products and quotients. We cannot yet differentiate compositions such as

$$h(x) = \sin\left(x^3\right) \qquad \text{or} \qquad k(x) = \sqrt{3x^2 + 1}\,.$$

A chain reaction

Consider $h(x) = \sin\left(x^3\right)$. As x changes, x^3 changes, which leads to a change in $\sin\left(x^3\right)$.



So the derivative of $\sin\left(x^3\right)$ should involve both the derivative of x^3 and the derivative of $\sin u$ with respect to u, where $u = x^3$.

Rates multiply. At $x = 1$: u changes 3 times as fast as x, and y changes $\cos 1 \approx 0.54$ times as fast as u. So y changes about $3 \times 0.54 \approx 1.62$ times as fast as x.

The derivative of $\sin\left(x^3\right)$ at $x = a$

By the definition of the derivative, multiplying and dividing by $x^3 - a^3$:

$$h'(a) = \lim_{x \to a} \frac{\sin\left(x^3\right) - \sin\left(a^3\right)}{x - a} = \lim_{x \to a} \frac{\sin\left(x^3\right) - \sin\left(a^3\right)}{x^3 - a^3} \cdot \frac{x^3 - a^3}{x - a}.$$

Second factor. By the definition of the derivative,

$$\lim_{x \to a} \frac{x^3 - a^3}{x - a} = \frac{d}{dx}\left(x^3\right)\Big|_{x=a} = 3a^2.$$

First factor. Let $u = x^3$. As $x \to a$, $u \to a^3$:

$$\lim_{u \to a^3} \frac{\sin u - \sin\left(a^3\right)}{u - a^3} = \frac{d}{du}(\sin u)\Big|_{u=a^3} = \cos\left(a^3\right).$$

Thus $h'(a) = \cos\left(a^3\right) \cdot 3a^2$.

The chain rule

Rule: The Chain Rule

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function $h(x) = (f \circ g)(x) = f\big(g(x)\big)$ is

$$h'(x) = \underbrace{f'\big(g(x)\big)}_{\substack{\text{outer derivative,} \\ \text{evaluated at the inner function}}} \cdot \underbrace{g'(x)}_{\substack{\text{derivative of} \\ \text{the inner function}}} . \tag{3.17}$$

Alternatively, if y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

For $h(x) = \sin\left(x^3\right)$: $f(x) = \sin x$ and $g(x) = x^3$, so $h'(x) = \cos\left(x^3\right) \cdot 3x^2$, as found above.

Problem-solving strategy: applying the chain rule

1. To differentiate $h(x) = f\big(g(x)\big)$, begin by identifying $f(x)$ and $g(x)$.
2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'\big(g(x)\big)$.
3. Find $g'(x)$.
4. Write $h'(x) = f'\big(g(x)\big) \cdot g'(x)$.

Keep in mind

- Work your way from the outside function in.
- The derivative of a composition of two functions has two parts; of three functions, three parts; and so on.
- We never evaluate a derivative at a derivative.

Power rule for composition of functions

To differentiate $h(x) = \big(g(x)\big)^n$, think of it as $f\big(g(x)\big)$ with $f(x) = x^n$. Then $f'(x) = nx^{n-1}$, so $f'\big(g(x)\big) = n\big(g(x)\big)^{n-1}$.

Rule: Power Rule for Composition of Functions

For all values of x for which the derivative is defined, if $h(x) = \big(g(x)\big)^n$, then

$$h'(x) = n\big(g(x)\big)^{n-1} g'(x). \tag{3.18}$$

Examples 3.48 & 3.49 — chain and power rules

3.48 — find the derivative of $h(x) = \frac{1}{(3x^2+1)^2}$

Rewrite $h(x) = (3x^2+1)^{-2}$ and apply the power rule with $g(x) = 3x^2+1$:

$$h'(x) = -2(3x^2+1)^{-3}(6x) = \frac{-12x}{(3x^2+1)^3}.$$

3.49 — find the derivative of $h(x) = \sin^3 x$

Recall $\sin^3 x = (\sin x)^3$. Apply the power rule with $g(x) = \sin x$:

$$h'(x) = 3(\sin x)^2 \cos x = 3\sin^2 x \cos x.$$

Example 3.50 — finding the equation of a tangent line

Find the equation of the line tangent to the graph of $h(x) = \dfrac{1}{(3x-5)^2}$ at $x = 2$.

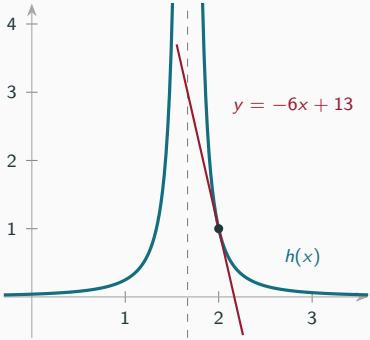
Point. $h(2) = \dfrac{1}{(3(2)-5)^2} = 1$, so the point is $(2, 1)$.

Slope. Rewrite $h(x) = (3x-5)^{-2}$:

$$h'(x) = -2(3x-5)^{-3}(3) = -6(3x-5)^{-3},$$

so $h'(2) = -6(3(2)-5)^{-3} = -6$.

Line. $y - 1 = -6(x-2)$, that is, $y = -6x + 13$.



Example 3.52 — using the chain rule on a cosine function

Find the derivative of $h(x) = \cos\left(5x^2\right)$.

Think of $h(x)$ as $f\big(g(x)\big)$, with outer function $f(x) = \cos x$ and inner function $g(x) = 5x^2$.

