Noncommutative Geometry and Semiclassical Analysis

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Abstract. Semiclassical analysis and noncommutative geometry are two active fields within Quantum theory. Its only recently that links between them have emerged. This article is devoted to explainig how to extend the semiclassical Weyls law and Connes' integral formulas to a large class of noncommutative manifolds (i.e., spectral triples). This improves and simplifies earlier results of McDonald-Sukochev-Zanin. We illustrate these results in several examples: spectral triples associated with Dirichlet and Neumann conditions on bounded domains in Euclidean spaces, spectral triples on quantum tori, and spectral triples associated with sub-Riemannian structures. In the case of quantum tori we solve a conjecture of Edward McDonald and the author.

Keywords: noncommutative geometry, semiclassical analysis, operator theory

1 Introduction

Noncommutative geometry and semiclassical analysis are two important fields within the broader area of quantum theory. Recently, bridges between them have been emerging (see, e.g., [62, 73, 74, 80, 93]). This has its roots, on the one hand, in the operator theoretic approach to semiclassical analysis of Schrödinger operators pioneered by Birman-Solomyak [11, 18] and Simon [85], and, on the other hand, in the paramount role of operator ideals in noncommutative geometry program of Connes [29].

In the framework of noncommutative geometry, the role of manifolds is played by spectral triples $(\mathscr{A}, \mathscr{H}, D)$, where $\mathscr{A}$ is a (unital) $*$-algebra represented by bounded operators on the Hilbert space $\mathscr{H}$ and D is an unbounded selfadjoint operator on $\mathscr{H}$ with compact resolvent which commutes with elements of $\mathscr{A}$ up to bounded operators. The notion of dimension is replaced by p-summability, in the sense that D^{-1} lies in the weak Schatten class $\mathscr{L}_{p,\infty}$[1]. The main prototype of a spectral triple is the Dirac spectral triple $(C^\infty(M), L^2(M, \not{S}), \not{D})$ associated with any closed Riemannian spin manifold. A closely related example is the square-root Laplacian spectral triple $(C^\infty(M), L^2(M), \sqrt{\Delta_g})$.

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[1] We refer to the appendix for the main definitions and properties of singular values and weak Schatten classes $\mathscr{L}_{p,\infty}$

In a recent article [62] McDonald-Sukochev-Zanin established a semiclassical Weyl law for spectral triples $(\mathscr{A},\mathscr{H},D)$, i.e., for the counting functions of Schrödinger operators h^2D^2+V under the semiclassical limit $h\rightarrow 0^+$. This semiclassical Weyl law is established as a consequence of a general Tauberian theorem for non-commuting pairs of operators on Hilbert space (see [62, Theorem 1.2]). In order to apply this Tauberian theorem three conditions for the spectral triple are required:

(a) p-summability with $p>2$.
(b) Lipschitz regularity, i.e., $[|D|,a]\in\mathscr{L}(\mathscr{H})$ for all $a\in\mathscr{A}$.
(c) A Tauberian condition for the zeta functions $\operatorname{Tr}[a^z|D|^{-z}]$, $a\geq 0$, $\Re z>p$ (see Condition (Z_0) below).

It would be desirable to remove the technical conditions (a) and (b), especially the former, since it prevents us from dealing with 1-dimensional and 2-dimensional examples. It is actually conjectured in [62] that the main results there should hold for $p\leq 2$.

In addition, it would be good to relate the Tauberian condition (c) to more standard Tauberian conditions in terms of zeta functions $\operatorname{Tr}[a|D|^{-z}]$ or heat traces $\operatorname{Tr}[ae^{-tD^2}]$, $t>0$, which do not involve powers of a. The need for this was pointed out by Alain Connes during an online seminar in May 2021.

The main aim of this paper is to show that, in the setup of spectral triples, results such as semiclassical Weyl laws and Connes' integration formula, are mere consequences of *classical* Weyl laws for some Laplace-type operators, namely, operators of the form aD^2a, where a ranges over positive invertible elements of $\mathscr{A}$. This simplifies the approach of [62] and leads to stronger results. In particular, we remove the conditions (a)–(b) in the approach of [62] and replace the Tauberian condition in (c) by a more general spectral theoretic condition (see Condition (W) below). That condition is easier to check in practice, which allows us to widen significantly the scope of applications of the results and cover various new examples.

All the proofs of the results mentioned here, along with further results, can be found in [75].

2 Main Results

In this section, we present the main result of this note.

2.1 Spectral Condition (W)

Let $(\mathscr{A},\mathscr{H},D)$ be a p-summable (unital) spectral triple, where p may be *any* positive number. We denote by $\overline{\mathscr{A}}$ the closure of $\mathscr{A}$ in $\mathscr{L}(\mathscr{H})$. We also let $\overline{\mathscr{A}}_{++}$ be the cone of invertible positive elements of $\overline{\mathscr{A}}$ (i.e., selfadjoint elements with positive spectrum), and set $\mathscr{A}_{++}=\mathscr{A}\cap\overline{\mathscr{A}}_{++}$. Moreover, given any $a\in\overline{\mathscr{A}}_{++}$, we let

$$\lambda_0(aD^2a)\leq\lambda_1(aD^2a)\leq\cdots \tag{1}$$

be the *positive* eigenvalue of aD^2a (counted with multiplicity).

The approach of this monograph relies on trading the Tauberian condition of [62] for some spectral condition (Condition (W) below) which does not involve any additional regularity conditions or restriction on the degree of summability for spectral triples. As we shall see this condition is actually weaker than the Tauberian condition of [62] and will allow us to cover a wealth of new examples. This condition is formulated as follows.

Definition 1 (Condition (W)). *We say that Condition (W) is satisfied if, for every* $a \in \mathscr{A}_{++}$, *we have*

$$\lim_{j\to\infty} j^{-\frac{2}{p}}\lambda_j\big(aD^2a\big) = \tau\big[a^{-p}\big]^{-\frac{2}{p}}, \tag{2}$$

where $\tau : \overline{\mathscr{A}} \to \mathbb{C}$ *is a given positive linear map.*

We can see Condition (W) as requiring to have Weyl laws for conformal deformations of the square D^2, where conformal deformation is meant in the sense of Connes-Moscovici [30].

2.2 Spectral asymptotics for Birman-Schwinger operators

As we shall see, Condition (W) allows us to get spectral asymptotics for Birman-Schwinger type operators $|D|^{-q/2}a|D|^{-q/2}$, $q > 0$, as a ranges over the C^*-algebra $\overline{\mathscr{A}}$. Once we have these spectral asymptotics, routine arguments enable us to get semiclassical Weyl laws and integration formulas (see below).

In what follows, given any $q > 0$, we denote by $\mu_j(|D|^{-q/2}a|D|^{-q/2})$, $j \geq 0$, the singular values of $|D|^{-q/2}a|D|^{-q/2}$ (i.e., the eigenvalues of the absolute value $||D|^{-q/2}a|D|^{-q/2}|$). If $a^* = a$, then $|D|^{-q/2}a|D|^{-q/2}$ is selfadjoint, and so we further denote by $\lambda_j^{\pm}(|D|^{-q/2}a|D|^{-q/2})$, $j \geq 0$, its positive/negative eigenvalues (counted with multiplicity).

More precisely, we establish the following spectral asymptotics,

Proposition 1 (Spectral Asymptotics). *Assume Condition* (W) *holds. Given any* $q > 0$, *for every* $a \in \overline{\mathscr{A}}$, *we have*

$$\lim_{j\to\infty} j^{\frac{q}{p}}\mu_j\big(|D|^{-\frac{q}{2}}a|D|^{-\frac{q}{2}}\big) = \tau\left[|a|^{\frac{p}{q}}\right]^{\frac{q}{p}}, \tag{3}$$

$$\lim_{j\to\infty} j^{\frac{q}{p}}\lambda_j^{\pm}\big(|D|^{-\frac{q}{2}}a|D|^{-\frac{q}{2}}\big) = \tau\left[\big(a_{\pm}\big)^{\frac{p}{q}}\right]^{\frac{q}{p}} \quad (\text{if } a^* = a). \tag{4}$$

Here $a_{\pm} = \frac{1}{2}(|a| \pm a)$ *are the positive/negative parts of* a.

