College Mathematics II
§2.8 Related Rates

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Pumping Air into a Balloon

Example

Air is being pumped into a spherical balloon so that its volume increases at a rate of $100\,\mathrm{cm}^3\,/s$. How fast is the radius of the balloon increasing when the diameter is $50\,\mathrm{cm}$?

Solution

Denote by V the volume and by r the radius.

- What are we looking for?

$$\frac{dr}{dt}.$$

- What do we know?

$$\frac{dV}{dt} = 100\,\mathrm{cm}^3\,/s.$$

- We use the relation between V and r. That is,

$$V = \frac{4}{3}\pi r^3.$$

Pumping Air into a Balloon

Solution (continued)

- We differentiate both sides of the equation, and use the Chain Rule to get

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi \frac{d}{dt}\left(r^3\right) = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) = 4\pi r^2\frac{dr}{dt}.$$

- This gives

$$\frac{dr}{dt} = \frac{1}{4\pi}r^{-2}\frac{dV}{dt}.$$

- We know that

$$\frac{dV}{dt} = 100 = 4 \times 25\,\mathrm{cm}^3/s.$$

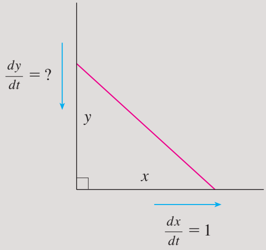
- If the diameter is $50\,\mathrm{cm}$, then $r = 25$. Thus,

$$\frac{dV}{dt} = \frac{1}{4\pi}\left(25^{-2}\right)\left(4 \times 25\right) = \frac{1}{25\pi} \simeq 0.0127\,\mathrm{cm}/s.$$

Sliding Ladder

Example

A ladder 5 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 m from the wall?



Sliding Ladder

Solution

- What are we looking for?

$$\frac{dy}{dt}.$$

- What do we know?

$$\frac{dx}{dt} = 1\,\mathrm{m/s}\,.$$

- By the Pythagorean Theorem we have

$$x^2 + y^2 = 5^2 = 25.$$

- Differentiating this equation gives

$$0 = \frac{d}{dt}(25) = \frac{d}{dt}\left(x^2 + y^2\right) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

- That is,

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \iff y\frac{dy}{dt} = -x\frac{dx}{dt} \iff \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

Sliding Ladder

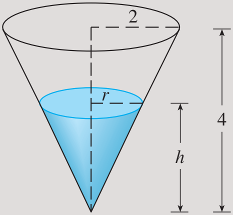
Solution (continued)

- Note that if $y \geq 0$, then
$$x^2 + y^2 = 25 \iff y^2 = 25 - x^2 \iff y = \sqrt{25 - x^2}.$$
- For $x = 3$ we get
$$y = \sqrt{25 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4\,\mathrm{m}\,.$$
- As $\frac{dx}{dt} = 1\,\mathrm{m/s}$, we get
$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{3}{4}(1) = -\frac{3}{4} = -0.75\,\mathrm{m/s}\,.$$

Water Tank

Example

A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.



Water Tank

Solution

Denote by V the volume of water.

- What are we looking for?

$$\frac{dh}{dt}.$$

- What do we know?

$$\frac{dV}{dt} = 2\,\mathrm{m}^3/\mathrm{min}\,.$$

- We have

$$V = \frac{1}{3}\pi r^2 h.$$

- If θ is the angle of the cone, then

$$\tan\theta = \frac{r}{h} = \frac{2}{4} = \frac{1}{2}.$$

- It follows that $r = \frac{1}{2}h$.

Solution (continued)

- This allows to express V as a sole function of h:

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)h = \frac{1}{12}\pi h^3.$$

- Differentiating gives

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{12}\pi h^3\right) = \frac{\pi}{12}\left(3h^2\frac{dh}{dt}\right) = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

- We then get

$$\frac{dh}{dt} = \frac{4}{\pi}h^{-2}\frac{dV}{dt}.$$

- As $\frac{dV}{dt} = 2\,\mathrm{m}^3/\mathrm{min}$, for $h = 3\,\mathrm{m}$ we get

$$\frac{dh}{dt} = \frac{4}{\pi}(3^{-2})\,2 = \frac{4}{\pi}\left(\frac{1}{9}\right)2 = \frac{8}{9\pi} \simeq 0.28\,\mathrm{m}^3/\mathrm{min}\,.$$

Problem Solving Strategy

Strategy

1. Read the problem carefully. Draw a diagram if possible.
2. Introduce notation. Assign symbols to all quantities that are functions of time.
3. Express the given information and the required rate in terms of derivatives.
4. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution (as in the 3rd example).
5. Use the Chain Rule to differentiate both sides of the equation with respect to t.
6. Substitute the given information into the resulting equation and solve for the unknown rate.