Since $f'(x) = -\sin x$, we have $f'\big(g(x)\big) = -\sin\left(5x^2\right)$; and $g'(x) = 10x$.

$$h'(x) = f'\big(g(x)\big)\, g'(x) = -\sin\left(5x^2\right) \cdot 10x = -10x \sin\left(5x^2\right).$$

Example 3.53 — the secant of a function

Find the derivative of $h(x) = \sec\left(4x^5 + 2x\right)$.

Apply the chain rule to $h(x) = \sec\left(g(x)\right)$:

$$h'(x) = \sec\left(g(x)\right)\tan\left(g(x)\right)g'(x).$$

Here $g(x) = 4x^5 + 2x$, so $g'(x) = 20x^4 + 2$. Therefore

$$\begin{aligned} h'(x) &= \sec\left(4x^5 + 2x\right)\tan\left(4x^5 + 2x\right)\left(20x^4 + 2\right) \\ &= \left(20x^4 + 2\right)\sec\left(4x^5 + 2x\right)\tan\left(4x^5 + 2x\right). \end{aligned}$$

Example 3.54 — combining the chain rule with the product rule

Find the derivative of $h(x) = (2x+1)^5(3x-2)^7$.

First apply the product rule, then the chain rule to each factor:

$$
\begin{aligned}
h'(x) &= \frac{d}{dx}\big((2x+1)^5\big)\cdot(3x-2)^7 + \frac{d}{dx}\big((3x-2)^7\big)\cdot(2x+1)^5 && \text{product rule}\\
&= 5(2x+1)^4\cdot 2\cdot(3x-2)^7 + 7(3x-2)^6\cdot 3\cdot(2x+1)^5 && \text{chain rule}\\
&= 10(2x+1)^4(3x-2)^7 + 21(3x-2)^6(2x+1)^5 && \text{simplify}\\
&= (2x+1)^4(3x-2)^6\big(10(3x-2)+21(2x+1)\big) && \text{factor out } (2x+1)^4(3x-2)^6\\
&= (2x+1)^4(3x-2)^6(72x+1) && \text{simplify}
\end{aligned}
$$

Example 3.56 — using the chain rule in a velocity problem

A particle moves along a coordinate axis. Its position at time t is

$$s(t) = \sin(2t) + \cos(3t).$$

What is the velocity of the particle at time $t = \dfrac{\pi}{6}$?

The velocity is the derivative of the position. By the chain rule,

$$v(t) = s'(t) = \cos(2t) \cdot 2 + \big(-\sin(3t)\big) \cdot 3 = 2\cos(2t) - 3\sin(3t).$$

Substituting $t = \dfrac{\pi}{6}$:

$$v\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{3}\right) - 3\sin\left(\frac{\pi}{2}\right) = 2 \cdot \frac{1}{2} - 3 \cdot 1 = -2.$$

An informal proof

For simplicity, assume $g(x) \neq g(a)$ for $x \neq a$ in some open interval containing a. Then

$$h'(a) = \lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{x - a} = \lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}.$$

- The second factor tends to $g'(a)$.
- For the first factor: since g is differentiable at a, it is continuous at a, so $\lim_{x \to a} g(x) = g(a)$. Substituting $y = g(x)$ and $b = g(a)$,

$$\lim_{x \to a} \frac{f\big(g(x)\big) - f\big(g(a)\big)}{g(x) - g(a)} = \lim_{y \to b} \frac{f(y) - f(b)}{y - b} = f'(b) = f'\big(g(a)\big).$$

Finally, $h'(a) = f'\big(g(a)\big)\, g'(a)$. $\square$

Example 3.57 — using the chain rule with functional values

Let $h(x) = f\big(g(x)\big)$. If $g(1) = 4$, $g'(1) = 3$, and $f'(4) = 7$, find $h'(1)$.

Use the chain rule, then substitute:

$$\begin{aligned} h'(1) &= f'\big(g(1)\big)\, g'(1) && \text{chain rule} \\ &= f'(4) \cdot 3 && \text{substitute } g(1) = 4 \text{ and } g'(1) = 3 \\ &= 7 \cdot 3 && \text{substitute } f'(4) = 7 \\ &= 21. \end{aligned}$$

The chain rule using Leibniz's notation

For $h(x) = f\big(g(x)\big)$, let $u = g(x)$ and $y = h(x) = f(u)$. Then

$$h'(x) = \frac{dy}{dx}, \qquad f'\big(g(x)\big) = f'(u) = \frac{dy}{du}, \qquad g'(x) = \frac{du}{dx}.$$

Consequently,

$$\frac{dy}{dx} = h'(x) = f'\big(g(x)\big)\, g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Rule: Chain Rule Using Leibniz's Notation

If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

This notation is used heavily in physics applications.

Example 3.58 — Leibniz's notation

Find the derivative of $y = \left(\dfrac{x}{3x+2}\right)^5$.

Let $u = \dfrac{x}{3x+2}$, so $y = u^5$. By the quotient rule and the power rule,

$$\frac{du}{dx} = \frac{(3x+2)\cdot 1 - x\cdot 3}{(3x+2)^2} = \frac{2}{(3x+2)^2} \qquad \text{and} \qquad \frac{dy}{du} = 5u^4.$$

Putting it all together:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 5u^4\cdot\frac{2}{(3x+2)^2} = 5\left(\frac{x}{3x+2}\right)^4\cdot\frac{2}{(3x+2)^2} = \frac{10x^4}{(3x+2)^6}.$$

Remember

With the Leibniz form of the chain rule, the final answer must be expressed entirely in terms of the original variable x.