Remark 1. In the special case $q = p/2$, a version of (3) for $a \geq 0$ is provided by [62, Theorem 1.4] and a version of (4) is given in [62, Theorem 1.5]. Both results are proved under assuming Lipschitz regularity and p-summability with $p > 2$. These two restrictions are removed in Proposition 1. The spectral asymptotics (4)–(3) for $q \neq p/2$ are new.

2.3 Semiclassical Weyl laws

Combining the spectral asymptotics (4) with the Birman-Schwinger principle enables us to get semiclassical Weyl laws for (fractional) Schrödinger operators,

$$H_V^{(q)}(h) := h^{2q}\left(D^2\right)^q + V, \qquad h > 0. \tag{5}$$

More precisely, in our setup the spectrum of $H_V^{(q)}(h)$ is discrete and bounded from below. Given any energy level $\lambda \in \mathbb{R}$, we denote by $N(H_V^{(q)}(h); \lambda)$ the number of eigenvalues $H_V^{(q)}(h)$ counted with multiplicity that are $< \lambda$. In particular, $N(H_V^{(q)}(h); 0)$ is the number of negative eigenvalues. We then have the following semiclassical Weyl law.

Theorem 1 (Semiclassical Weyl law). *Assume Condition* (W) *holds. Given any $q > 0$ and $V^* = V \in \overline{\mathscr{A}}$, for all energy levels $\lambda \in \mathbb{R}$, we have*

$$\lim_{h\to 0^+} h^p N\big(H_V^{(q)}(h); \lambda\big) = \tau\Big[\,(V-\lambda)_-^{\frac{p}{2q}}\,\Big]. \tag{6}$$

Remark 2. For $q = 1$ we recover from (6) the semiclassical Weyl law of [62] without any extra regularity or summability assumption whatsoever, and so this extends this result to all values of p. The semiclassical Weyl law (6) for $q \neq 1$ are new (even for $p > 2$).

Remark 3. It is known that semiclassical Weyl laws for $q = 2$ imply a classical Weyl law for the eigenvalue distribution of D^2. Therefore, we may interpret Theorem 1 as some kind of converse of this statement.

2.4 Integration formulas

The Weyl laws (3)–(4) have further implications in noncommutative geometry. In the framework of noncommutative geometry the role of the integral is played by positive traces in $\mathscr{L}_{1,\infty}$ (see [29]). An important class of such traces is provided by Dixmier traces [36]. Following Connes [29] an operator $A \in \mathscr{L}_{1,\infty}$ is called *measurable* if it takes the same value on all Dixmier traces. Equivalently, A is measurable if and only if

$$⨍ A := \lim_{N\to\infty} \sum_{j<N} \lambda_j(A) \text{ exists}, \tag{7}$$

where $\{\lambda_j(A)\}$ is an eigenvalue sequence for A (see [57, 74]). The limit $⨍ A$ is precisely the NC integral of A. A stronger notion of measurability (called *strong measurability* or *positive measurability*) requires A to take the same value on all positive (normalized) traces.

Given a closed Riemannian manifold (M^n, g), Connes' integration formula asserts that, if $f \in C^\infty(M)$, then the operator $f\Delta_g^{-n/2}$ is strongly measurable, and we have

$$⨍ f\Delta_g^{-\frac{n}{2}} = c(n)\int_M f(x)d\nu_g(x), \qquad c(n) := (2\pi)^{-n}|\mathbb{B}^n|, \tag{8}$$

where $\nu_g(x)$ is the Riemannian measure (see [28, 49]). This shows that the NC integral recaptures the Riemannnian measure.

Spectral asymptotics of form (4) with $q/p = 1$ implies an even stronger form of measurability. If $A^* = A$ it can be shown that if $\lim_{j\to\infty} j\lambda_j^{\pm}(A)$ exists, then A is strongly measurable, and $⨍ A$ agrees with the difference of these limits (see, e.g., [74]). This is extended to non-selfadjoint operators by looking at real and imaginary parts. If these conditions are satisfied we shall say that A is *spectrally measurable.* Thus, spectral measurability is an even stronger form of measurability than strong measurability.

It actually follows from results of Birman-Solomyak [14–16] that, on any closed manifold M^n, $\varPsi$DOs of order $-n$ are spectrally measurable. Birman-Solomyak's result actually predates by a decade Connes' integration formula (8) and implies a stronger form of this result.

In the light of this, the spectral asymptotics (3)–(4) imply far-reaching extensions to spectral triples of Connes' integration formula.

Theorem 2 (Integration Formula). *Assume Condition* (W) *holds. For all* $a \in \overline{\mathscr{A}}$*, the operators* $a|D|^{-p}$*,* $|D|^{-p/2}a|D|^{-p/2}$*, and* $||D|^{-p/2}a|D|^{-p/2}|$ *are spectrally measurable, and we have*

$$⨍ a|D|^{-p} = ⨍ |D|^{-\frac{p}{2}}a|D|^{-\frac{p}{2}} = \tau\big[a\big], \tag{9}$$

$$⨍ \big||D|^{-\frac{p}{2}}a|D|^{-\frac{p}{2}}\big| = \tau\big[|a|\big]. \tag{10}$$

Remark 4. The integration formula for $a|D|^{-p}$ is alluded to in [62, p. 77].

Remark 5. The spectral asymptotics (3)–(4) further imply that, for any $q > 0$, the operators $||D|^{-q/2}a|D|^{-q/2}|^{p/q}$ and $(|D|^{-q/2}a|D|^{-q/2})_{\pm}^{p/q}$ (if $a^* = a$), are spectrally measurable, and we have

$$⨍ \big||D|^{-\frac{q}{2}}a|D|^{-\frac{q}{2}}\big|^{\frac{p}{q}} = \tau\left[|a|^{\frac{p}{q}}\right], \qquad ⨍ \big(|D|^{-\frac{q}{2}}a|D|^{-\frac{q}{2}}\big)_{\pm}^{\frac{p}{q}} = \tau\left[(a_{\pm})^{\frac{p}{q}}\right]. \tag{11}$$

2.5 Extensions to unbounded potentials

It is important to understand to which extent the spectral asymptotics (3)–(4), semiclassical Weyl law (6) and integration formulas (9)–(10) above continue to hold whenever the corresponding r.h.s. make sense. Note that, from a spectral theoretic perspective, this involves considering *unbounded* potentials. On $\mathbb{R}^n$ or bounded domains in $\mathbb{R}^n$, the extension of the semiclassical Weyl laws to optimal L^r-spaces of potentials was made possible by the CLR inequality of Cwikel [33], Lieb [55, 56], and Rozenblum [78, 79].

The approach of Cwikel [33] relied on proving some weak-Schatten quasinorm estimates for Birman-Schwinger operators that were conjectured by Simon [85]. Combining these estimates with the perturbation theory of Birman-Solomyak further enables us to extends the spectral asymptotics for eigenvalues of Birman-Schwinger operators and the integration formulas to the relevant L^r-classes. We refer to [54, 80, 81, 88, 91], and the references therein, for various extensions of the Cwikel estimates.

In this paper, we lay down a general paradigm for extending the previous results to L^r-potentials. Conceptually, this is a mere elaboration of the approach of Birman-Solomyak (see, e.g., [13]) to semiclassical Weyl laws for L^r-potentials (see also Simon [85]). The only technical difference here is that, as we are in a noncommutative setting, we need to work with noncommutative L^r-spaces.

