College Mathematics IV

Chapter 4: Mathematical Expectation

§4.1 Mean of a Random Variable

Sichuan University, Spring 2026

Expected Value

Definition

Let X be a discrete random variable with probability distribution $f(x)$. The expected value (or mean) is

$$\mu = E(X) = \sum_x x f(x).$$

Definition

Let X be a continuous random variable with probability distribution $f(x)$. The expected value (or mean) is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx.$$

Example

A lot containing 7 components is sampled by a quality inspector. The lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

Example

Let X be the random variable that denotes the life in hours of some electronic device. Its probability density function is

$$f(x) = \begin{cases} \frac{20{,}000}{x^3} & \text{for } x > 100, \\ 0 & \text{elsewhere.} \end{cases}$$

Find the expected life of this type of device.

Expected Value of $g(X)$

Remark

Let X be a random variable defined on a sample space S.

- By definition X is a function on S.
- We can compose it with any function $g(x)$ defined on the range of X.
- The composition is denoted $g(X)$. This is a function on S.
- We thus get a new random variable.

Expected Value of $g(X)$

Example

Suppose that the range of X is $\{-1, 0, 1, 2\}$ and $g(x) = x^2$.

- The range of $g(X)$ is
$$\{g(-1), g(0), g(1), g(2)\} = \{0, 1, 4\}.$$
- If $y \in \{0, 1, 4\}$, then
$$g(X) = y \iff X = x \text{ with} x^2 = y \text{ and } x \in \{0, \pm 1, 2\}.$$
- Thus,
$$\{g(X) = 0\} = \{X = 0\}, \qquad \{g(X) = 4\} = \{X = 2\},$$
$$\{g(X) = 1\} = \{X = -1\} \cup \{X = 1\}.$$
- If $f(x)$ is the probability distribution of X, then
$$P[g(X) = 0] = P[X = 0] = f(0),$$
$$P[g(X) = 1] = P[X = -1] + P[X = 1] = f(-1) + f(1),$$
$$P[g(X) = 4] = P[X = 2] = f(2).$$

Expected Value of $g(X)$

Example (continued)

- We can now compute the expected value of $g(X)$.
- We obtain:

$$\begin{aligned}
E[g(X)] &= \sum_{y \in \{0,1,4\}} g(y) P[g(X) = y] \\
&= g(0) P[g(X) = 0] + g(1) P[g(X) = 1] \\
&\quad + g(4) P[g(X) = 4] \\
&= 0 f(0) + 1 (f(-1) + f(1)) + 4 f(2) \\
&= g(0) f(0) + g(-1) f(1) + g(1) f(1) + g(2) f(2) \\
&= \sum_{x \in \{0, \pm 1, 2\}} g(x) f(x).
\end{aligned}$$

Expected Value of $g(X)$

Theorem

Let X be a <u>discrete</u> random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is given by

$$\mu_{g(X)} = E\left[g(X)\right] = \sum_x g(x)f(x).$$

Theorem

Let X be a <u>continuous</u> random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is given by

$$\mu_{g(X)} = E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)f(x)dx.$$

Example

Let X be a (continuous) random variable with density function,

$$f(x) = \begin{cases} \frac{1}{3}x^2 & \text{for } -1 < x < 2, \\ 0 & \text{elsewhere.} \end{cases}$$

Find the expected value of $g(X) = 4X + 3$.

Remark

Let X and Y be (discrete or continuous) random variables on a sample space S.

- By definition X and Y are functions on S.
- If $g(x, y)$ is a function defined on $\mathrm{range}(X) \times \mathrm{range}(Y)$, then we define another function $g(X, Y)$ on S by
$$g(X, Y)(s) = g\left(X(s), Y(s)\right), \qquad s \in S.$$
- We thus get a new random variable on S.

Expected Value of $g(X, Y)$

Theorem

Let X and Y be <u>discrete</u> random variables with joint probability distribution $f(x, y)$. The expected value of the random variable $g(X, Y)$ is given by

$$\mu_{g(X,Y)} = E[g(X, Y)] = \sum_x \sum_y g(x, y) f(x, y).$$

Theorem

Let X and Y be <u>continuous</u> random variable with probability distribution $f(x, y)$. The expected value of the random variable $g(X, Y)$ is given by

$$\mu_{g(X,Y)} = E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dxdy.$$

Expected Value of $g(X, Y)$

Example

Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. Let X be the number of blue pens selected and Y the number of red pens selected. In Section 3.4 we saw that the probability distribution $f(x, y)$ of X and Y is given by the following table:

	$f(x, y)$	$x = 0$	$x = 1$	$x = 2$	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Find the expected value of $g(X, Y) = XY$.

Expected Value of $g(X, Y)$

Example

Let X and Y be (continuous) random variables with joint density,

$$f(x, y) = \begin{cases} \frac{1}{4}x(1 + 3y^2), & 0 < x < 2,\ 0 < y < 1, \\ 0 & \text{elsewhere}. \end{cases}$$

Find the expected value of $g(X, Y) = Y/X$.