Summary

- **Chain rule:** $\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big)\, g'(x)$, or $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.
- **With the power rule:** $\frac{d}{dx} \big(g(x)\big)^n = n\big(g(x)\big)^{n-1} g'(x)$.
- **With trigonometric functions:** $\frac{d}{dx} \sin\big(g(x)\big) = \cos\big(g(x)\big)\, g'(x)$, and so on.
- Work from the outside in, one derivative at a time; never evaluate a derivative at a derivative.

§3.7 Derivatives of Inverse Functions

Learning objectives

By the end of this section you should be able to:

- Calculate the derivative of an inverse function.
- Recognize the derivatives of the standard inverse trigonometric functions.

Why it matters

The relationship between the derivative of a function and the derivative of its inverse lets us differentiate inverses without the limit definition. We use it to

- extend the power rule to rational exponents, such as $\sqrt[3]{x} = x^{1/3}$;
- differentiate the inverse trigonometric functions, such as $\sin^{-1} x$ and $\tan^{-1} x$.

The inverse function theorem

If f is invertible and differentiable, we find the derivative of f^{-1} with the chain rule. Recall that $x = f\big(f^{-1}(x)\big)$. Differentiating both sides (chain rule on the right), then solving for $\big(f^{-1}\big)'(x)$:

$$1 = f'\big(f^{-1}(x)\big)\,\big(f^{-1}\big)'(x), \qquad \text{so} \qquad \big(f^{-1}\big)'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}. \tag{3.19}$$

Theorem 3.11 (Inverse Function Theorem)

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'\big(f^{-1}(x)\big) \neq 0$,

$$\frac{dy}{dx} = \frac{d}{dx}\big(f^{-1}(x)\big) = \big(f^{-1}\big)'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}.$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then $g'(x) = \dfrac{1}{f'\big(g(x)\big)}$.

Example 3.61 — the derivative of $\sqrt[3]{x}$

Use the inverse function theorem to find the derivative of $g(x) = \sqrt[3]{x}$.

The function $g(x) = \sqrt[3]{x}$ is the inverse of the function $f(x) = x^3$. Since $g'(x) = \dfrac{1}{f'\big(g(x)\big)}$, begin by finding $f'(x)$:

$$f'(x) = 3x^2 \qquad \text{and} \qquad f'\big(g(x)\big) = 3\big(\sqrt[3]{x}\big)^2 = 3x^{2/3}.$$

Finally,

$$g'(x) = \frac{1}{3x^{2/3}} = \frac{1}{3}\,x^{-2/3}.$$

This is the power rule $\frac{d}{dx}x^n = nx^{n-1}$ with $n = \frac{1}{3}$: $\frac{1}{3}\,x^{\frac{1}{3}-1} = \frac{1}{3}\,x^{-2/3}$.

Extending the power rule to rational exponents

Example 3.61 suggests that the power rule extends to exponents of the form $\frac{1}{n}$, and then to any rational exponent.

Theorem 3.12 (Extending the Power Rule to Rational Exponents)

If n is a positive integer, then

$$\frac{d}{dx}\left(x^{1/n}\right) = \frac{1}{n}x^{(1/n)-1}. \tag{3.20}$$

Also, if n is a positive integer and m is an arbitrary integer, then

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n}x^{(m/n)-1}. \tag{3.21}$$

Proof of Theorem 3.12

The function $g(x) = x^{1/n}$ is the inverse of the function $f(x) = x^n$. Since $g'(x) = \dfrac{1}{f'\big(g(x)\big)}$, begin by finding $f'(x)$:

$$f'(x) = nx^{n-1} \qquad \text{and} \qquad f'\big(g(x)\big) = n\big(x^{1/n}\big)^{n-1} = nx^{(n-1)/n}.$$

Finally,

$$g'(x) = \frac{1}{nx^{(n-1)/n}} = \frac{1}{n}\,x^{(1-n)/n} = \frac{1}{n}\,x^{(1/n)-1}.$$

To differentiate $x^{m/n}$, rewrite it as $\big(x^{1/n}\big)^m$ and apply the chain rule:

$$\frac{d}{dx}\big(x^{m/n}\big) = \frac{d}{dx}\Big(\big(x^{1/n}\big)^m\Big) = m\big(x^{1/n}\big)^{m-1} \cdot \frac{1}{n}\,x^{(1/n)-1} = \frac{m}{n}\,x^{(m/n)-1}. \qquad \square$$

Example 3.62 — applying the power rule to a rational power

Find an equation of the line tangent to the graph of $y = x^{2/3}$ at $x = 8$.

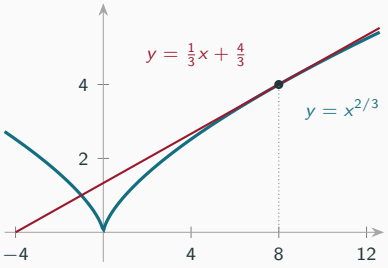
Slope. Since

$$\frac{dy}{dx} = \frac{2}{3}x^{-1/3} \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{x=8} = \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3},$$

the slope of the tangent line at $x = 8$ is $\frac{1}{3}$.

Point. Substituting $x = 8$ into the original function gives $y = 8^{2/3} = 4$, so the point is $(8, 4)$.

Line. By the point-slope formula, $y - 4 = \frac{1}{3}(x - 8)$, that is, $y = \frac{1}{3}x + \frac{4}{3}$.



The derivative of the inverse tangent

Use the inverse function theorem to find the derivative of $g(x) = \tan^{-1} x$.

