

College Mathematics II

§3.3 How Derivatives Affect the Shape of a Graph

Sichuan University, Spring 2026

What Does f' say about f ?

Theorem (Increasing/Decreasing Test)

Let f be continuous on $[a, b]$ and differentiable on (a, b) .

- 1 If $f' > 0$ on (a, b) , then f is increasing on $[a, b]$.
- 2 If $f' < 0$ on (a, b) , then f is decreasing on $[a, b]$.

What Does f' say about f ?

Proof.

Suppose that $f' > 0$ on $[a, b]$.

- Let x_1 and x_2 be point of $[a, b]$ with $x_1 < x_2$.
- We apply the Mean Value Theorem to f on $[x_1, x_2]$.
- Therefore, there is c in (x_1, x_2) such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

- As $x_2 > x_1$ and $f'(c) > 0$, we get

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0.$$

- This shows that if $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- That is, f is increasing.

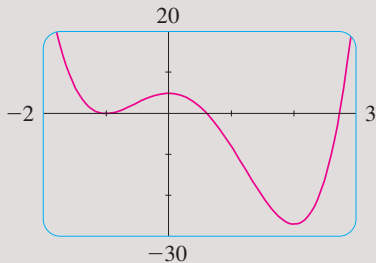
If $f' < 0$ on $[a, b]$, then it can be similarly shown that f is decreasing. □

What Does f' Say About f ?

Example

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	-	-	-	-	decreasing on $(-\infty, -1)$
$-1 < x < 0$	-	-	+	+	increasing on $(-1, 0)$
$0 < x < 2$	+	-	+	-	decreasing on $(0, 2)$
$x > 2$	+	+	+	+	increasing on $(2, \infty)$

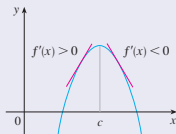


Local Extreme Values

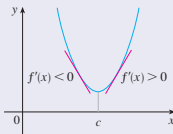
Theorem (First Derivative Test)

Let c be a critical number of a continuous function f .

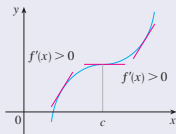
- 1 If f' changes from positive to negative at c , then f has a **local maximum** at c .
- 2 If f' changes from negative to positive at c , then f has a **local minimum** at c .
- 3 If f' is either positive on both sides of c , or negative on both sides of c , then f has **no local extremum** at c .



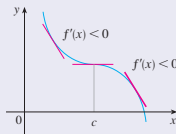
(a) Local maximum



(b) Local minimum



(c) No maximum or minimum



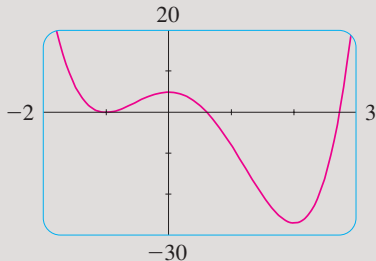
(d) No maximum or minimum

Local Extreme Values

Example

Find the local extrema of the function in the previous example.

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	-	-	-	-	decreasing on $(-\infty, -1)$
$-1 < x < 0$	-	-	+	+	increasing on $(-1, 0)$
$0 < x < 2$	+	-	+	-	decreasing on $(0, 2)$
$x > 2$	+	+	+	+	increasing on $(2, \infty)$

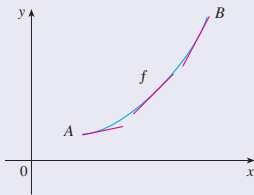


What Does f'' Say About f ?

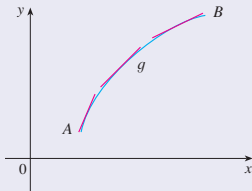
Definition

We say that on interval I a function f is:

- **concave upward** if its graph lies above all the tangents on I .
- **concave downward** if its graph lies below all the tangents on I .



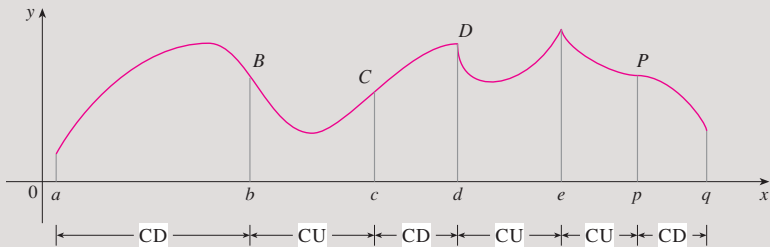
(a) Concave upward



(b) Concave downward

What Does f'' Say About f ?

Example



What Does f'' Say About f ?

Theorem (Concavity Test)

- 1 If $f''(x) < 0$ on I , then f is *concave upward* on I .
- 2 If $f''(x) > 0$ on I , then f is *concave downward* on I .

Definition (Inflection Point)

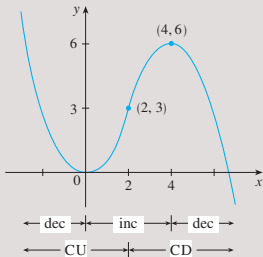
A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there, and at P the curve changes either from concave upward to concave downward, or from concave downward to concave upward.

What Does f'' Say About f ?

Example

Sketch a possible graph for a function f with the following properties:

- (i) $f(0) = 0$, $f(2) = 3$, $f(4) = 6$, and $f'(0) = f'(4) = 0$.
- (ii) $f'(x) > 0$ for $0 < x < 4$, and $f'(x) < 0$ for $x < 0$ and $x > 4$.
- (iii) $f''(x) > 0$ for $x < 2$, and $f''(x) < 0$ for $x > 2$.



What Does f'' Say About f ?

Theorem (Second Derivative Test)

Suppose that f'' exists and is continuous near c .

- 1 If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c .
- 2 If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c .

What Does f'' Say About f ?

Example

Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Interval	$f''(x) = 12x(x - 2)$	Concavity
$(-\infty, 0)$	+	upward
$(0, 2)$	-	downward
$(2, \infty)$	+	upward

