College Mathematics II

§3.4 Limits at Infinity; Horizontal Asymptotes

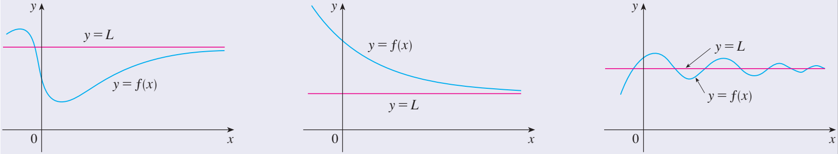
Sichuan University, Spring 2026

Definition (Limit at ∞)

Let f be a function defined on some interval $[a, \infty)$. We say that $f(x)$ has limit L at ∞, and we write

$$\lim_{x \to \infty} f(x) = L,$$

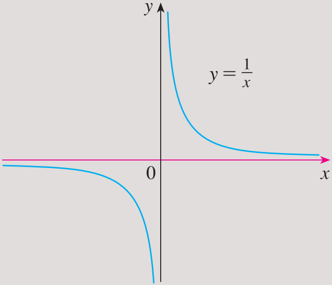
if $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.



Example

We have

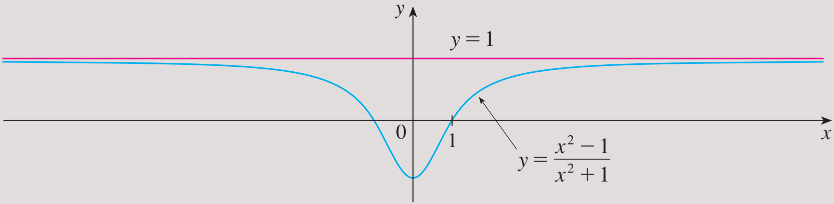
$$\lim_{x\to\infty} \frac{1}{x} = \infty.$$



Limits at Infinity

Example

Find the limit at ∞ of $f(x) = \frac{x^2-1}{x^2+1}$.

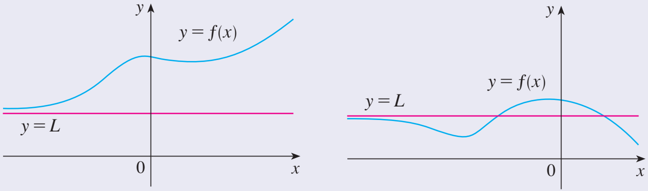


Definition (Limit at $-\infty$)

Let f be a function defined on some interval $(-\infty, a]$. We say that $f(x)$ has limit L at $-\infty$, and we write

$$\lim_{x \to -\infty} f(x) = L,$$

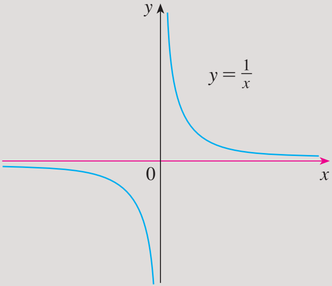
if $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.



Example

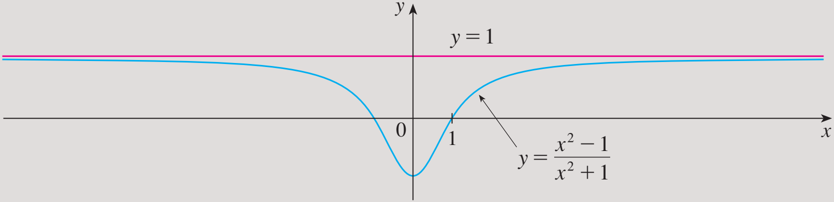
We have

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$



Example

Find the limit at $-\infty$ of $f(x) = \frac{x^2-1}{x^2+1}$.



Horizontal Asymptotes

Definition

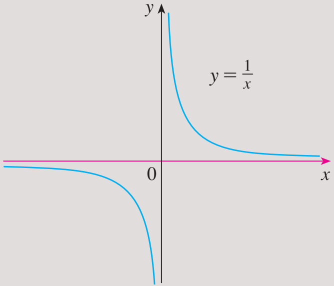
We say that the (horizontal) line $y = L$ is a horizontal asymptote of the curve $y = f(x)$ if

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

Horizontal Asymptotes

Example

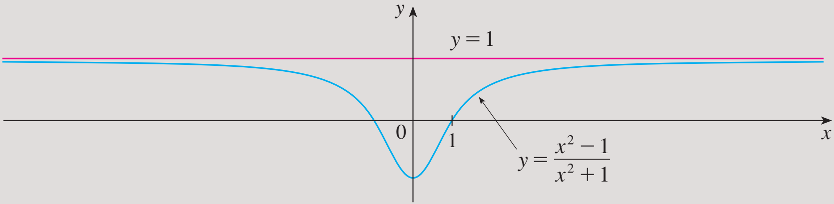
The line $y = 0$ is a horizontal asymptote of the curve $y = 1/x$.



