Exam 1 Review

Calculus 1 — MATH 2200

Limits · continuity · derivatives · local linear approximations

How your work is marked

Each problem gets one mark

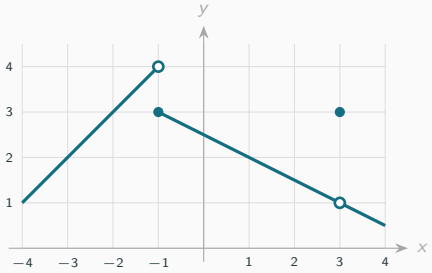
- ✓ **Convincing**: central reasoning right (small slips are fine)
- $\sim$ **Partial**: real work, important gap
- $\times$ **Not yet**: little or no usable work

Level	Needs
2	Problems 1–2: no $\times$, at least one ✓
3	Problems 3–6: at least three ✓, *no* $\times$
4	Level 3 *and* ✓ on Problem 7

What this means for you

- **Never leave a core problem blank.** One $\times$ on Problems 3–6 rules out Level 3. A meaningful partial attempt can earn $\sim$; a blank is always $\times$.
- **Explain why**, in words or pictures, not just *what*.
- **Answer both halves** of "what is right, and what is wrong?"
- Use $=$ only for exact statements, $\approx$ for estimates, and always give **units**.

Limits and continuity from a graph



The graph of g on $-4 \le x \le 4$.

- $g(3) = 3$
- $\lim_{x \to 3} g(x) = 1$ (the filled dot does not matter)
- $\lim_{x \to -1^-} g(x) = 4, \quad \lim_{x \to -1^+} g(x) = 3$
- $\lim_{x \to -1} g(x)$: does not exist, since $4 \neq 3$
- Not continuous at: $x = -1$ and $x = 3$
- Removable one: redefine $g(3) = 1$

The jump at $x = -1$ cannot be removed by changing a single value.

Rates, slopes, and local lines

Setting

Water drains from a tank. $V(t)$ is the volume in liters, t in minutes:

$$V(4) = 500 \text{ L}, \qquad V(6) = 440 \text{ L}, \qquad V'(4) = -25 \text{ L/min}.$$

1. Average rate of change on $[4, 6]$: $\dfrac{440 - 500}{6 - 4} = -30$ L/min.
2. Local line (linear approximation) at $t = 4$: $L(t) = 500 - 25(t - 4)$.
3. Estimate $V(4.4)$: $V(4.4) \approx L(4.4) = 500 - 25(0.4) = 490$ L.
4. If in fact $V(4.4) = 491$ L, the miss is $|491 - 490| = 1$ L.
5. Slope of the tangent line at $t = 4$: $V'(4) = -25$ L/min.

Note the $\approx$ for the estimate, $=$ for exact values, and units on every answer.

1. A limit is exact, and ignores the value at the point

Let $h(x) = \dfrac{x^2 - 9}{x - 3}$, which is undefined at $x = 3$.

- A table ($h(2.99) = 5.99$, $h(3.01) = 6.01$) *suggests* 6, but a table alone cannot prove it.
- Algebra *proves* it: $h(x) = x + 3$ for every $x \neq 3$. The limit only looks at $x \neq 3$, so

$$\lim_{x \to 3} h(x) = \lim_{x \to 3} (x + 3) = 6 \quad \text{exactly.}$$

- Redefining $h(3) = 0$ does *not* change the limit: the limit never uses $h(3)$.
- h is continuous at 3 exactly when $h(3) = 6$: value = limit.

A convincing sentence

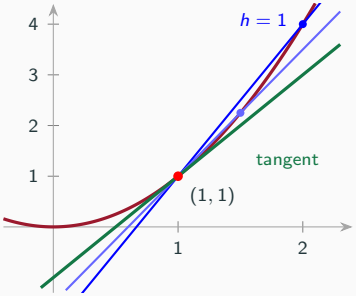
"The limit is decided by the values of h *near* 3, not *at* 3. Near 3, h agrees with $x + 3$, so the limit is exactly 6."

2. The derivative is the limit of average rates

For $f(x) = x^2$ at $x = 1$, the average rate on $[1, 1+h]$ is

$$\frac{(1+h)^2 - 1}{h} = \frac{2h + h^2}{h} = 2 + h \qquad (h \neq 0).$$

- "A slope needs two points." That is true of an *average* rate, which is why we use the interval $[1, 1+h]$.
- $2 + h$ is never 2, but as $h \to 0$ the average rates close in on 2: $2.1,\ 2.01,\ 2.001, \ldots$
- The derivative is that *one number*: $f'(1) = \lim_{h \to 0}(2 + h) = 2$, exactly.



