College Mathematics II
§2.7 Derivatives in the Natural and Social Sciences

Sichuan University, Spring 2026

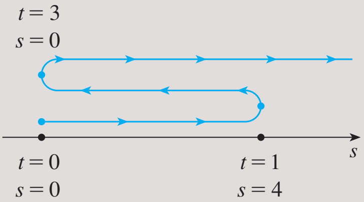
Physics Examples

Example (Moving Particle)

The position of a particle is given by

$$s = f(t) = t^3 - 6t^2 + 9t.$$

- (a) Find the velocity at time t.
- (b) What is the velocity after $2s$? After $4s$?
- (c) When is the particle at rest?
- (d) When is the particle moving forward (that is, in the positive direction)?



Physics Examples

Example (Wire/Rod Mass Density)

- If a rod or piece of wire is homogeneous, then its linear density is uniform and is defined as the mass per unit length ($\rho = m/\ell$) and measured in kilograms per meter.
- Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point x is $m = f(x)$.

- The mass of the rod that lies between $x = x_1$ and $x = x_2$ is $\Delta m = f(x_2) - f(x_1)$. The average density of that part then is
$$\frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$
- The linear density ρ is the limit of this average as $\Delta x \to 0$. That is,
$$\rho(x) = \lim_{x_1 \to x} \frac{f(x) - f(x_1)}{x - x_1} = \frac{dm}{dx}.$$

Physics Examples

Example

- For instance, suppose that $m = f(x) = \sqrt{x}$.
- The average density for the part of the rod between $x_1 = 1$ and $x_2 = 1.2$ then is

$$\frac{\Delta m}{\Delta x} = \frac{f(1.2) - f(1)}{1.2 - 1} = \frac{\sqrt{1.2} - \sqrt{1}}{1.2 - 1} = \frac{\sqrt{1.2} - 1}{0.2} \simeq 0.48\,\mathrm{kg} \cdot \mathrm{m}^{-1}\,.$$

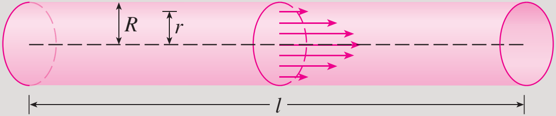
- The linear density at $x = 1$ is

$$\rho(1) = \left.\frac{dm}{dx}\right|_{x=1} = \left.\frac{d}{dx}\sqrt{x}\right|_{x=1} = \left.\frac{1}{2\sqrt{x}}\right|_{x=1} = \frac{1}{2} = 0.50\,\mathrm{kg} \cdot \mathrm{m}^{-1}\,.$$

Biology Examples

Example (Blood Vessel)

- When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius R and length l.



- Because of friction at the walls of the tube, the velocity v of the blood is greatest along the central axis of the tube.
- The velocity decreases as the distance r from the axis increases, until v becomes 0 at the wall.
- The relationship between v and r is given by the law of laminar flow (discovered by Poiseuille in 1840).

Biology Examples

Example (Blood Vessel; Continued)

- The law of laminar flow states that

  $$v = \frac{P}{4\eta\ell}\left(R^2 - r^2\right),$$

  where η is the viscosity of the blood and P is the pressure difference between the ends of the tube.
- If P and ℓ are constant, then v is a function of r with domain $[0, R]$.
- The average rate of change of the velocity v as we move from $r = r_1$ outward to $r = r_2$ is

  $$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}.$$
- Letting $\Delta r \to 0$ we get the velocity gradient,

  $$\text{velocity gradient} = \lim_{r_1 \to r} \frac{v(r) - v(r_1)}{r - r_1} = \frac{dv}{dr}.$$

Biology Examples

Example (Blood Vessel; Continued)

- Using the law of laminar flow (and assuming P and ℓ constant) we get

$$\frac{dv}{dr} = \frac{d}{dr}\left(\frac{P}{4\eta\ell}\left(R^2 - r^2\right)\right) = \frac{P}{4\eta\ell}(0 - 2r) = -\frac{Pr}{2\eta\ell}.$$

- For a small human artery we may take $\eta = 0.027$, $R = 0.008\,\mathrm{cm}$, $\ell = 2\,\mathrm{cm}$, and $P = 4K\ \mathrm{dynes}/\mathrm{cm}^2$. We get

$$v = \frac{4,000}{4(0.027)2}\left((0.008)^2 - r^2\right) \simeq \left(1.85\times 10^4\right)\left(6.4\times 10^{-5} - r^2\right)\mathrm{cm}/s.$$

- For $r = 0.002\,\mathrm{cm}$ we obtain

$$v(0.002) \simeq \left(1.85\times 10^4\right)\left(6.4\times 10^{-5} - (0.002)^2\right) = 1.11\,\mathrm{cm}/s$$

- The gradient velocity for this value of r then is

$$\left.\frac{dv}{dr}\right|_{r=0.002} = -\frac{4,000(0.002)}{2(0.027)2} \simeq -74(\mathrm{cm}/s)/\mathrm{cm}.$$

Economics Example

Example (Marginal Cost)

- Suppose that $C(x)$ is the total cost that a company incurs producing x units of a certain commodity.
- $C(x)$ is called a cost function.
- The average rate of change of the cost between $x = x_1$ and $x = x_2$ is
$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1}.$$
- Let $\Delta x \to 0$ yields the marginal cost,
$$\text{marginal cost} = \lim_{x_1 \to x} \frac{C(x) - C(x_1)}{x - x_1} = \frac{dC}{dx}.$$
- If $x_1 = n$ and $x_2 = n + 1$, then
$$C'(n) \simeq \frac{\Delta C}{\Delta x} = \frac{C(n+1) - C(n)}{(n+1) - n} = C(n+1) - C(n).$$

Economics Example

Example (Marginal Cost; Continued)

- It is often appropriate to represent a total cost function by a polynomial,

  $$C(x) = a + bx + cx^2 + dx^3,$$

  where a is the overhead cost (rent, heat, etc.), and the other coefficients represent the cost of raw materials, labor, etc..
- For instance, suppose that the cost (in US\$) is given by

  $$C(x) = 10,000 + 5x + 0.01x^2.$$
- The marginal cost function then is

  $$\frac{dC}{dx} = \frac{d}{dx}\left(10,000 + 5x + 0.01x^2\right) = 5 + 0.02x.$$
- For $x = 500$ the marginal cost is

  $$C'(500) = 5 + 0.02(500) = 5 + 10 = \$15/\text{ item}.$$

Economics Example

Example

- Note that

$$\begin{aligned} C(501) - C(500) &= (10,000 + 5(501) + (0.01)(501)^2) \\ &\quad - (10,000 + 5(500) + (0.01)(500)^2) \\ &= \$15.01 \\ &\simeq \$15 = C'(500). \end{aligned}$$

Remark

Economists also study marginal demand, marginal revenue, and marginal profit, which are the derivatives of the demand, revenue, and profit functions.