College Mathematics II

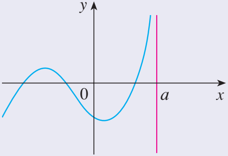
§3.5 Summary of Curve Sketching

Sichuan University, Spring 2026

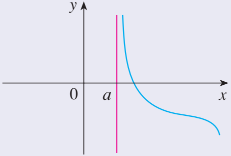
Reminder (Vertical Asymptotes; see Section 1.5)

The vertical line $x = a$ is a vertical asymptote of the curve $y = f(x)$ if at least one the following situations occurs:

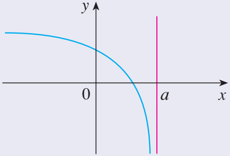
$$\lim_{x \to a} f(x) = \pm\infty, \qquad \lim_{x \to a^-} f(x) = \pm\infty, \qquad \lim_{x \to a^+} f(x) = \pm\infty.$$



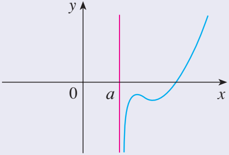
(a) $\lim_{x \to a^-} f(x) = \infty$



(b) $\lim_{x \to a^+} f(x) = \infty$



(c) $\lim_{x \to a^-} f(x) = -\infty$



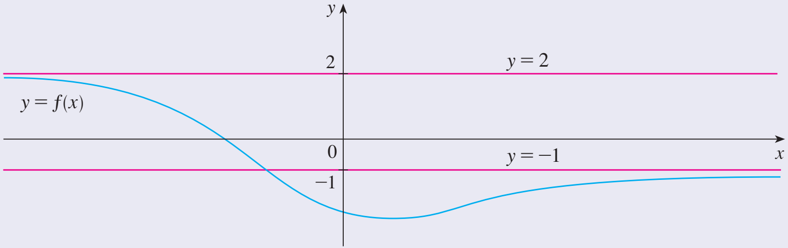
(d) $\lim_{x \to a^+} f(x) = -\infty$

Asymptotes

Reminder (Horizontal Asymptotes; see Section 3.4)

We say that the (horizontal) line $y = L$ is a horizontal asymptote of the curve $y = f(x)$ if

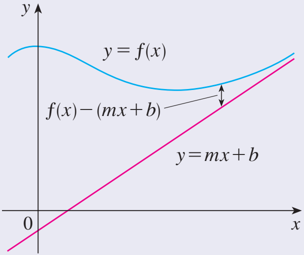
$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L.$$



Definition (Slant Asymptotes)

We say that the line $y = mx + b$ (with $m \neq 0$) is a slant asymptote of the curve $y = f(x)$ if

$$\lim_{x\to\infty} [f(x) - (mx + b)] = 0 \quad \text{or} \quad \lim_{x\to-\infty} [f(x) - (mx + b)] = 0.$$



Asymptotes

Example

Show that $y = x$ is a slant asymptote for $y = \frac{x^3}{x^2+1}$.

Guidelines for Sketching a Curve

- The following checklist describes the main steps for sketching by hand a curve $y = f(x)$.
- Not every item is relevant depending on the function.
- However, this provides all the info that is need to make a sketch that displays the main aspects of the function.

A. Domain

- Determine the domain D of f.

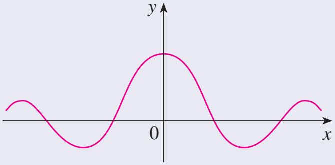
B. Intercepts

- Find the y-intercept, i.e., the point $(0, f(0))$.
- Find the x-intercept, i.e, the points $(x, f(x))$ with $f(x) = 0$.
- The previous step can be skipped if the equation $f(x) = 0$ is too hard to solve.

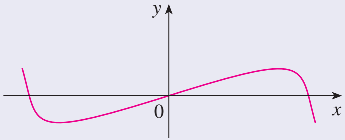
Guidelines for Sketching a Curve

C. Symmetry (Part 1)

- Check for symmetries.
- If $f(-x) = f(x)$, then the function is even and its graph is symmetric about the y-axis.
- If $f(-x) = -f(x)$, then the function is odd and its graph is symmetric about the origin.



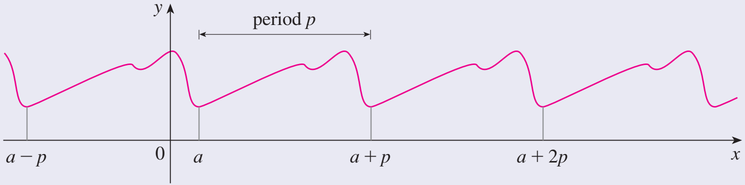
(a) Even function: reflectional symmetry



(b) Odd function: rotational symmetry

C. Symmetry (Part 2)

- If $f(x+p) = f(x)$ for some $p > 0$, then the function is p-periodic. Its graph is obtained by translations along the x-axis of its graph over any interval of length p (i.e., $[-p/2, p/2]$ or $[0, p]$).



Guidelines for Sketching a Curve

D. Asymptotes

- Look for asymptotes.
- Horizontal and slant asymptotes give the shape of the graph as x approaches ∞ or $-\infty$.
- A vertical asymptote $x = a$ gives the shape of the graph as x approaches a.

Guidelines for Sketching a Curve

E. Intervals of Increase or Decrease

- Use the Increasing/Decreasing Test.
- Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive or negative.
- The function is increasing where $f'(x) > 0$ and is decreasing where $f'(x) < 0$.

F. Local Extrema

- Find the critical points of f (i.e., points c such that $f'(c)$ does not exist or $f'(c) = 0$).
- Use the First Derivative Test if $f'(c) = 0$.
- If $f'(x)$ changes from positive to negative, then there is a local maximum at $x = c$.
- If $f'(x)$ changes from negative to positive, then there is a local minimum at $x = c$.
- The 2nd Derivative Test can also be used.

Guidelines for Sketching a Curve

G. Concavity and Inflection Points

- Compute $f''(x)$ and use the Concavity Test.
- The curve is concave upward where $f''(x) > 0$.
- The curve is concave downward where $f''(x) < 0$.
- Inflection points at points at which concavity changes.

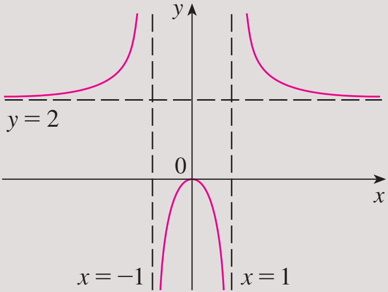
Guidelines for Sketching a Curve

H. Sketch the Graph

- Draw the graph by using the info from A–G.
- Sketch the asymptotes as dashed lines.
- Plot the the intercepts, local maximum and local minimum points, and inflection points.
- Make the curve pass through these points, rising or falling according to E, with concavity according to G, and approaching the asymptotes.

Example

Use the guidelines to sketch the graph of $y = \frac{2x^2}{x^2-1}$.



Example

Sketch the graph of $y = \frac{x^3}{x^2+1}$.

Interval	x	$3 - x^2$	$(x^2 + 1)^3$	$f''(x)$	f
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$	CU on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	$-$	$+$	$+$	$-$	CD on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	$+$	$+$	$+$	$+$	CU on $(0, \sqrt{3})$
$x > \sqrt{3}$	$+$	$-$	$+$	$-$	CD on $(\sqrt{3}, \infty)$