Horizontal Asymptotes

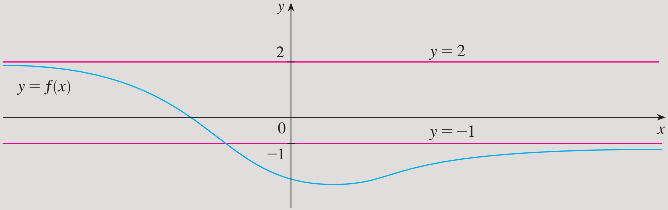
Example

The line $y = 1$ is a horizontal asymptote of the curve $y = \frac{x^2-1}{x^2+1}$.



Example

The lines $y = -1$ and $y = 2$ are horizontal asymptotes of the curve $y = f(x)$ below.



Here we have

$$\lim_{x \to \infty} f(x) = -1 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2.$$

Limit Laws

The laws for limits of function in Section 1.6 continue to holds for limits at infinity.

Theorem (Limit Laws)

1. *If* $\lim_{x\to\infty} f(x)$ *and* $\lim_{x\to\infty} g(x)$ *both exist, then*

$$\lim_{x\to\infty}[f(x)\pm g(x)] = \left[\lim_{x\to\infty} f(x)\right] \pm \left[\lim_{x\to\infty} g(x)\right],$$

$$\lim_{x\to\infty}[cf(x)] = c\lim_{x\to\infty} f(x) \quad (c \textit{ any constant}),$$

$$\lim_{x\to\infty}[f(x)g(x)] = \lim_{x\to\infty} f(x)\cdot\lim_{x\to\infty} g(x),$$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)} \qquad \textit{if } \lim_{x\to\infty} g(x)\neq 0.$$

2. *The same rules hold if* $\lim_{x\to-\infty} f(x)$ *and* $\lim_{x\to-\infty} g(x)$ *exist.*

Remark

- If q is a positive integer, then by definition $x^{1/q}$ is the unique solution of the equation $y^q = x$.
- If q is even, then $x^{1/q}$ is defined only for $x \geq 0$.
- If q is odd, then $x^{1/q}$ is defined for all real values of x.
- If p is another positive integer, then $x^{p/q}$ is defined by
$$x^{\frac{p}{q}} = (x^p)^{\frac{1}{q}}.$$
- If p is even or q is odd, then $x^{p/q}$ is defined for all real values of x.
- If p is odd and q is even, then $x^{p/q}$ is defined only for $x \geq 0$.
- We further define
$$x^{-\frac{p}{q}} = \frac{1}{x^{\frac{p}{q}}} = \left[\left(\frac{1}{x}\right)^p\right]^{\frac{1}{q}}.$$
provided $x \neq 0$ and $x^{p/q}$ is well defined.

Power Functions

Remark

- The power function x^r can also be defined if r is irrational.
- If $r > 0$, its domain is $[0, \infty)$.
- If $r < 0$, its domain is $(0, \infty)$.
- We always have

$$x^{r_1 + r_2} = x^{r_1} x^{r_2}, \qquad x^{-r} = \frac{1}{x^r},$$

whenever $x^{r_1 + r_2}$, x^{r_1}, x^{r_2} and $x^{\pm r}$ are well defined.

Summary

1. If $x > 0$, then x^r is always well defined for any value of r.
2. If $x = 0$, then $0^r = 0$ if $r \geq 0$.
3. If $x < 0$, then x^r make sense if r is rational, $r = \pm p/q$ with p even or q odd.

Power Functions

Theorem (Power Law)

Let r be any real number.

1. *Suppose that $\lim_{x\to\infty} f(x)$ exists. We have*

$$\lim_{x\to\infty} f(x)^r = \left[\lim_{x\to\infty} f(x)\right]^r,$$

   provided $f(x)^r$ and $[\lim_{x\to\infty} f(x)]^r$ are well defined.

2. *The same rule applies to $\lim_{x\to-\infty} f(x)$ if it exists.*

Power Functions

Theorem

Suppose that $r > 0$.

1. *We have*
$$\lim_{x \to \infty} \frac{1}{x^r} = 0.$$
2. *If* r *is a rational number,* $r = p/q$ *with* p *even or* q *odd, then*
$$\lim_{x \to -\infty} \frac{1}{x^r} = 0.$$
3. *In any case* $y = 0$ *is a horizontal asymptote of the curve* $y = 1/x^r$.