For x in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $f(x) = \tan x$ is the inverse of $g(x) = \tan^{-1} x$. Begin by finding $f'(x)$, using $\frac{d}{dx}(\tan x) = \tan^2 x + 1$:

$$f'(x) = \tan^2 x + 1 \qquad \text{and} \qquad f'\big(g(x)\big) = \tan^2\left(\tan^{-1} x\right) + 1 = x^2 + 1,$$

since $\tan\left(\tan^{-1} x\right) = x$.

As $x^2 + 1 \neq 0$ for every x, the inverse function theorem gives, for all real x,

$$g'(x) = \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{f'\big(g(x)\big)} = \frac{1}{1 + x^2}.$$

A surprise: the derivative of an inverse trigonometric function is an algebraic function. The derivative of a function need not be of the same type as the function.

Example 3.67 — applying the inverse tangent function

The position of a particle at time t is given by $s(t) = \tan^{-1}\left(\frac{1}{t}\right)$ for $t \geq \frac{1}{2}$. Find the velocity of the particle at time $t = 1$.

Begin by differentiating $s(t)$ to find $v(t)$. By the chain rule, with $u = \frac{1}{t}$ and $\frac{du}{dt} = -\frac{1}{t^2}$:

$$v(t) = s'(t) = \frac{1}{1 + \left(\frac{1}{t}\right)^2} \cdot \frac{-1}{t^2}.$$

Simplifying, we have

$$v(t) = -\frac{1}{t^2\left(1 + \frac{1}{t^2}\right)} = -\frac{1}{t^2 + 1}.$$

Thus, $v(1) = -\frac{1}{2}$.

Example 3.63 — the derivative of the inverse sine

Use the inverse function theorem to find the derivative of $g(x) = \sin^{-1} x$.

For x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $f(x) = \sin x$ is the inverse of $g(x) = \sin^{-1} x$. Begin by finding $f'(x)$:

$$f'(x) = \cos x \qquad \text{and} \qquad f'\big(g(x)\big) = \cos\big(\sin^{-1} x\big) = \sqrt{1 - x^2} \quad \text{(next slide)}.$$

We see that

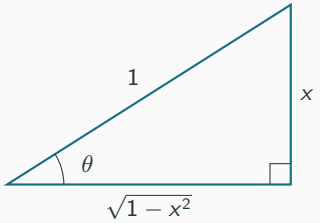
$$g'(x) = \frac{d}{dx}\big(\sin^{-1} x\big) = \frac{1}{f'\big(g(x)\big)} = \frac{1}{\sqrt{1 - x^2}}.$$

Why $\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$ (Figure 3.29)

Set $\sin^{-1} x = \theta$. Then $\sin\theta = x$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

- If $0 < \theta < \frac{\pi}{2}$: in a right triangle with acute angle θ, hypotenuse 1 and opposite side x, the adjacent side is $\sqrt{1-x^2}$ (Pythagoras). So $\cos\theta = \sqrt{1-x^2}$.
- If $-\frac{\pi}{2} < \theta < 0$: then $0 < -\theta < \frac{\pi}{2}$, and $\cos\theta = \cos(-\theta) = \sqrt{1-x^2}$.
- If $\theta = \pm\frac{\pi}{2}$: then $x = \pm 1$, and both sides equal 0. If $\theta = 0$: then $x = 0$, and both sides equal 1.

Consequently, in all cases, $\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$.



Example 3.64 — the chain rule and the inverse sine

Find the derivative of $h(x) = \sin^{-1}\left(2x^3\right)$.

Apply the chain rule, with the derivative of $\sin^{-1}$ from Example 3.63 as the outer derivative:

$$h'(x) = \frac{1}{\sqrt{1-\left(2x^3\right)^2}} \cdot \frac{d}{dx}\left(2x^3\right)$$

$$= \frac{1}{\sqrt{1-4x^6}} \cdot 6x^2 = \frac{6x^2}{\sqrt{1-4x^6}}.$$

Derivatives of inverse trigonometric functions

The derivatives of the remaining inverse trigonometric functions may also be found by using the inverse function theorem.

Theorem 3.13 (Derivatives of Inverse Trigonometric Functions)

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \quad (3.22) \qquad \frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}} \quad (3.23)$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \quad (3.24) \qquad \frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2} \quad (3.25)$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}} \quad (3.26) \qquad \frac{d}{dx}\csc^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}} \quad (3.27)$$

Summary

- **Inverse function theorem:** $\big(f^{-1}\big)'(x) = \dfrac{1}{f'\big(f^{-1}(x)\big)}$, or $g'(x) = \dfrac{1}{f'\big(g(x)\big)}$ when g is the inverse of f.
- **Rational exponents:** $\dfrac{d}{dx}\big(x^{m/n}\big) = \dfrac{m}{n}\,x^{(m/n)-1}$.
- **Inverse trigonometric functions:** $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$, $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$, and so on.

§3.8 Implicit Differentiation

Learning objectives

By the end of this section you should be able to:

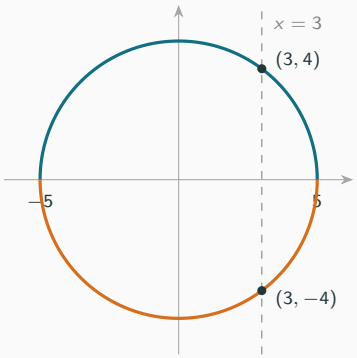
- Find the derivative of a complicated function by using implicit differentiation.
- Use implicit differentiation to determine the equation of a tangent line.

Why it matters

So far we had an explicit formula $y = f(x)$ to differentiate. To find the tangent line to an arbitrary curve, such as the circle $x^2 + y^2 = 25$, we find the derivatives of functions that the equation defines implicitly, without solving for y.