We proceed as follows. Suppose that Condition (W) holds and τ is the restriction of a positive faithful finite trace $\tau : \mathscr{M} \rightarrow \mathbb{C}$, where $\mathscr{M} \subset \mathscr{L}(\mathscr{H})$ is the von Neumann algebra generated by $\mathscr{A}$ (i.e., its weak closure in $\mathscr{L}(\mathscr{H})$). As τ is finite, for any $r \in [1, \infty)$ we may define the noncommutative L^r-space $L_r(\mathscr{M})$[2] as the closure of $\mathscr{M}$ with respect to the Banach norm $x \rightarrow \|x\|_r := \big(\tau\big[|x|^r\big]\big)^{\frac{1}{r}}$.

Definition 2 (Condition (C$_r$)). *Let $r > 0$, and set $\hat{r} = \max(r, 1)$. We say that Condition* (C$_r$) *holds if there is a continuous $*$-invariant norm $\|\cdot\|_{(r)}$ on $\overline{\mathscr{A}}$ such that*

(i) The inclusion of $\overline{\mathscr{A}}$ into $L_{\hat{r}}(\mathscr{M})$ is continuous with respect to the $\|\cdot\|_{(r)}$-topology.
(ii) There is a constant $C_r > 0$ such that

$$\big\|(1+D^2)^{-\frac{p}{4r}} a (1+D^2)^{-\frac{p}{4r}}\big\|_{r,\infty} \leq C_r \|a\|_{(r)} \qquad \forall a \in \overline{\mathscr{A}}. \tag{12}$$

If Condition (C$_r$) holds, then we denote by $\mathscr{V}_r$ the Banach space completion of $\overline{\mathscr{A}}$ with respect to the norm $\|\cdot\|_{(r)}$. For instance, for bounded domains $\Omega \subset \mathbb{R}^n$ we have $\mathscr{M} = L^\infty(\Omega)$ and we can take

$$\mathscr{V}_r = L^r(\Omega) \quad (r > 1), \qquad \mathscr{V}_1 = L^s(\Omega),\ s > 1, \qquad \mathscr{V}_r = L^1(\Omega) \quad (r < 1). \tag{13}$$

In fact, in the critical case $r = 1$ we may even take $\mathscr{V}_1$ be the Orlicz space $L\log L(\Omega)$ if n is even (see [81, 87, 88]; see also § 3).

If $r = pq^{-1}$ and $x \in \mathscr{V}_r$, then the estimate (12) ensures that $|D|^{-q/2} x |D|^{-q/2}$ is in the weak Schatten class $\mathscr{L}_{r,\infty}$. By using Birman-Solomyak's perturbation theory [11, 17] we then obtain the following extension of Proposition 1 to $\mathscr{V}_r$-potentials.

Proposition 2 (Spectral Asymptotics; $\mathscr{V}_r$-version). *Assume Condition* (W) *is satisfied. Let $q > 0$, and assume further that Condition* (C$_r$) *holds with $r := pq^{-1}$. Given any $x \in \mathscr{V}_r$, we have*

$$\lim_{j\rightarrow\infty} j^{\frac{q}{p}} \mu_j\big(|D|^{-\frac{q}{2}} x |D|^{-\frac{q}{2}}\big) = \tau\Big[|x|^{\frac{p}{q}}\Big]^{\frac{q}{p}}, \tag{14}$$

$$\lim_{j\rightarrow\infty} j^{\frac{q}{p}} \lambda_j^{\pm}\big(|D|^{-\frac{q}{2}} x |D|^{-\frac{q}{2}}\big) = \tau\Big[\big(x_{\pm}\big)^{\frac{p}{q}}\Big]^{\frac{q}{p}} \quad (\text{if } x^* = x). \tag{15}$$

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[2] Throughout this paper we shall use subscripts for the exponents of NC L^p-spaces.

Moreover, if $V = V^* \in \mathscr{V}_r$ the compactness of $|D|^{-q/2}V|D|^{-q/2}$ ensures that V is $(D^2)^q$-form compact, and so the fractional Scrödinger operator $H_V^{(q)}(h) = h^{2q}(D^2)^q + V$, $h > 0$, make sense as form sums and are bounded from below operators with pure district spectrum. We then obtain the following extension of the semiclassical Weyl law (6).

Theorem 3 (Semiclassical Weyl law; $\mathscr{V}_r$-version). *Suppose that Condition* (W) *is satisfied. Let $q > 0$ and assume that Condition* (C_r) *holds with $r = p(2q)^{-1}$. Then, the semiclassical Weyl law (6) holds for all $V = V^* \in \mathscr{V}_r$, i.e., for all energy levels $\lambda \in \mathbb{R}$, we have*

$$\lim_{h\to 0^+} h^p N\big(H_V^{(q)}(h); \lambda\big) = \tau\Big[\,(V-\lambda)_-^{\frac{p}{2q}}\,\Big]. \tag{16}$$

We further have the following extension of the integration formula (9).

Theorem 4 (Integration Formulas; $\mathscr{V}_1$-version). *Suppose that Condition* (W) *is satisfied. Assume further that Condition* (C_1) *holds. Then, for every $x \in \mathscr{V}_1$, the operators $|D|^{-p/2}x|D|^{-p/2}$ and $|D|^{-p/2}x|D|^{-p/2}$ are spectrally measurable, and we have*

$$\fint |D|^{-\frac{p}{2}}x|D|^{-\frac{p}{2}} = \tau\big[x\big], \qquad \fint \big||D|^{-\frac{p}{2}}x|D|^{-\frac{p}{2}}\big| = \tau\big[|x|\big]. \tag{17}$$

2.6 Tauberian conditions

There are various ways to establish Weyl laws. They are often deduced from Tauberian theorems. The approach of [62] relies on some new Tauberian condition. As mentioned above, the need for a more standard Tauberian condition was pointed out by Alain Connes. From this perspective, we establish the following Tauberian criteria for Condition (W).

Theorem 5 (Tauberian Theorem). *Assume there exists a Fréchet subalgebra $\mathscr{B} \subseteq \overline{\mathscr{A}}$ which contains $\mathscr{A}$ and is closed under holomorphic functional calculus such that one of the following conditions is satisfied:*

- **Condition (Z)**. *For every $a \in \mathscr{B}$, the function*

  $$\mathrm{Tr}\big[a|D|^{-s}\big] - p\tau(a)(s-p)^{-1}, \qquad \Re s > p, \tag{18}$$

  has a unique continuous extension to the halfplane $\Re s \geq p$.
- **Condition (H)**. *There is $\delta > 0$ such that, for all $a \in \mathscr{B}$, as $t \to 0^+$ we have*

  $$\mathrm{Tr}\big[ae^{-tD^2}\big] = \Gamma\left(1 + \frac{p}{2}\right) t^{-\frac{p}{2}} \left[\tau(a) + \mathrm{O}\big(t^{\delta}\big)\right]. \tag{19}$$

Then the Weyl law (2) is satisfied for all $a \in \mathscr{A}_{++}$, i.e., Condition (W) *holds.*

Theorem 5 is a refinement of the Tauberian theorem of [62]. First, all the extra summability and summability assumptions in [62] are removed from the picture. Second, the proof of Theorem 5 involves showing that Condition (Z) and Condition (H) implies a weaker version of the Tauberian condition of [62] and showing that this weaker version implies Condition (W). Therefore, we see that we get the Weyl law (2) under a weaker form of the Tauberian condition of [62].

Condition (Z) and Condition (H) are the types of Tauberian conditions sought for by Alain Connes in his comments mentioned above.

In practice, they are easier to check than the Tauberian condition of [62]. As we shall see in the second half of this monograph, they allow us to deal with a wealth of new examples.

Corollary 1. *All the results above, i.e., Proposition 1 through Theorem 4, hold true if we replace Condition* (W) *by Condition* (H) *or Condition* (Z).

2.7 Operator Theory – Proofs of Proposition 1 and Theorem 5

Proposition 1 and Theorem 5 are the main new technical tools that allow to get the main results of this paper. Their proofs involve a significant amount of operator theory.