Example

Find

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}.$$

Example

Find

$$\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right).$$

Infinite Limits at Infinity

Definition (Infinite Limits at ∞)

Let f be a function defined on some interval $[a, \infty)$.

1. We say that $f(x)$ has limit ∞ at ∞, and we write
$$\lim_{x \to \infty} f(x) = \infty,$$
if $f(x)$ can be made arbitrarily large by requiring x to be sufficiently large.

2. We say that $f(x)$ has limit $-\infty$ at ∞, and we write
$$\lim_{x \to \infty} f(x) = -\infty,$$
if $f(x)$ can be made arbitrarily large negative by requiring x to be sufficiently large.

Infinite Limits at Infinity

Definition (Infinite Limits at $-\infty$)

Let f be a function defined on some interval $(-\infty, a]$.

1. We say that $f(x)$ has limit ∞ at $-\infty$, and we write

$$\lim_{x \to -\infty} f(x) = \infty,$$

   if $f(x)$ can be made arbitrarily large by requiring x to be sufficiently large negative.

2. We say that $f(x)$ has limit $-\infty$ at $-\infty$, and we write

$$\lim_{x \to -\infty} f(x) = -\infty,$$

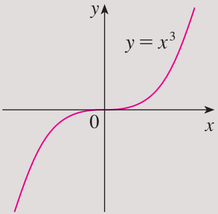
   if $f(x)$ can be made arbitrarily large negative by requiring x to be sufficiently large negative.

Infinite Limits at Infinity

Example

We have

$$\lim_{x\to\infty} x^3 = \infty, \qquad \lim_{x\to-\infty} x^3 = -\infty.$$



Infinite Limits at Infinity

Theorem

Let $r > 0$.

1. *We have*
$$\lim_{x\to\infty} x^r = \infty.$$

2. *Assume that* r *is a rational,* $r = p/q$, *with* p *even or* q *odd. Then, we have*
$$\lim_{x\to-\infty} x^r = \begin{cases} \infty & \text{if } p \text{ is even,} \\ -\infty & \text{if } p \text{ and } q \text{ are odd.} \end{cases}$$

Remark

Several (but not all) laws for finite limits continue to hold for infinite limits by using the following rules:

$$\infty + \infty = \infty - (-\infty) = \infty,$$

$$-\infty - \infty = -\infty + (-\infty) = -\infty,$$

$$c(\pm\infty) = \pm\infty \quad (c > 0), \qquad c(\pm\infty) = \pm\infty \quad (c < 0),$$

$$\infty \cdot (\pm\infty) = \pm\infty, \qquad -\infty \cdot (\pm\infty) = \mp\infty,$$

$$\frac{c}{\pm\infty} = 0 \quad (c \text{ any constant}), \qquad \frac{1}{0^{\pm}} = \pm\infty,$$

$$(\infty)^r = \infty, \qquad (-\infty)^r = \pm\infty \quad (\text{depending on } r).$$

Here 0^+ (resp., 0^-) means that $\lim f(x) = 0$ with $f(x) > 0$ (resp., $f(x) < 0$).

Remark (continued)

For instance:

- If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then

$$\lim_{x\to\infty}[f(x)+g(x)] = \left[\lim_{x\to\infty} f(x)\right] + \left[\lim_{x\to\infty} g(x)\right] = \infty + \infty = \infty,$$

$$\lim_{x\to\infty} f(x)g(x) = \lim_{x\to\infty} f(x) \cdot \lim_{x\to\infty} g(x) = \infty \cdot \infty = \infty,$$

$$\lim_{x\to\infty} \frac{1}{f(x)} = \frac{1}{\infty} = 0, \qquad \lim_{x\to\infty} f(x)^r = (\infty)^r = \infty \ (r > 0).$$

- If $\lim_{x\to-\infty} f(x)$ is finite and $\lim_{x\to-\infty} g(x) = \pm\infty$, then

$$\lim_{x\to-\infty}[f(x)+g(x)] = \pm\infty.$$

- If $\lim_{x\to\infty} f(x)$ is finite and > 0 and $\lim_{x\to\infty} g(x) = \pm\infty$, then

$$\lim_{x\to\infty} f(x)g(x) = \pm\infty.$$

Remark (Indeterminate Forms)

The following situations are indeterminate:

$$\infty - \infty, \qquad -\infty + \infty, \qquad \frac{\pm\infty}{\pm\infty}, \qquad 0 \cdot (\pm\infty).$$

In those situations any outcome is possible: finite limit, infinite limit, no limit.