The circle $x^2 + y^2 = 25$: two half circles

- The circle is not the graph of a function: the vertical line $x = 3$ meets it twice, at $(3, 4)$ and $(3, -4)$.
- Solving for y: $y^2 = 25 - x^2$, so $y = \pm\sqrt{25 - x^2}$.
- The circle is made of two half circles, each the graph of a function on $[-5, 5]$:
  - upper half: $y = \sqrt{25 - x^2}$, through $(3, 4)$;
  - lower half: $y = -\sqrt{25 - x^2}$, through $(3, -4)$.
- We say that the equation $x^2 + y^2 = 25$ defines these functions implicitly.



The tangent line to $x^2 + y^2 = 25$ at $(3, 4)$

Near $(3, 4)$ the circle is the graph of a function $y = y(x)$; we do not need its formula.

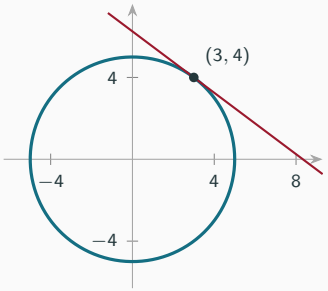
Differentiate both sides with respect to x, treating y as a function of x (chain rule: $\frac{d}{dx}(y^2) = 2y\,\frac{dy}{dx}$):

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25) \quad \Longrightarrow \quad 2x + 2y\,\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$: $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

At $(3, 4)$ the slope is $-\frac{3}{4}$, so the tangent line is

$$y - 4 = -\frac{3}{4}(x - 3), \quad \text{that is,} \quad y = -\frac{3}{4}x + \frac{25}{4}.$$



The circle $x^2 + y^2 = 25$ and its tangent line at $(3, 4)$.

Implicit differentiation: the method

Implicit differentiation

To find $\frac{dy}{dx}$ when an equation in x and y defines y as a function of x:

1. Differentiate both sides of the equation with respect to x, treating y as a function of x (use the chain rule for terms containing y).
2. Put all terms containing $\frac{dy}{dx}$ on the left, all other terms on the right.
3. Factor out $\frac{dy}{dx}$.
4. Solve for $\frac{dy}{dx}$.
5. For the tangent line at a point (x_0, y_0) on the curve: substitute $x = x_0$, $y = y_0$ into $\frac{dy}{dx}$ to get the slope m; then the tangent line is $y - y_0 = m(x - x_0)$.

Example 3.68 — using implicit differentiation

Assuming that y is defined implicitly by the equation $x^2 + y^2 = 25$, find $\frac{dy}{dx}$ and the tangent line at the point $(3, -4)$.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25) \qquad \text{Step 1: differentiate both sides}$$

$$2x + 2y\frac{dy}{dx} = 0 \qquad \text{chain rule: } \frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$$

$$2y\frac{dy}{dx} = -2x \qquad \text{Step 2: keep the terms with } \frac{dy}{dx} \text{ on the left}$$

$$\frac{dy}{dx} = -\frac{x}{y} \qquad \text{Step 4: divide by } 2y \text{ (Step 3 not needed)}$$

Step 5 (tangent line at $(3, -4)$). The slope is $\left.\frac{dy}{dx}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$, so the tangent line is

$$y + 4 = \frac{3}{4}(x - 3), \quad \text{that is,} \quad y = \frac{3}{4}x - \frac{25}{4}.$$

Example 3.69 — implicit differentiation and the product rule

Assuming that y is defined implicitly by $x^3 \sin y + y = 4x + 3$, find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(x^3 \sin y + y\right) = \frac{d}{dx}(4x + 3)$$ Step 1: differentiate both sides

$$\frac{d}{dx}\left(x^3 \sin y\right) + \frac{d}{dx}(y) = 4$$ sum rule on the left

$$\left(\frac{d}{dx}(x^3) \cdot \sin y + \frac{d}{dx}(\sin y) \cdot x^3\right) + \frac{dy}{dx} = 4$$ product rule; $\frac{d}{dx}(y) = \frac{dy}{dx}$

$$3x^2 \sin y + \left(\cos y \, \frac{dy}{dx}\right) \cdot x^3 + \frac{dy}{dx} = 4$$ chain rule: $\frac{d}{dx}(\sin y) = \cos y \, \frac{dy}{dx}$

$$x^3 \cos y \, \frac{dy}{dx} + \frac{dy}{dx} = 4 - 3x^2 \sin y$$ Step 2: terms with $\frac{dy}{dx}$ on the left

$$\frac{dy}{dx}\left(x^3 \cos y + 1\right) = 4 - 3x^2 \sin y$$ Step 3: factor out $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{4 - 3x^2 \sin y}{x^3 \cos y + 1}$$ Step 4: divide by $x^3 \cos y + 1$

Example 3.72 — the tangent line to the folium of Descartes

Find an equation of the line tangent to the graph of $y^3 + x^3 - 3xy = 0$ at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Begin by finding $\frac{dy}{dx}$:

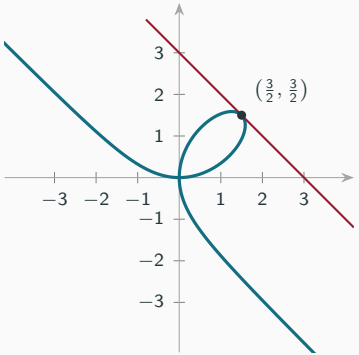
$$\frac{d}{dx}\left(y^3 + x^3 - 3xy\right) = \frac{d}{dx}(0)$$

$$3y^2 \frac{dy}{dx} + 3x^2 - \left(3y + \frac{dy}{dx}\, 3x\right) = 0$$

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}.$$

Substitute $\left(\frac{3}{2}, \frac{3}{2}\right)$: $\left.\frac{dy}{dx}\right|_{\left(\frac{3}{2}, \frac{3}{2}\right)} = \frac{\frac{9}{2} - \frac{27}{4}}{\frac{27}{4} - \frac{9}{2}} = -1.$

Point-slope: $y - \frac{3}{2} = -\left(x - \frac{3}{2}\right)$, that is, $y = -x + 3$.



