College Mathematics IV

§3.4 Joint Probability Distributions

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Joint Probability Distribution (Discrete Case)

Definition (Joint Probability Distribution)

Suppose that X and Y are discrete random variables. A joint probability distribution for X and Y is a function $f(x, y)$ such that

(i) $f(x, y) \geq 0$ for all x, y.

(ii) $\sum_x \sum_y f(x, y) = 1$.

(iii) $P(X = x,\ Y = y) = f(x, y)$ for all x, y.

Remark

Joint probability distributions are also called probability mass functions.

Joint Probability Distribution (Discrete Case)

Fact

If $f(x, y)$ is a joint probability distribution for X and Y, then,a for any region A in the xy-plane, we have

$$\begin{aligned} P\left((X, Y) \in A\right) &= \sum_{(x,y)\in A} P(X = x, Y = y) \\ &= \sum_{(x,y)\in A} f(x, y). \end{aligned}$$

Proof.

- We have
$$\{(X,Y)\in A\} = \bigcup_{(x,y)\in A} \{X=x, Y=y\}.$$
- For $(x,y)\neq(x',y')$ the events $\{X=x, Y=y\}$ and $\{X=x', Y=y'\}$ are disjoints.
- Therefore, by the Additive Rule we have
$$P\Bigg(\bigcup_{(x,y)\in A}\{X=x, Y=y\}\Bigg) = \sum_{(x,y)\in A} P\left(\{X=x, Y=y\}\right).$$
- That is,
$$P\left((X,Y)\in A\right) = \sum_{(x,y)\in A} P(X=x, Y=y) = \sum_{(x,y)\in A} f(x,y).$$

$\square$

Joint Probability Distribution (Discrete Case)

Example (Example 3.14)

Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. Let X be the number of blue pens selected and Y the number of red pens selected. Find

(a) Find the joint probability distribution $f(x, y)$ of X and Y.

(b) Compute $P[(X, Y) \in A]$, where $A = \{(x, y); x + y \leq 1\}$.

	$f(x,y)$	x = 0	x = 1	x = 2	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Joint Density Function (Continuous Case)

Definition (Joint Density Function)

Suppose that X and Y are <u>continuous</u> random variables. A joint density function for X and Y is a function $f(x, y)$ such that

1. $f(x, y) \geq 0$ for all x, y.
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy = 1$.
3. For every region A is the xy-plane, we have

$$\iint_A f(x, y) dxdy = P\left((X, Y) \in A\right).$$

Example (Example 3.15)

A privately owned business operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y be the respective proportions of the time that the drive-in and the walk-in facilities are in use. Their joint density function is given by

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & \text{for } 0 \le x, y \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Verify that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$.

(b) Compute $P[(X,Y) \in A]$, where $A = (0, 1/2) \times (1/4, 1/2)$.

Marginal Distributions

Definition (Marginal Distribution (Discrete Case))

Let X and Y be discrete random variables with joint probability distribution $f(x, y)$.

1. The marginal distribution of X alone is the function,

$$g(x) = \sum_{y} f(x, y).$$

2. The marginal distribution of Y alone is the function,

$$h(y) = \sum_{x} f(x, y).$$

Marginal Distributions

Remark

- For all x, we have

$$P(X = x) = g(x).$$

  That is, $g(x)$ is the probability distribution of X.

- In particular, we have

$$\sum_x g(x) = 1.$$

- Likewise, we have

$$P(Y = y) = h(y) \qquad \text{and} \qquad \sum_y h(y) = 1.$$

Marginal Distributions

Example

Compute the marginal distribution $h(x)$ of the random variable Y alone in Example 3.14.

Marginal Distributions

Definition (Marginal Distribution (Continuous Case))

Let X and Y be continuous random variables with joint probability density function $f(x, y)$.

1. The marginal distribution of X alone is the function,
$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy.$$

2. The marginal distribution of Y alone is the function,
$$h(y) = \int_{-\infty}^{\infty} f(x, y) dx.$$

Remark

We always have
$$\int_{-\infty}^{\infty} g(x) dx = \int_{-\infty}^{\infty} h(y) dy = 1.$$

Marginal Distributions

Example

Find the marginal distribution $g(x)$ of the random variable X alone in Example 3.15.

Conditional Distribution

Definition

Let X and Y be discrete or continuous random variables with joint probability distribution/density $f(x, y)$.

1. The conditional distribution of Y given that $X = x$ is
$$f(y|x) = \frac{f(x, y)}{g(x)} \qquad \text{(provided } g(x) > 0\text{).}$$
2. The conditional distribution of X given that $Y = y$ is
$$f(x|y) = \frac{f(x, y)}{h(y)} \qquad \text{(provided } h(y) > 0\text{).}$$

Conditional Distribution

Remark

Suppose that X and Y are discrete random variables.

- By definition,

$$P(X = x|Y = y) = \frac{P(\{X = x\} \cap \{Y = y\})}{P(Y = y)} = \frac{P(X = x, Y = y)}{P(Y = y)}.$$

- As $P(X = x, Y = y) = f(x, y)$ and $P(Y = y) = h(y)$, we get

$$P(X = x|Y = y) = \frac{f(x, y)}{h(y)} = f(x|y).$$

- Likewise, we have

$$P(Y = y|X = x) = \frac{f(x, y)}{g(x)} = f(y|x).$$

Conditional Distribution

Example

In Example 3.14 find the conditional distribution of X given that $Y = 1$. Use it to compute $P(X = 0|Y = 1)$.