Infinite Limits at Infinity

Example (Indeterminate Form $\infty - \infty$)

Let r_1 and r_2 be positive numbers.

$$\lim_{x\to\infty} \left(x^{r_1} - x^{r_2}\right) = \begin{cases} \infty & \text{if } r_1 > r_2, \\ 0 & \text{if } r_1 = r_2, \\ -\infty & \text{if } r_1 < r_2. \end{cases}$$

Example (Indeterminate Form $\frac{\infty}{\infty}$)

Let r_1 and r_2 be positive numbers.

$$\lim_{x\to\infty} \frac{x^{r_1}}{x^{r_2}} = \begin{cases} \infty & \text{if } r_1 > r_2, \\ 1 & \text{if } r_1 = r_2, \\ 0 & \text{if } r_1 < r_2. \end{cases}$$

Infinite Limits at Infinity

Example

Find
$$\lim_{x \to \infty} \left(x^2 - x \right).$$

Example

Find
$$\lim_{x \to \infty} \frac{x^2 + x}{3 - x}.$$

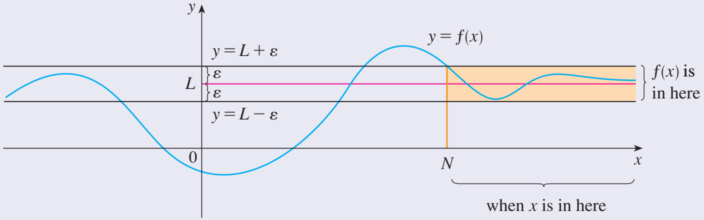
Definition (Informal Definition)

We say that $f(x)$ has limit L at ∞, and write $\lim_{x\to\infty} f(x) = L$, if $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Definition (Precise Definition)

We say that $f(x)$ has limit L at ∞, and write $\lim_{x\to\infty} f(x) = L$, if for every $\epsilon > 0$ there is a number N such that

$$|f(x) - L| < \epsilon \qquad \text{for all } x > N.$$



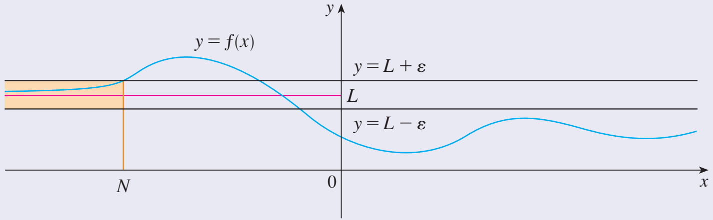
Definition (Informal Definition)

We say that $f(x)$ has limit L at $-\infty$, and write $\lim_{x\to-\infty} f(x) = L$, if $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

Definition (Precise Definition)

We say that $f(x)$ has limit L at $-\infty$, and write $\lim_{x\to-\infty} f(x) = L$, if for every $\epsilon > 0$ there is a number N such that

$$|f(x) - L| < \epsilon \qquad \text{for all } x < N.$$



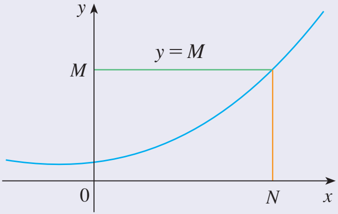
Definition (Informal Definition)

We say that $f(x)$ has limit ∞ at ∞, and write $\lim_{x\to\infty} f(x) = \infty$, if $f(x)$ can be made arbitrarily large by requiring x to be sufficiently large.

Definition (Precise Definition)

We say that $f(x)$ has limit ∞ at ∞, and write $\lim_{x\to\infty} f(x) = \infty$, if for every $M > 0$ there is a number N such that

$$f(x) > M \qquad \text{for all } x > N.$$



Precise Definitions

Definition (Informal Definition)

We say that $f(x)$ has limit $-\infty$ at ∞, and write $\lim_{x\to\infty} f(x) = -\infty$, if $f(x)$ can be made arbitrarily large negative by requiring x to be sufficiently large.

Definition (Precise Definition)

We say that $f(x)$ has limit $-\infty$ at ∞, and write $\lim_{x\to\infty} f(x) = -\infty$, if for every $M < 0$ there is a number N such that

$$f(x) < M \qquad \text{for all } x > N.$$

Precise Definitions

We also have precise definitions of infinite limits at $-\infty$.

Definition (Precise Definition)

We say that $f(x)$ has limit ∞ at $-\infty$, and write $\lim_{x\to-\infty} f(x) = \infty$, if for every $M > 0$ there is a number N such that

$$f(x) > M \qquad \text{for all } x < N.$$

Definition (Precise Definition)

We say that $f(x)$ has limit $-\infty$ at $-\infty$, and write $\lim_{x\to-\infty} f(x) = -\infty$, if for every $M < 0$ there is a number N such that

$$f(x) < M \qquad \text{for all } x < N.$$