The folium of Descartes $y^3 + x^3 - 3xy = 0$

Example 3.73 — applying implicit differentiation

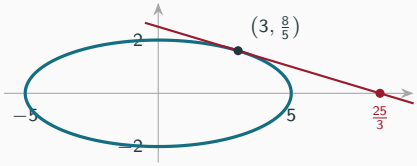
In a video game, a rocket travels along the ellipse $4x^2 + 25y^2 = 100$ and fires missiles along lines tangent to its path. If it fires when it is at $\left(3, \frac{8}{5}\right)$, where does the missile cross the x-axis?

Differentiating, $8x + 50y\,\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\dfrac{4x}{25y}$.

The slope at $\left(3, \frac{8}{5}\right)$ is $-\dfrac{4 \cdot 3}{25 \cdot \frac{8}{5}} = -\dfrac{12}{40} = -\dfrac{3}{10}$,

so the tangent line is $y = -\frac{3}{10}x + \frac{5}{2}$.

Solving $0 = -\frac{3}{10}x + \frac{5}{2}$ gives $x = \frac{25}{3}$: the missile crosses the x-axis at $\left(\frac{25}{3}, 0\right)$.



Summary

- An equation in x and y may define y implicitly as one or several functions of x.
- **Implicit differentiation:** differentiate both sides, treating y as a function of x (chain rule: $\frac{d}{dx}\left(y^n\right) = ny^{n-1}\frac{dy}{dx}$, $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$), then solve for $\frac{dy}{dx}$.
- The result usually involves both x and y.
- **Tangent lines:** substitute the point into $\frac{dy}{dx}$ to get the slope, then use the point-slope equation.

§3.9 Derivatives of Exponential and Logarithmic Functions

Learning objectives

By the end of this section you should be able to:

- Find the derivative of exponential functions.
- Find the derivative of logarithmic functions.
- Use logarithmic differentiation to determine the derivative of a function.

Our route

We start from the natural logarithm, characterized by the way it turns products into sums. We deduce its derivative from its tangent line at $x = 1$, and then obtain e^x as its inverse function.

The natural logarithm

Definition

The natural logarithm (or Napierian logarithm, after John Napier, 1614) is the unique differentiable function $\ln : (0, \infty) \to \mathbb{R}$ such that

$$\ln(xy) = \ln x + \ln y \quad \text{for all } x, y > 0, \qquad \ln 1 = 0, \qquad \ln'(1) = 1.$$

- A logarithm turns products into sums: this is why logarithm tables were used to multiply large numbers by hand.

Properties of the natural logarithm

Properties

From $\ln(xy) = \ln x + \ln y$ and $\ln 1 = 0$ we get, for all $x, y > 0$:

$$\ln\left(\frac{1}{x}\right) = -\ln x, \qquad \ln\left(\frac{x}{y}\right) = \ln x - \ln y, \qquad \ln\left(x^q\right) = q \ln x \text{ for every rational number } q.$$

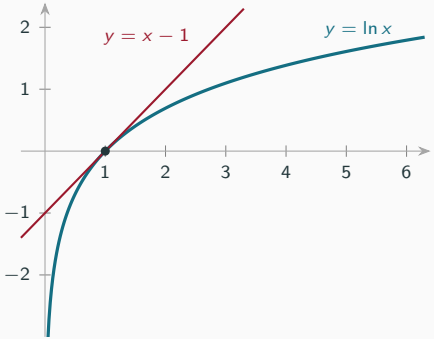
- $\ln x + \ln\left(\frac{1}{x}\right) = \ln\left(x \cdot \frac{1}{x}\right) = \ln 1 = 0$.
- For a positive integer n: $\ln\left(x^n\right) = \ln(x \cdots x) = \ln x + \cdots + \ln x = n \ln x$.
- Then $n \ln\left(x^{1/n}\right) = \ln\left((x^{1/n})^n\right) = \ln x$, so $\ln\left(x^{1/n}\right) = \frac{1}{n} \ln x$.
- For positive integers m, n: $\ln\left(x^{m/n}\right) = \ln\left((x^{1/n})^m\right) = m \ln\left(x^{1/n}\right) = \frac{m}{n} \ln x$.
- Negative exponents follow from $\ln\left(\frac{1}{x}\right) = -\ln x$.

The graph of $\ln x$

- $\ln x$ is defined for $x > 0$ only.
- $\ln 1 = 0$: the graph passes through $(1, 0)$.
- Since $\ln 1 = 0$ and $\ln'(1) = 1$, the tangent line at $x = 1$ is

$$y = x - 1.$$

- Near $x = 1$: $\ln x \approx x - 1$.