As in [62] the approach relies on the integral representation of [31, 92] for the differences $(A^{1/2}BA^{1/2})^s - B^sA^s$, $\Re s > 1$, where A and B are arbitrary positive operators. However, we make a few observations that simplify matters significantly and lead to stronger results. They partly rely on well known results of Birman-Solomyak [10, 12, 19] on double integral operators. In particular, degree summability issues disappear and we do not need to appeal to the difficult and deep results of [47, 76] that were used in [62]. Thus, even if we get stronger results, with respect to the approaches of [31, 62, 92], our approach remains fairly elementary.

Our approach relies on the following two lemmas.

Lemma 1. *Let $q > 0$, and set $\epsilon_0 = \min(q, 1)$. For all $a \in \mathscr{A}_{++}$, we have*

$$\big(a^{\frac{1}{2}}|D|^{-1}a^{\frac{1}{2}}\big)^q - a^{\frac{q}{2}}|D|^{-q}a^{\frac{q}{2}} \in \mathscr{L}_{(q+\epsilon)^{-1}p,\infty} \qquad \forall \epsilon \in (0, \epsilon_0). \tag{20}$$

Remark 6. In the setup of spectral triples, the above lemma is a refinement of a result of Connes-Sukochev-Zanin [31, Lemma 5.3] (which was stated without proof by Connes [29, Lemma IV.3.11]). This lemma is instrumental in removing from the approach of [62] the restriction $p > 2$ on the summability degree of $(\mathscr{A}, \mathscr{H}, D)$.

Lemma 2. *Let $\alpha \in (0, 1]$ be such that $\delta := \min(1, p - \alpha) > 0$. Given any $a \in \mathscr{A}_{++}$, the function*

$$\operatorname{Tr}\big[\big(a^{\frac{1}{2}}|D|^{-\alpha}a^{\frac{1}{2}}\big)^s\big] - \operatorname{Tr}\big[|D|^{-\alpha s}a^s\big], \qquad \Re s > \alpha^{-1}p, \tag{21}$$

has an analytic extension to the half-plane $\Re s > \alpha^{-1}(p - \delta)$.

Remark 7. In the setup of (unital) spectral triples, Lemma 2 is a significant improvement of a result of Sukochev-Zanin [92, Theorem 5.4.2], which is proved for $p > 2$ under very strong regularity assumptions (their result actually holds for non-unital spectral triples). All these regularity conditions are removed in Lemma 2. In addition, the assumptions for Lemma 2 do not induce any restrictions on the summability degree p, since we always can find $\alpha \in (0,1]$ small enough so that $\min(1, p-\alpha) > 0$.

3 Bounded Domains in Euclidean Spaces

In this section, we explain how the results from the previous section enables us to recover well known spectral asymptotics for Laplacians on bounded domains in Euclidean space. Those results date back from the 60s and 70s. The only novelty here is recovering those results from heat kernel asymptotics that were established by Carleman [22] and Minakshisundaram [68] in the 30s and 40s. We also state integration formulas for bounded Euclidean domains, which do not seem to have appeared elsewhere.

Throughout this section we let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega$. We also let Δ_Ω be either the Dirichlet Laplacian or the Neumann Laplacian on Ω. In either case we get an essentially selfadjoint strongly elliptic 2nd order differential operator whose spectrum is non-negative and purely discrete (we have a positive spectrum in the Dirichlet case).

To get a spectral triple we consider the $*$-algebra,

$$\dot{C}^\infty(\Omega) := \mathbb{C}1 + C_c^\infty(\Omega). \tag{22}$$

That is, $\dot{C}^\infty(\Omega)$ is the algebra of C^∞-functions on Ω that are constant near the boundary $\partial\Omega$. This is a $*$-subalgebra of the C^*-algebra of bounded continuous functions on Ω, and so it is represented in $L^2(\Omega)$ by bounded multiplication operators. Its norm closure is the C^*-algebra,

$$\dot{C}(\Omega) := \mathbb{C} + C_0(\Omega), \tag{23}$$

where $C_0(\Omega)$ is the (closed) ideal of continuous functions on Ω that converges to 0 near the boundary $\partial\Omega$. In other words, $\dot{C}(\Omega)$ is the space of continuous functions on $\overline{\Omega}$ that are constant on the boundary $\partial\Omega$.

We define the square-root Laplacian $\Delta_\Omega^{1/2}$ by Borel functional calculus. Results of Seeley [82, 84] and Grubb [43, §4.4] enable us to show that we get a spectral triple,

$$\left(\dot{C}^\infty(\Omega), L^2(\Omega), \sqrt{\Delta_\Omega}\right). \tag{24}$$

This spectral triple satisfies Condition (W). In fact, old results of Carleman [22] and Minakshisundaram [68] (see also [83]) allow us to check that Condition (Z) holds. Alternatively, it follows from results of Grubb [43, §4.5] that, for all $0 < f \in \dot{C}^\infty(\Omega)$, as $j \to \infty$ we have the Weyl law,

$$\lambda_j\big(f\Delta_\Omega f\big) \sim j^{\frac{2}{n}} \left(c(n)\int_\Omega f(x)^{-n}dx\right)^{-\frac{2}{n}}, \qquad c(n) := (2\pi)^{-n}|\mathbb{B}^n|. \tag{25}$$

In any case, Condition (W) holds with

$$\tau(f) = c(n)\int_{\Omega} f(x)dx, \qquad f\in \dot{C}^{\infty}(\Omega). \tag{26}$$

Furthermore, Condition (C_r) is a consequence of well known Cwikel-type estimates. It holds as follows:

(i) For $r>1$ with $\mathscr{V}_r = L_r(\Omega)$ (see [78, 79]).
(ii) For $r<1$ with $\mathscr{V}_r = L^1(\Omega)$ (see [13]).
(iii) For $r=1$ with $\mathscr{V}_1 = L\log L(\Omega)$ if n is even (see [81, 87, 88]) and $\mathscr{V}_1 = L_{r'}(\Omega)$ for any $r'>1$ if n is odd (see [13]).

Here $L\log L(\Omega)$ is the space of $L\log L$-Orlicz functions on Ω, i.e., measurable functions $f(x)$ such that

$$\int_{\Omega} |f(x)|\log(1+|f(x)|)dx < \infty. \tag{27}$$

In particular, $L^{r'}(\Omega) \subsetneq L\log L(\Omega) \subsetneq L^1(\Omega)$ for all $r'>1$.

For the spectral triple (24) the spectral asymptotics (3)–(4) and the semiclassical Weyl law (16) are well known (see [11, 13, 78, 79, 81, 88]). The only novelty here is the observation that these results can be deduced from the results of Carleman and Minakshisundaram mentioned above. In particular, we recover the following semiclassical Weyl law.

Theorem 6 ([11, 13, 78, 79, 81, 88]). *Assume we are in one the following situations:*

(i) $q\neq n/2$ *and* $V(x)$ *is a real-valued potential in* $L^r(\Omega)$ *with* $r=\max(n(2q)^{-1},1)$.
(ii) $q=n/2$ *and* $V(x)$ *is a real-valued potential in* $L\log L(\Omega)$ *(n even) or is in* $L_{r'}(\Omega)$ *with* $r'>1$ *(n odd).*

Then, for all energy levels $\lambda\in\mathbb{R}$*, we have*

$$\lim_{h\to 0^+} h^n N\left(h^{2q}\Delta_{\Omega}^{q} + V;\lambda\right) = c(n)\int_{\Omega}(V(x)-\lambda)_-^{\frac{n}{2q}}dx. \tag{28}$$

In this setting the integration formulas (17) do not seem to have been mentioned elsewhere, and so, in this sense, they are new.

