College Mathematics II
§2.5 The Chain Rule

Sichuan University, Spring 2026

The Chain Rule

Chain Rule

If $g(x)$ *is differentiabl at* a *and* $f(u)$ *is differentiable at* $g(a)$*, then* $F(x) = f(g(x))$ *is differentiable at* a*, and we have*

$$F'(a) = f'(g(a))\,g'(a).$$

Chain Rule (2nd Version; Leibniz Notation)

If $y = f(u)$ *and* $u = g(x)$ *are differentiable functions, then*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

The Chain Rule

Example

Find $F'(x)$, if $F(x) = \sqrt{x^2 + 1}$.

Example

Differentiate $y = \sin(x^2)$ and $y = \sin^2(x)$.

Power Rule + Chain Rule

Power + Chain Rule

If $u = g(x)$ *is differentiable, and* r *is any real number, then*

$$\frac{d}{dx}\left[u^r\right] = ru^{r-1}\frac{du}{dr}.$$

Equivalently,

$$\frac{d}{dx}\left[g(x)^r\right] = rg(x)^{r-1}g'(x).$$

Example

Differentiate $y = (x^3 - 1)^{100}$.

Example

Find $f'(x)$ if

$$f(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}}.$$

Chain Rule + Trigonometric Derivatives

Example

If $f(x) = \sin(\cos(\tan x))$, then find $f'(x)$.

Example

Differentiate $y = \sqrt{\sec(x^3)}$.

Idea of Proof of the Chain Rule

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Assume that $g(x) \neq g(a)$ on $(a-\epsilon, a) \cup (a, a+\epsilon)$ for some $\epsilon > 0$.

- We then may write

$$\frac{f(g(x)) - f(g(a))}{x-a} = \frac{f(g(x)) - f(g(a))}{g(x) - a} \cdot \frac{g(x) - g(a)}{x-a}.$$

- As g is differentiable at a, we have

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} = \lim_{x \to a} \frac{g(x) - g(a)}{x-a}.$$

- Likewise, we have

$$f'(g(a)) = \lim_{h \to 0} \frac{f(g(a)+h) - f(g(a))}{h} = \lim_{u \to g(a)} \frac{f(u) - f(g(a))}{u - g(a)}.$$

□

Idea of Proof of the Chain Rule

Idea of Proof of the Chain Rule.

- As x approaches a, $u = g(x)$ converges to $g(a)$. Thus,

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = \lim_{u \to g(a)} \frac{f(u) - f(g(a))}{u - g(a)} = f'(g(a)).$$

- Combining all this gives

$$\begin{aligned}\lim_{x \to a} & \frac{f(g(x)) - f(g(a))}{x - a} \\ &= \left(\lim_{x \to g(a)} \frac{f(g(x)) - f(g(a))}{g(x) - a} \right) \cdot \left(\lim_{x \to a} \frac{g(x) - g(a)}{x - a} \right) \\ &= f'(g(a))g'(a).\end{aligned}$$

- This shows that $F(x) = f(g(x))$ is differentiable at a and $F'(a) = f'(g(a))g'(a)$.

□

Idea of Proof of the Chain Rule

Remark

- In general, we cannot always find $\epsilon > 0$ such that $g(x) \neq g(a)$ on $(a - \epsilon, a) \cup (a, a + \epsilon)$.
- In this case, we cannot divide by $g(x) - g(a)$, and so the proof above is invalid.
- In this case the Chain Rule can be proved by interpreting the derivative in terms of "linear approximation".
- This concept will be explained in Section 2.9.