The derivative of $\ln x$

Fix $x > 0$. Since $\ln(x+h) - \ln x = \ln\left(\frac{x+h}{x}\right) = \ln\left(1 + \frac{h}{x}\right)$,

$$\begin{aligned}
\frac{d}{dx}\ln x &= \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{h} \\
&= \lim_{k\to 0}\frac{\ln(1+k)}{kx} && \text{substitute } k = \tfrac{h}{x} \quad (k \to 0 \text{ as } h \to 0) \\
&= \frac{1}{x}\lim_{k\to 0}\frac{\ln(1+k)-\ln 1}{k} = \frac{1}{x}\ln'(1) = \frac{1}{x} && \text{the slope of the tangent at } x = 1 \text{ is } 1
\end{aligned}$$

Theorem 3.15 (Derivative of the Natural Logarithm)

If $x > 0$, then

$$\frac{d}{dx}\ln x = \frac{1}{x}. \tag{3.30}$$

Theorem 3.15 (continued)

Let $g(x)$ be a differentiable function. For all x with $g(x) > 0$,

$$\frac{d}{dx}\ln\big(g(x)\big) = \frac{g'(x)}{g(x)}. \tag{3.31}$$

Proof. Write $h(x) = \ln\big(g(x)\big) = f\big(g(x)\big)$ with outer function $f(x) = \ln x$ and inner function $g(x)$.

Since $f'(x) = \dfrac{1}{x}$, we have $f'\big(g(x)\big) = \dfrac{1}{g(x)}$.

By the chain rule,

$$h'(x) = f'\big(g(x)\big)\,g'(x) = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}. \qquad \square$$

Example 3.77 — taking a derivative of a natural logarithm

Find the derivative of $f(x) = \ln\left(x^3 + 3x - 4\right)$.

Use Equation (3.31) with $g(x) = x^3 + 3x - 4$, so $g'(x) = 3x^2 + 3$:

$$f'(x) = \frac{1}{x^3 + 3x - 4} \cdot \left(3x^2 + 3\right) = \frac{3x^2 + 3}{x^3 + 3x - 4}.$$

The exponential function

Definition

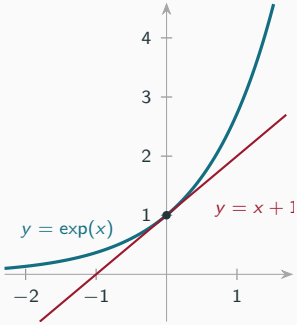
Since ln is increasing ($\ln'(x) = \frac{1}{x} > 0$) and takes every real value, it has an inverse function. The exponential function $\exp : \mathbb{R} \to (0, \infty)$ is the inverse function of ln:

$$y = \exp(x) \iff \ln y = x.$$

Equivalently, exp is the unique differentiable function such that

$$\exp(x + y) = \exp(x)\exp(y), \quad \exp(0) = 1, \quad \exp'(0) = 1.$$

Indeed, $\ln\big(\exp(x)\exp(y)\big) = \ln\exp(x) + \ln\exp(y) = x + y$;
$\ln 1 = 0$; and by the inverse function theorem,
$\exp'(0) = \frac{1}{\ln'(\exp(0))} = \frac{1}{\ln'(1)} = 1$.



Tangent line at $x = 0$: $y = x + 1$.

The number e and the notation e^x

Definition

$$e = \exp(1), \qquad \text{that is,} \qquad \ln e = 1. \qquad \text{Numerically, } e \approx 2.71828.$$

- For every rational number q: $\ln\left(e^q\right) = q \ln e = q$, so $\exp(q) = e^q$.
- This justifies the notation: for every real x we write

$$e^x = \exp(x).$$

In this notation

$$e^{x+y} = e^x e^y, \qquad e^0 = 1, \qquad \ln\left(e^x\right) = x \ \ (x \in \mathbb{R}), \qquad e^{\ln x} = x \ \ (x > 0).$$

The derivative of e^x

By the inverse function theorem (§3.7), with $g(x) = e^x$ the inverse of $f(x) = \ln x$ and $f'(x) = \frac{1}{x}$:

$$g'(x) = \frac{1}{f'\big(g(x)\big)} = \frac{1}{1/e^x} = e^x.$$

Alternatively, using $e^{x+h} = e^x e^h$ and $\exp'(0) = 1$:

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} = e^x \lim_{h\to 0}\frac{e^h - 1}{h} = e^x \exp'(0) = e^x.$$

Theorem 3.14 (Derivative of the Natural Exponential Function)

For all real x,

$$\frac{d}{dx}e^x = e^x.$$

The derivative of $e^{g(x)}$

Chain rule

If $g(x)$ is differentiable, then

$$\frac{d}{dx} e^{g(x)} = e^{g(x)}\, g'(x).$$

Example 3.75 — find the derivative of $y = \frac{e^{x^2}}{x}$

Use the derivative of e^x, the quotient rule, and the chain rule:

$$y' = \frac{\left(e^{x^2} \cdot 2x\right) x - 1 \cdot e^{x^2}}{x^2} = \frac{e^{x^2}\left(2x^2 - 1\right)}{x^2}.$$

Example 3.76 — applying the natural exponential function

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'(t)$, to the population, $A(t)$, is constant.

First find $A'(t)$. By the chain rule,

$$A'(t) = 1000e^{0.3t} \cdot 0.3 = 300e^{0.3t}.$$

Thus the ratio of the rate of change of the population to the population is

$$\frac{A'(t)}{A(t)} = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3.$$

Other exponential functions

Let $b > 0$.

Definition

Since $b = e^{\ln b}$, we define, for every real number x,

$$b^x = e^{x \ln b}.$$

- This agrees with the usual powers: for every rational q, $\ln\left(b^q\right) = q \ln b$, so $b^q = e^{q \ln b}$.
- Then $\ln\left(b^x\right) = x \ln b$ for every real x.

Derivative of b^x

$$\frac{d}{dx} b^x = b^x \ln b.$$

Proof: by the chain rule, $\frac{d}{dx} e^{x \ln b} = e^{x \ln b} \cdot \ln b = b^x \ln b$.