Theorem 7. *Assume that, either* n *is even and* $f\in L\log L(\Omega)$*, or* n *is odd and* $f\in L_{r'}(\Omega)$ *with* $r'>1$*. Then, the operators* $\Delta_{\Omega}^{-n/4} f\Delta_{\Omega}^{-n/4}$ *and* $|\Delta_{\Omega}^{-n/4} f\Delta_{\Omega}^{-n/4}|$ *are both spectrally measurable, and we have*

$$⨍ \Delta_{\Omega}^{-\frac{n}{4}} f\Delta_{\Omega}^{-\frac{n}{4}} = c(n)\int_{\Omega} f(x)dx, \tag{29}$$

$$⨍ \left|\Delta_{\Omega}^{-\frac{n}{4}} f\Delta_{\Omega}^{-\frac{n}{4}}\right| = c(n)\int_{\Omega} |f(x)|dx. \tag{30}$$

Remark 8. The results of this section continue to hold for more general local boundary conditions. At least for continuous potentials, they hold on compact manifolds with boundary conditions as those considered in [43], since in this case the above mentioned results of [43, §4.4–5] apply. This includes the spectral triple of Levy-Iochum [48], which is using the Dirac operator under a (local) chiral boundary condition (see also [5] for a generalization for this construction).

4 Quantum Tori

In this section, we look at spectral asymptotics and semiclassical Weyl laws for quantum tori. The semiclassical Weyl law that we obtain in this context gives a positive answer to a conjecture of Edward McDonald and the author in [58]. We refer to [44, 97], and the references therein, for background on quantum tori.

Let $\theta = (\theta_{jk})$ be a real anti-symmetric $n \times n$-matrix. The *quantum torus* $\mathbb{T}^n_\theta$ is meant to be the noncommutative space represented by a C^*-algebra $C(\mathbb{T}^n_\theta)$ generated by n-unitaries $U_1, \ldots, U_n$ and subject to the relations,

$$U_k U_j = e^{2i\pi\theta_{jk}} U_j U_k, \qquad j,k = 1, \ldots, n. \tag{31}$$

For $\theta = 0$ we recover the C^*-algebra $C(\mathbb{T}^n)$ of continuous functions on the ordinary torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ with $U_j = e^{ix_j}$.

In general, a complete independent set of $C(\mathbb{T}^n_\theta)$ is provided by the unitaries,

$$U^m := U^{m_1} \cdots U^{m_n}, \qquad m \in \mathbb{Z}^n. \tag{32}$$

The *standard trace* $\tau_0 : C(\mathbb{T}^n_\theta) \to \mathbb{C}$ is given by

$$\tau_0(1) = 1, \qquad \tau_0\big(U^m\big) = 0 \quad \text{if } m \neq 0. \tag{33}$$

This is a faithful positive trace. We then define the space $L^2(\mathbb{T}^n_\theta)$ as the Hilbert space completion of $C(\mathbb{T}^n_\theta)$ with respect to the inner product

$$\langle x|y\rangle = \tau_0\big[y^* x\big], \qquad x, y \in C(\mathbb{T}^n_\theta). \tag{34}$$

Note that $\{U^m;\ m \in \mathbb{Z}^n\}$ is an orthonormal basis of $L^2(\mathbb{T}^n_\theta)$. Moreover, the action of $C(\mathbb{T}^n_\theta)$ on itself uniquely extends to a $*$-representation of $C(\mathbb{T}^n_\theta)$ in $L^2(\mathbb{T}^n_\theta)$; this is just the GNS representation associated with τ. More generally, for any $r \in [1, \infty)$ we define the NC L^r-space $L_r(\mathbb{T}^n_\theta)$ as in Section 2, which is possible since τ_0 extends to a continuous trace on the von Neuman algebra $L_\infty(\mathbb{T}^n_\theta)$ generated by $U_1, \ldots, U_n$.

We have a natural C^*-action of $\mathbb{R}^n$ on $C(\mathbb{T}^n_\theta)$ given by

$$\alpha_s(U^m) = e^{is\cdot m} U^m, \qquad m \in \mathbb{Z}^n,\ s \in \mathbb{R}^n. \tag{35}$$

For $\theta = 0$ this is just the action arising from the action of $\mathbb{R}^n$ on $\mathbb{T}^n$ by translation. The canonical derivations $\partial_1, \ldots, \partial_n$ are the infinitesimals generators of this action with the convention that

$$\partial_j\big(U_j\big) = iU_j, \qquad \partial_j\big(U_k\big) = 0, \quad k \neq j. \tag{36}$$

The *smooth quantum torus* $C^\infty(\mathbb{T}^n_\theta)$ is precisely the $*$-subalgebra of smooth vectors for this action. Equivalently,

$$C^\infty(\mathbb{T}^n_\theta) = \Big\{ u = \sum u_m U^m ; (u_m)_{m\in\mathbb{Z}^n} \in \mathscr{S}(\mathbb{Z}^n) \Big\}, \tag{37}$$

where $\mathscr{S}(\mathbb{Z}^n)$ is the space of complex-valued sequences with rapid decay. For $\theta = 0$ we recover the description of smooth functions on $\mathbb{T}^n$ in terms of the rapid decay of their Fourier coefficients. Note that $C^\infty(\mathbb{T}^n_\theta)$ is closed under holomorphic functional calculus.

In addition, given any $s \geq 0$, the Sobolev space $W_2^s(\mathbb{T}^n_\theta)$ (see [89, 97] is defined by

$$W_2^s(\mathbb{T}^n_\theta) := \Big\{ u = \sum_{k\in\mathbb{Z}^n} u_k U^k \in L_2(\mathbb{T}^n_\theta); \sum_{k\in\mathbb{Z}^n} (1+|k|^2)^s |u_k|^2 < \infty \Big\}. \tag{38}$$

Our main focus is the (flat) Laplacian

$$\varDelta := -(\partial_1^2 + \cdots + \partial_n^2).$$

This is a non-negative selfadjoint operator on $L_2(\mathbb{T}^n_\theta)$ with domain $W_2^2(\mathbb{T}^n_\theta)$. We have

$$\varDelta\big(U^k\big) = |k|^2 U^k, \qquad k \in \mathbb{Z}^n. \tag{39}$$

Thus, $\varDelta$ is isospectral to the Laplacian on the ordinary torus $\mathbb{T}^n$. In particular, the partial inverse $\varDelta^{-1}$ is in the weak Schatten class $\mathscr{L}_{n/2,\infty}$.

In fact, we have an n-summable spectral triple,

$$\Big(C^\infty(\mathbb{T}^n_\theta), L_2(\mathbb{T}^n_\theta), \sqrt{\varDelta} \Big). \tag{40}$$

A big impetus for the results of this note is proving the following semiclassical Weyl law for (flat) quantum tori, which was conjectured by Edward McDonald and the author in [58]

Conjecture 1 ([58, Conjecture 8.8]). Let $q > 0$, set $r = 2nq^{-1}$, and suppose that either $r \neq 1$ and $r' = \max(r, 1)$, or $r = 1 < r'$. Given any $V = V^* \in L_{r'}(\mathbb{T}^n_\theta)$, for every energy level $\lambda \in \mathbb{R}$, we have

$$\lim_{h\to 0^+} h^n N\big(h^{2q}\varDelta^q + V; \lambda\big) = \hat{c}(n)\tau_0\big[(V-\lambda)_-^{\frac{n}{2q}}\big], \qquad \hat{c}(n) := |\mathbb{B}^n|. \tag{41}$$

A special case of this conjecture was proved in [62] for $n \geq 3$, $q = 1$ and potentials $V = V^* \in C(\mathbb{T}^n_\theta)$. We stress that the approach of $n \geq 3$ does not apply to quantum 2-tori due to the fact that it requires p-summability with $p > 2$. This excludes quantum 2-tori from that approach, since in this case the spectral triple (40) is 2-summable. Once again being able to include 2-dimensional examples is one of the main motivation for the results of this paper.