Example (Example 3.20)

Let X and Y be (continuous) random variable with joint density function,

$$f(x,y) = \begin{cases} \frac{1}{4}x(1+3y^2) & \text{for } (x,y) \in (0,2) \times (0,1), \\ 0 & \text{elsewhere.} \end{cases}$$

Determine $g(x)$, $h(y)$,$f(x|y)$, and $P(1/4 < X < 1/2|Y = 1/3)$.

Statistical Independence

Fact

If $f(x|y)$ does not depend on y, then

$$f(x|y) = g(x) \qquad \textit{and} \qquad f(x,y) = g(x)h(y).$$

Proof (discrete case).

- We have

$$f(x,y) = f(x|y)h(y) \qquad \text{and} \qquad g(x) = \sum_x f(x,y).$$

- If $f(x|y)$ does not depend on y, then

$$g(x) = \sum_y f(x|y)h(y) = f(x|y)\sum_y h(y).$$

- As $\sum_y h(y) = 1$, we get

$$g(x) = f(x|y) \quad \text{and} \quad f(x,y) = f(x|y)h(y) = g(x)h(y).$$

$\square$

Statistical Independence

Definition (Statistical Independence)

Let X and Y be (discrete or continuous) random variables with probability distribution/density $f(x, y)$ and marginal distributions $g(x)$ and $h(y)$. We say that X and Y are statistically independent if we have

$$f(x, y) = g(x)h(y) \qquad \text{for all } x, y.$$

Statistical Independence

Remark

Suppose that X and Y are <u>discrete</u> random variables.

- We have

$$P(\{X = x\} \cap \{Y = y\}) = P(X = x, Y = y) = f(x, y),$$
$$P(X - x = g(x), \qquad P(Y = y) = h(y).$$

- Thus,

$$P(\{X = x\} \cap \{Y = y\}) = P(x = x)P(Y = y)$$
$$\iff f(x, y) = g(x)h(y).$$

- Therefore, X and Y are statistically independent if and only if $\{X = x\}$ and $\{Y = y\}$ are independent events for all x, y.

Remark

Suppose that X and Y are continuous random variables.

- If $f(x,y) = g(x)h(y)$, then
$$P(\{a < X < b\} \cap \{c < Y < d\}) = P[(X,Y) \in (a,b) \times (c,d)] = \int_a^b \int_c^d f(x,y)dxdy.$$
- Thus, if $f(x,y) = g(x)h(y)$, then
$$\begin{aligned} P(\{a < X < b\} \cap \{c < Y < d\}) &= \int_a^b \int_c^d g(x)h(y)dxdy \\ &= \int_a^b g(x)dx \int_c^d h(y)dy \\ &= P(a < X < b)P(c < Y < d). \end{aligned}$$
- Therefore, if X and Y are statistically independent, then

$\{a < X < b\}$ and $\{c < Y < d\}$ are independent events for all a, b, c, d.

Statistical Independence

Example

Show that the random variables in Example 3.20 are statistically independent.

Example

Show that the random variables in Example 3.14 are not statistically independent.

Extension to Multi Random Variables

All the previous notions can be extended to random variables $X_1, X_2, \ldots, X_n$ with $n \geq 3$.

Definition (Discrete Case)

Suppose that $X_1, X_2, \ldots, X_n$ are discrete random variables with joint distribution $f(x_1, x_2, \ldots, x_n)$.

- The marginal distribution of X_1 is

$$g(x_1) = \sum_{x_2} \cdots \sum_{x_n} f(x_1, x_2, \ldots, x_n).$$

- The joint marginal distribution of X_1 and X_2 is

$$g(x_1, x_2) = \sum_{x_3} \cdots \sum_{x_n} f(x_1, x_2, \ldots, x_n).$$

Extension to Multi Random Variables

Definition (Continuous Case)

Suppose that $X_1, X_2, \ldots, X_n$ are continuous random variables with joint distribution $f(x_1, x_2, \ldots, x_n)$.

- The marginal distribution of X_1 is

$$g(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n) dx_2 \cdots dx_n.$$

- The joint marginal distribution of X_1 and X_2 is

$$g(x_1, x_2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n) dx_3 \cdots dx_n.$$

Extension to Multi Random Variables

Definition

Let $X_1, X_2, \ldots, X_n$ be (discrete or continuous) random variables with joint distribution $f(x_1, x_2, \ldots, x_n)$. The joint conditional distribution of X_1, X_2, X_3 given that $X_4 = x_4, \ldots, X_n = x_n$ is

$$f(x_1, x_2, x_3 | x_4, x_5, \ldots, x_n) = \frac{f(x_1, x_2, \ldots, x_n)}{g(x_4, x_5, \ldots, x_n)},$$

where $g(x_4, x_5, \ldots, x_n)$ is the joint marginal distribution of $X_4, X_5, \ldots, X_n$.

Extension to Multi Random Variables

Definition

Let $X_1, X_2, \ldots, X_n$ be (discrete or continuous) random variables with joint distribution $f(x_1, x_2, \ldots, x_n)$ and marginal distributions $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$. We say that $X_1, X_2, \ldots, X_n$ are statistically mutually independent if and only if

$$f(x_1, x_2, \ldots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n) \quad \text{for all } x_1, x_2, \ldots, x_n.$$

Extension to Multi Random Variables

Example

Suppose that the shelf life, in years, of a certain perishable food product packaged in cardboard containers is a random variable whose probability density function is given by

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

Let X_1, X_2, X_3 represent the shelf lives for three of these containers selected independently. Compute $P(X_1 < 2, 1 < X_2 < 3, X_3 > 2)$.