Secants (blue) close in on the tangent (green).

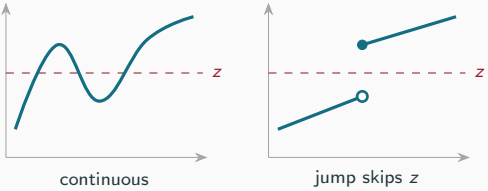
3. The Intermediate Value Theorem: what it says, and what it doesn't

IVT

If f is continuous on $[a, b]$ and z is between $f(a)$ and $f(b)$, then $f(c) = z$ for at least one c in $[a, b]$.

A hiker's altitude changes continuously from 1200 m at 8:00 to 2100 m at 12:00.

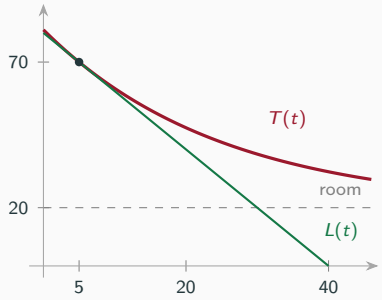
- Was she at exactly 1500 m at some time? Yes, by the IVT.
- "At 9:20 (a proportional guess), and only then" is unsupported. The IVT says nothing about *when*, and she may pass 1500 m several times.
- Without continuity (a reading that jumps), the IVT does not apply, and a value can be skipped.



4. What a local line can, and cannot, tell you

Coffee cools: $T(t)$ in °C, t in minutes, with $T(5) = 70$ and $T'(5) = -2$ °C/min.

- **Meaning:** at 5 minutes, the coffee is cooling at about 2 °C per minute.
- **Near $t = 5$:** the graph looks like its tangent line, so $T(5.5) \approx 70 - 2(0.5) = 69$ °C is trustworthy.
- **Far from $t = 5$:** the line reaches 0 °C at $t = 40$, which is impossible in a 20 °C room. The rate itself changes, and one good nearby estimate does not validate the line far away.
- **Better forecast:** more readings, later rates $T'(t)$, or a model of how cooling slows.



Judging claims from data

A reservoir's water level $W(t)$, in meters, is read every 2 hours:

t (h)	0	2	4	6
$W(t)$ (m)	12.0	12.6	13.0	13.2

average rates: 0.3, 0.2, 0.1 m/h

1. "The average rate from $t = 0$ to $t = 2$ was 0.3 m/h."
   Supported: $(12.6 - 12.0)/2 = 0.3$.
2. "The level was exactly 12.8 m at some moment between $t = 2$ and $t = 4$."
   Needs an assumption: W continuous; then the IVT applies ($12.6 < 12.8 < 13.0$).
3. "$W'(3) = 0.2$ m/h exactly."
   Not supportable: 0.2 is an *average* rate on $[2, 4]$. At best $W'(3) \approx 0.2$.
4. "The level rose at every moment from $t = 0$ to $t = 6$."
   Needs an assumption about what happens between readings: W could dip in between.
5. "At 0.1 m/h, the level will reach 15 m by $t = 24$."
   Not supportable: this extrapolates far past the data, and the rates are already changing.

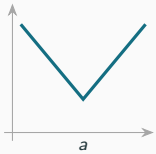
Differentiable $\Rightarrow$ continuous, but not the other way around

Theorem

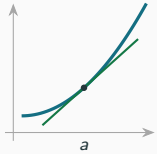
If f is differentiable at a, then f is continuous at a. So if f is *not* continuous at a, it cannot be differentiable there.



Not continuous, so not differentiable: no single line through $(a, f(a))$ fits.



Continuous but not differentiable: a corner never straightens out when you zoom in.



Differentiable: zooming in, the graph looks like the tangent line, with slope $f'(a)$.

Sentences worth practising

- "The limit depends on the values of f *near* a, not *at* a."
- "$f'(a)$ is how fast [quantity] is changing per [unit of input] at [moment], in [units]."
- "The derivative is the exact number the average rates approach as the interval shrinks."
- "The IVT guarantees at least one such c, but not where it is or how many there are."
- "The local line is reliable only near a; farther away the rate may change."
- "Differentiable $\Rightarrow$ continuous; a corner is continuous but not differentiable."

On exam day

Attempt every problem · explain *why* · $=$ vs. $\approx$ · units.