As it turns out, the conjecture is now an easy consequence of the main results of this paper and the Cwikel-type estimates for (flat) quantum tori of [58, 60]. It can be shown that Condition (H) holds. In fact, for all $a \in C^\infty(\mathbb{T}^n_\theta)$, we have

$$\operatorname{Tr}\big[ae^{-t\Delta}\big] = \pi^{\frac{n}{2}} t^{-\frac{n}{2}} \Big[\tau_0(a) + \mathrm{O}\big(e^{-\frac{\pi^2}{t}}\big)\Big] \qquad \text{as } t \to 0^+. \tag{42}$$

This heat-trace asymptotic is an elementary consequence of Poisson's summation formula. It follows that Condition (W) is satisfied with

$$\tau(a) := \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}\tau_0(a) = \hat{c}(n)\tau_0(a). \tag{43}$$

Moreover, the Cwikel-type estimates of [58, 60] ensure us that Condition (C_r) is satisfied, with

$$\mathscr{V}_r = L_r(\mathbb{T}^n_\theta)\ (r > 1), \quad \mathscr{V}_r = L_{r'}(\mathbb{T}^n_\theta)\ (r = 1 < r'), \quad \mathscr{V}_r = L_1(\mathbb{T}^n_\theta)\ (r < 1). \tag{44}$$

It follows that the assumptions of Theorem 3 hold, and so we arrive at the following result.

Theorem 8. *Conjecture 1 holds true.*

We may also apply Theorem 4. This gives the following integration formulas.

Theorem 9. *For every $x \in L_r(\mathbb{T}^n_\theta)$, $r > 1$, the operators $\Delta^{-n/4}x\Delta^{-n/4}$ and $|\Delta^{-n/4}x\Delta^{-n/4}|$ are spectrally measurable, and we have*

$$\fint \Delta^{-\frac{n}{4}}x\Delta^{-\frac{n}{4}} = \hat{c}(n)\tau_0\big[x\big], \tag{45}$$

$$\fint \big|\Delta^{-\frac{n}{4}}x\Delta^{-\frac{n}{4}}\big| = \hat{c}(n)\tau\big[|x|\big]. \tag{46}$$

Remark 9. In the above level of generality the integration formula (45) is known (see [59, Corollary 5.3]; see also [61, 72]). The improvement is getting a stronger form of measurability. In [59, 61, 72] the focus is on strong measurability. The above result provides spectral measurability, which is a stronger property.

Remark 10. The integration formula (46) is new. Note that, even for $x \in C^\infty(\mathbb{T}^n_\theta)$, it does not follow from the trace theorems for ΨDOs in [61, 72], since $|\Delta^{-n/4}x\Delta^{-n/4}|$ need not be a ΨDO.

5 Sub-Riemannian Geometry

In this section, we apply the main results of this paper to get semiclassical Weyl laws and integration formulas for Schrödinger operators associated with fractional powers of sub-Laplacians on sub-Riemannian manifolds. These results are new and connect nicely with a string of recent results on spectral asymptotics and quantum limits in sub-Riemannian geometry (see, e.g., [4, 24, 26, 37–39, 52, 63]).

5.1 Sub-Riemannian manifolds

Let (M, H, g_H) be a closed sub-Riemannian manifold, where M is an n-dimensional closed manifold, H is a Lie-bracket generating subbundle of the tangent bundle TM of rank m, and g_H is a sub-Riemannian metric on H, i.e., a smooth field of positive-definite inner products on its fibers. The Lie-bracket generating condition H then ensures that, at every point $x_0 \in M$, we have a filtration,

$$H(x_0) =: H_1(x_0) \subseteq H_2(x_0) \subseteq \cdots \subseteq H_r(x_0) = TM(x_0), \tag{47}$$

where $H_w(x_0)$, $w \geq 2$, is the subspace spanned by iterated Lie brackets,

$$[X_{i_1}, [X_{i_2}, \cdots [X_{i_{k-1}}, X_{i_k}] \cdots]](x_0), \qquad k \leq w, \tag{48}$$

with $X_{i_1}, \ldots, X_{i_k}$ ranging over all H-valued vector fields near x_0.

Sub-Riemannian structures and Lie-bracket generating distributions naturally occur in a variety of settings. This includes control theory of non-holonomic systems of vector fields as those that comes into play in computer vision and image processing, Asian option in finance, dynamics of polymers, and distributions of clusters in space in astronomy (see [20]). Sub-Riemannian structures also appear on (graded) nilpotent Lie groups and their quotients by lattices (i.e., nilmanifolds).

Such structures also appear in the setting of parabolic geometry [21], including boundary-geometry of symmetric spaces of rank ≥ 2. Contact manifolds (e.g., cosphere bundles of positive line bundles) and their various generalizations yield additional examples. A related class of examples is provided by boundaries of complex domains (e.g., odd-dimensional Euclidean spheres), and more generally finite-type Cauchy-Riemann manifolds. In addition, Engel manifolds are well known step 3 examples. We refer to [1, 23, 69] and the various references therein for more background on sub-Riemannian geometry and more substantial discussions of the examples mentioned above.

We further assume that M is equipped with a smooth measure ν; it allows us to define an inner product on $L^2(M)$. In this setting, the analogue of the Laplacian is provided by the *sub-Laplacian* $\Delta_H : C_c^\infty(M) \to C_c^\infty(M)$. This is a formally selfadjoint 2nd order differential operator. In the Riemannian setting, i.e., $H = TM$, in which case g_H is a Riemannian metric, we recover the Laplacian associated with ν and this Riemannian metric.

In general, if $X_1, \ldots, X_m$ is a local orthonormal frame of H, then, locally, we have

$$\Delta_H = \sum_{1 \leq j \leq m} X_j^* X_j = \sum_{1 \leq j \leq m} \left(-X_j^2 + \operatorname{div}_\nu(X_j) X_j \right). \tag{49}$$

where div_ν is the divergence of vector fields with respect to the measure ν. In particular, if $m < n$, then Δ_H is not elliptic. However, by a celebrated result of Hörmander [46], the Lie-bracket generating condition ensures that Δ_H and the heat operator $\Delta_H + \partial_t$ on $M \times \mathbb{R}$ are both hypoelliptic with loss of derivatives.

5.2 Semiclassical Weyl law and integration formulas

As the manifold M is closed, the hypoellipticity of M ensures it is essentially selfadjoint and Fredholm. Its spectrum then consists of isolated non-negative eigenvalues with finite multiplicity. They thus can be listed as a non-decreasing sequence,

$$\lambda_0(\Delta_H) \leq \lambda_1(\Delta_H) \leq \lambda_2(\Delta_H) \leq \cdots, \tag{50}$$

where each eigenvalue is repeated according to multiplicity.

We shall further assume that the sub-Riemannian structure is *equiregular*. This means that in (47) the step r can be chosen independently of x_0 and the dimensions $\dim H_w(x_0)$, $w \geq 2$, do not vary with x_0. This condition is satisfied in many examples of sub-Riemannian structures. For $j = 2, \ldots, r$, the vector spaces $H_j(x_0)$, $x_0 \in M$, can then be organized as smooth subbundles $H_w \subseteq TM$. We thus get a filtration by subbundles,

$$\{0\} = H_0 \subseteq H_1 = H \subseteq H_2 \subseteq \cdots \subseteq H_r = TM. \tag{51}$$

This filtration is compatible with the Lie bracket of vector fields, i.e., $[H_j, H_k] \subseteq H_{j+k}$. Manifolds equipped with such filtrations of their tangent bundles are called *filtered manifolds* (see, e.g., [21]). The *sub-Riemannian dimension* (or *sR-dimension*) of M then is

$$N := \sum_{1 \leq j \leq r} j \dim\left(H_j/H_{j-1}\right). \tag{52}$$

This is also the Hausdorff dimension with respect to the Carathéodory metric defined by (H, g_H) (see, e.g., [1]).