Example 3.79 — applying the derivative of b^x

Find the derivative of $h(x) = \dfrac{3^x}{3^x + 2}$.

Use the quotient rule and $\frac{d}{dx} 3^x = 3^x \ln 3$:

$$h'(x) = \frac{3^x \ln 3 \left(3^x + 2\right) - 3^x \ln 3 \left(3^x\right)}{\left(3^x + 2\right)^2} = \frac{2 \cdot 3^x \ln 3}{\left(3^x + 2\right)^2}.$$

Logarithms in other bases

Definition

Let $b > 0$, $b \neq 1$. The function b^x is one-to-one with range $(0, \infty)$. The logarithm in base b, $\log_b$, is its inverse function: for $x > 0$,

$$y = \log_b x \iff b^y = x.$$

Since $b^y = x \iff \ln\left(b^y\right) = \ln x \iff y \ln b = \ln x$, we get $\log_b x = \dfrac{\ln x}{\ln b}$.

Derivative of $\log_b x$

$$\frac{d}{dx} \log_b x = \frac{1}{x \ln b} \qquad (x > 0).$$

Proof: $\ln b$ is a constant, so $\frac{d}{dx} \frac{\ln x}{\ln b} = \frac{1}{\ln b} \cdot \frac{1}{x}$.

Example 3.80 — finding the slope of a tangent line

Find the slope of the line tangent to the graph of $y = \log_2(3x+1)$ at $x = 1$.

By the derivative of $\log_2$ and the chain rule,

$$\frac{dy}{dx} = \frac{1}{(3x+1)\ln 2} \cdot 3 = \frac{3}{(3x+1)\ln 2}.$$

So the tangent line at $x = 1$ has slope

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{3}{4\ln 2} = \frac{3}{\ln 16}.$$

Logarithmic differentiation

We still cannot differentiate functions such as $y = x^x$ or $y = x^\pi$. Logarithmic differentiation handles them.

Problem-solving strategy: using logarithmic differentiation

1. To differentiate $y = h(x)$, take the natural logarithm of both sides: $\ln y = \ln\big(h(x)\big)$.
2. Use properties of logarithms to expand $\ln\big(h(x)\big)$ as much as possible.
3. Differentiate both sides. On the left we have $\frac{1}{y}\,\frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h(x)$.

Example 3.81 — using logarithmic differentiation

Find the derivative of $y = \left(2x^4 + 1\right)^{\tan x}$.

$$\ln y = \ln \left(2x^4 + 1\right)^{\tan x}$$ Step 1: take ln of both sides

$$\ln y = \tan x \ln \left(2x^4 + 1\right)$$ Step 2: expand

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln \left(2x^4 + 1\right) + \frac{8x^3}{2x^4 + 1} \cdot \tan x$$ Step 3: differentiate (product rule)

$$\frac{dy}{dx} = y \cdot \left(\sec^2 x \ln \left(2x^4 + 1\right) + \frac{8x^3}{2x^4 + 1} \cdot \tan x\right)$$ Step 4: multiply by y

$$\frac{dy}{dx} = \left(2x^4 + 1\right)^{\tan x}\left(\sec^2 x \ln \left(2x^4 + 1\right) + \frac{8x^3}{2x^4 + 1} \cdot \tan x\right)$$ Step 5: substitute y

The power rule for real exponents

- So far: $\frac{d}{dx}x^n = nx^{n-1}$ for integers n (§3.3), and $\frac{d}{dx}x^{m/n} = \frac{m}{n}x^{(m/n)-1}$ for rational exponents (§3.7).
- For $x > 0$ and any real number r, the power x^r is defined: $x^r = e^{r \ln x}$ (the base-b exponential with $b = x$).

Power rule for real exponents

For every real number r and every $x > 0$,

$$\frac{d}{dx}x^r = rx^{r-1}.$$

For example, $\frac{d}{dx}x^\pi = \pi x^{\pi-1}$ and $\frac{d}{dx}x^{\sqrt{2}} = \sqrt{2}\,x^{\sqrt{2}-1}$.

Proof of the power rule (Example 3.83)

Let $y = x^r$, where r is an arbitrary real number and $x > 0$. We use logarithmic differentiation.

$$\ln y = \ln x^r \qquad \text{Step 1: take ln of both sides}$$

$$\ln y = r \ln x \qquad \text{Step 2: expand}$$

$$\frac{1}{y}\frac{dy}{dx} = r\frac{1}{x} \qquad \text{Step 3: differentiate}$$

$$\frac{dy}{dx} = y\frac{r}{x} \qquad \text{Step 4: multiply by } y$$

$$\frac{dy}{dx} = x^r\frac{r}{x} = rx^{r-1} \qquad \text{Step 5: substitute } y = x^r\text{; simplify}$$

Alternatively, $x^r = e^{r \ln x}$, so by the chain rule $\frac{d}{dx}x^r = e^{r \ln x} \cdot \frac{r}{x} = x^r \cdot \frac{r}{x} = rx^{r-1}$.

Summary

Key results

- $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$, $\dfrac{d}{dx}\ln\big(g(x)\big) = \dfrac{g'(x)}{g(x)}$.
- $\dfrac{d}{dx}e^x = e^x$, $\dfrac{d}{dx}e^{g(x)} = e^{g(x)}g'(x)$.
- Other bases: $\dfrac{d}{dx}b^x = b^x \ln b$, $\dfrac{d}{dx}\log_b x = \dfrac{1}{x\ln b}$.
- Logarithmic differentiation: take ln of both sides, expand, differentiate, multiply by y.
- Power rule for real exponents: $\dfrac{d}{dx}x^r = rx^{r-1}$ (every real r, $x > 0$).