Under the equiregularity assumption, Métivier [67] obtained a Weyl law for the sub-Laplacian Δ_H. Namely, as $j \to \infty$ we have

$$\lambda_j(\Delta_H) \sim j^{\frac{2}{N}} \left(\int_M \gamma(\Delta_H)(x) d\nu(x) \right)^{-\frac{2}{N}}, \tag{53}$$

where $\gamma(\Delta_H)(x)$ is some positive smooth function on M (see also [65, 66] in the contact case). In particular, the order of the r.h.s. is provided by the sR-dimension N, and not $\dim M$ as we have in the Riemannian case.

Alternatively, the hypoellipticity of $\Delta_H + \partial_t$ ensures that the heat semigroup $e^{-t\Delta_H}$, $t > 0$, has a smooth kernel $k_t(x,y)$, i.e.,

$$e^{-t\Delta_H} u(x) = \int_M k_t(x,y) u(y) d\nu(y), \qquad u \in L^2_\nu(M). \tag{54}$$

As $t \to 0^+$, and uniformly on M, we have a short time asymptotic expansion,

$$k_t(x,x) \sim t^{-\frac{N}{2}} \sum_{j \geq 0} t^{\frac{j}{2}} a_j(\Delta_H)(x), \tag{55}$$

where the coefficients $a_j(\Delta_H)(x)$ are smooth (see [8, 25, 34, 51, 94]; see also [7] for the step 2 case). Combining this with Karamata's Tauberian theorem (see, e.g., [9, Theorem 2.42]) then gives the Weyl law (53), with

$$\gamma(\Delta_H)(x) = \Gamma\left(\frac{N}{2}+1\right)^{-1} a_0(\Delta_H)(x). \tag{56}$$

In the step 2 case (i.e., $r = 2$) the relevant pseudodifferential calculus to study sub-Laplacians is the Heisenberg calculus of Beals-Greiner [6] and Taylor [95]. A version of this calculus for arbitrary filtered manifolds was presented by Melin in the unpublished preprint [64]. As shown by van Erp-Yuncken [96] this can understood in terms of the pseudodifferential calculus on Lie groupoids (see also [2] for an extension of this approach to singular Lie filtrations).

Results of Dave-Haller [34, 35] (see also [71] in the step 2 case) ensure that the square root $\sqrt{\Delta_H}$ is a pseudodifferential operator of order 1 in the aforementioned pseudodifferential calculus. As a result we get a spectral triple,

$$\left(C^\infty(M), L^2_\nu(M), \sqrt{\Delta_H}\right), \tag{57}$$

where $C^\infty(M)$ acts by bounded multiplication operators on $L^2_\nu(M)$. The Weyl law (53) implies that the above spectral triple is N-summable. Moreover, the heat kernel asymptotics (55) ensures that Condition (H) and Condition (W) hold with

$$\tau(f) := \int_M f(x)\gamma(\Delta_H)d\nu(x), \qquad f \in C(M). \tag{58}$$

In this setting Proposition 1 yields that, given any $q > 0$ and any function $f \in C(M)$, we have the spectral asymptotics,

$$\lim_{j\to\infty} j^{\frac{q}{N}} \mu_j\big(\Delta_H^{-\frac{q}{4}} f \Delta_H^{-\frac{q}{4}}\big) = \left(\int_M |f(x)|^{\frac{N}{q}} \gamma(\Delta_H)(x)d\nu(x)\right)^{\frac{q}{N}}, \tag{59}$$

$$\lim_{j\to\infty} j^{\frac{q}{N}} \lambda_j^{\pm}\big(\Delta_H^{-\frac{q}{4}} f \Delta_H^{-\frac{q}{4}}\big) = \left(\int_M f_\pm(x)^{\frac{N}{q}} \gamma(\Delta_H)(x)d\nu(x)\right)^{\frac{q}{N}} \quad \text{(if f is real valued).} \tag{60}$$

Applying Theorem 1 yields the following semiclassical result.

Theorem 10. *Let $q > 0$. Given any potential $V \in C(M,\mathbb{R})$, for all energy levels $\lambda \in \mathbb{R}$, we have*

$$\lim_{h\to 0^+} h^N N\big(h^{2q}\Delta_H^q + V; \lambda\big) = \int_M (V(x)-\lambda)_-^{\frac{N}{2q}} \gamma(\Delta_H)(x)d\nu(x). \tag{61}$$

Remark 11. Using a different approach, Fisher-Mikkelsen [39] recently obtained semiclassical Weyl laws for nilmanifolds. Note that for nilmanifolds the density $\gamma(\Delta_H)$ is constant. We also refer to [63] for a semiclassical Weyl law on graded nilpotent Lie group of sub-Riemannian dimension $N \geq 4$ (see also [32, 53] for related results). The semiclassical Weyl law (61) seems to be the first instance of such a result for sub-Riemannian manifolds that are not locally equivalent to graded nilpotent Lie groups.

Applying Theorem 2 yields integration formulas in sub-Riemannian geometry. In particular, we have the following result.

Theorem 11. *For every $f \in C(M)$, the operators $f\Delta_H^{-N/2}$, $\Delta_H^{-N/4} f \Delta_H^{-N/4}$, and $|\Delta_H^{-N/4} f \Delta_H^{-N/4}|$ are spectrally measurable, and we have*

$$\ooalign{$\int$\cr$-$} f\Delta_H^{-\frac{N}{2}} = \ooalign{$\int$\cr$-$} \Delta_H^{-\frac{N}{4}} f \Delta_H^{-\frac{N}{4}} = \int_M f(x)\gamma(\Delta_H)(x)d\nu(x), \tag{62}$$

$$\ooalign{$\int$\cr$-$} \left|\Delta_H^{-\frac{N}{4}} f \Delta_H^{-\frac{N}{4}}\right| = \int_M |f(x)|\gamma(\Delta_H)(x)d\nu(x). \tag{63}$$

Remark 12. The integration formulas (62)–(63) are new. In the step 2 case the integration formula (63) is known (see [70]; see also [50] for the contact case). However, even in that case, the above result provides a stronger form of measurability. In fact, in [50, p. 18] it is conjectured that we should have spectral measurability and spectral asymptotics of the form (59). Therefore, our results establish these conjectures in the far more general setting of (equiregular) sub-Riemannian geometry. Moreover, our approach bypasses the technical considerations in [50] of the C^*-approach to the principal symbol. Note that it is unclear how the approach of [50] can be extended to sub-Riemannian manifolds (or more generally filtered manifolds) that not locally equivalent to graded nilpotent Lie groups.

Remark 13. In the contact case, the density $\gamma(\Delta_H)$ can be computed explicitly thanks to the results of [7] (see also [95]). This gives explicit semiclassical Weyl law and integration formulas for contact manifolds.

Remark 14. The results of [34, 35, 71] actually apply to all positive selfadjoint Rockland operators on general filtered manifolds. Therefore, our approach applies *mutatis standis* to replacing Δ_H by such operators.

5.3 Extension to singular sub-Riemannian manifolds

The semiclassical Weyl law and integration formulas provided by Theorem 10 and Theorem 11 can be extended to singular sub-Riemannian manifolds.

We still have spectral triple $(C^\infty(M), L^2_\nu(M), \sqrt{\Delta_H})$. This can be seen by applying the results of [2] to the heat operator $\Delta_H + \partial_t$. It admits a parametrix in the class of pseudodifferential operators of [2]. Arguing as in [34, 71] we then can show that $\sqrt{\Delta_H}$ is pseudodifferential operator of order 1 in this class of pseudodifferential operators. Such operators commute with functions modulo operators of order 0. It then follows that the commutators $[\sqrt{\Delta_H}, f]$, $f \in C^\infty(M)$ are bounded, and so $(C^\infty(M), L^2_\nu(M), \sqrt{\Delta_H})$ is a spectral triple.

The validity of Condition (H) is a bit more subtle. It is ensured by the local Weyl laws of [26, Theorem 7.1] for equisingular sub-Riemannian manifolds. Roughly speaking this means that the sub-Riemannian structure induces a regular filtered structure on its singular set (see [26] for the precise definition).

For instance, the Baouendi-Grushin and Martinet examples fall in this class of singular sub-Riemannian manifolds (see [26]).

In general, the singular set and the regular set have different sR-dimensions. The leading terms in the local Weyl laws is then provided by the greatest of these sR-dimensions, which then gives the degree of summability of our spectral triple. In case this the sR-dimension of the singular set, then this yields to a singular Weyl measure that is supported on the singular set. If the two srR-dimensions agrees, then the leading term has a logarithmic terms, and so this falls out of the scope of this paper.

We illustrate the type of results we expect to get in the special case of the Baouendi-Grushin example[3] on the 2-torus $\mathbb{T}^2$, with

$$H = \langle X, Y\rangle, \qquad \Delta_H = X^2 + Y^2, \tag{64}$$

where

$$X := \partial_x, \qquad Y := 2(1-\cos x)\partial_y. \tag{65}$$

The singular set is $S = \{0\}\times\mathbb{T}$. Near $x=0$ we have $Y = x^2\partial_y + \mathrm{O}(x^4)$.Thus,

$$H_1 = \langle X, Y\rangle, \quad H_2 = \langle X, Y, [X,Y]\rangle, \quad H_3 = \langle X, Y, [X,Y], [X,[X,Y]]\rangle. \tag{66}$$

In particular, on $\mathbb{T}^2\setminus S$, we have $H_1 = H_2 = H_3 = T(\mathbb{T}^2\setminus S)$, while on S we have

$$H_{1|TS} = H_{2|TS} = 0, \qquad H_{3|TS} = TS.$$

Thus, the sR-dimension of $\mathbb{T}^2\setminus S$ is 2, while S has sR-dimension 3. In this example the local Weyl law [26, Theorem 7.1] and the translation invariance with respect to y ensure that Condition (H) holds with $p=3$, and

$$\tau(f) = c\int_{\mathbb{T}} f(0,y)dy, \qquad f\in C^\infty(M), \tag{67}$$

where c is some positive constant. It follows that Theorem 10 and Theorem 11 continue holds by taking $N=3$ and replacing the Weyl measure $\gamma(\Delta_H)dv(x)$ by the *singular* measure $c\delta_S$. Namely, we have the following semiclassical Weyl law and integration formula.

Theorem 12. *Let $q>0$. For all potentials $V\in C(\mathbb{T}^2,\mathbb{R})$, we have*

$$\lim_{h\to 0^+} h^3 N^-\big(h^{2q}\Delta_H^q + V\big) = c\int_{\mathbb{T}} V_-(0,y)^{\frac{3}{2q}}dy. \tag{68}$$

Theorem 13. *For all $f\in C(\mathbb{T}^2)$, the operators $f\Delta_H^{-3/2}$, $\Delta_H^{-3/4}f\Delta_H^{-3/4}$, and $|\Delta_H^{-3/4}f\Delta_H^{-3/4}|$ are spectrally measurable, and we have*

$$⨍ f\Delta_H^{-\frac{3}{2}} = ⨍ \Delta_H^{-\frac{3}{4}} f\Delta_H^{-\frac{3}{4}} = c\int_{\mathbb{T}} f(0,y)dy. \tag{69}$$

$$⨍ \left|\Delta_H^{-\frac{3}{4}} f\Delta_H^{-\frac{3}{4}}\right| = c\int_{\mathbb{T}} |f(0,y)|dy. \tag{70}$$

Remark 15. The semiclassical Weyl law (68) and the integration formulas (69)–(70) are the first instances of such results in the singular sub-Riemannian setting.

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[3] In the terminology of [26] this is the 2-Baouendi-Grushin example.

Appendix: Singular Values and Weak Schatten Classes

In this appendix, we collect some basic definitions and facts regarding singular values of compact operators and weak Schatten classes. We refer to [17, 42, 57] for more details on weak Schatten classes.

Throughout this appendix we let $\mathscr{H}$ be a separable Hilbert space, and we denote by $\mathscr{K}$ its ideal of compact operators. Recall that $\mathscr{K}$ is a closed ideal of the C^*-algebra $\mathscr{L}(\mathscr{H})$ of bounded operators on $\mathscr{H}$.

Given any operator $T \in \mathscr{K}$, we let

$$\mu_0(T) \geq \mu_1(T) \geq \mu_2(T) \geq \cdots \tag{71}$$

be its sequence of *singular values*. That is, $\mu_j(T)$ is the $(j+1)$-th eigenvalue counted with multiplicity of the absolute value $|T| = \sqrt{T^*T}$. By the *min-max principle* we have

$$\mu_j(T) = \min\left\{\|T_{|E^\perp}\|;\ \dim E = j\right\}. \tag{72}$$

We have the following properties of singular values,

$$\mu_j(T) = \mu_j(T^*) = \mu_j(|T|), \tag{73}$$

$$\mu_j(ATB) \leq \|A\|\mu_j(T)\|B\| \qquad \forall A, B \in \mathscr{L}(\mathscr{H}), \tag{74}$$

$$\mu_{j+k}(S+T) \leq \mu_j(S) + \mu_k(T), \tag{75}$$

The *weak Schatten classe* $\mathscr{L}_{p,\infty}$, $p > 0$, is defined by

$$\mathscr{L}_{p,\infty} := \left\{T \in \mathscr{K};\ \mu_j(T) = \mathrm{O}(j^{-\frac{1}{p}})\right\}. \tag{76}$$

This is a two-sided ideal of $\mathscr{L}(\mathscr{H})$. This is even a quasi-Banach ideal (in the sense of [57]) with respect to the quasi-norm,

$$\|T\|_{p,\infty} := \sup_{j \geq 0}(j+1)^{\frac{1}{p}}\mu_j(T), \qquad T \in \mathscr{L}_{p,\infty}. \tag{77}$$

In particular, $\mathscr{L}_{p,\infty}$ is a TVS for which a basis of neighborhoods of the origin is provided by the balls $B(0,\epsilon) := \left\{T \in \mathscr{L}_{p,\infty};\ \|T\|_{p,\infty}\right\}$, $\epsilon > 0$. For $p > 1$, the quasi-norm is equivalent to the norm, and so in this case $\mathscr{L}_{p,\infty}$, is a Banach space. In particular, it is locally convex. For $p \leq 1$, we don't have local convexity anymore.

If $(\mathscr{L}_{p,\infty})_0$ is the closure in $\mathscr{L}_{p,\infty}$ of the ideal of finite-rank operators, then we have

$$\left(\mathscr{L}_{p,\infty}\right)_0 = \left\{T \in \mathscr{K};\ \mu_j(T) = \mathrm{o}\big(j^{-\frac{1}{p}}\big)\right\}. \tag{78}$$

If $\mathscr{L}_q := \{T \in \mathscr{K}; \mathrm{Tr}[|T|^q] < \infty\}$, $q > 0$, are the usual Schatten classes, we then have strict inclusions,

$$\mathscr{L}_p \subsetneq \left(\mathscr{L}_{p,\infty}\right)_0 \subsetneq \mathscr{L}_{p,\infty} \subsetneq \mathscr{L}_q, \qquad 0 < p < q. \tag{79}$$

In particular, the strictness of the inclusion $(\mathscr{L}_{p,\infty})_0 \subsetneq \mathscr{L}_{p,\infty}$ implies that $\mathscr{L}_{p,\infty}$ is not separable.

In addition, we have the following version of Hölder's inequality for weak Schatten classes. Suppose that $p^{-1}+q^{-1}=r^{-1}$. If $S \in \mathscr{L}_{p,\infty}$ and $T \in \mathscr{L}_{q,\infty}$, then $ST \in \mathscr{L}_{r,\infty}$, and we have

$$\|ST\|_{r,\infty} \leq C_{pq}\|S\|_{p,\infty}\|T\|_{q,\infty}. \tag{80}$$

where $C_{pq} = p^{-\frac{1}{q}}q^{-\frac{1}{p}}(p+q)^{\frac{1}{p}+\frac{1}{q}}$. This inequality is sharp (see [90]).

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