

College Mathematics IV
§2.3 Counting Sample Points

Sichuan University, Spring 2026

Multiplication Rule

- In statistics we may need to evaluate the chances of occurrence of some events during an experiment.
- In many cases we may be able to count the number of points of a sample space without listing its elements.

Rule (Multiplication Rule)

If an operation can be performed in n_1 ways, and for each way there are n_2 ways to perform a second operation, then the two operations can be performed together in $n_1 n_2$ ways.

Example (Tossing a Die Twice)

- 6 ways for the first tossing ($n_1 = 6$).
- 6 ways for the 2nd tossing ($n_2 = 6$).
- The total number of ways is equal to

$$n_1 \times n_2 = 6 \times 6 = 36.$$

Multiplication Rule

Example

A 22-member club needs to elect a president and a treasurer. They cannot be the same person.

- 22 possibilities for the president ($n_1 = 22$).
- 21 possibilities for the treasurer, since they cannot be the president ($n_2 = 21$).
- The number of possibilities is

$$n_1 \times n_2 = 22 \times 21 = 462.$$

Generalized Multiplication Rule

Example (Flipping a Coin 3 Times)

- 2 possibilities for the first flipping ($n_1 = 2$).
- 2 possibilities for the second flipping ($n_2 = 2$).
- 2 possibilities for the third flipping ($n_3 = 2$).
- The number of possibilities is

$$n_1 \times n_2 \times n_3 = 2 \times 2 \times 2 = 8.$$

Generalized Multiplication Rule

Rule (Generalized Multiplication Rule)

If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \cdots n_k$ ways.

Generalized Multiplication Rule

Example

How many **even** 4-digit positive numbers can be made $a_4a_3a_2a_1$ from **1, 2, 4, 5, 6** if each digit can be used only once?

- The digit a_1 can be only **2, 4, 6** since the number must be even, and so $n_1 = 3$.
- There are **4** choices for the digit a_2 (and so $n_2 = 4$).
- There are **3** choices for the digit a_3 (and so $n_3 = 3$).
- There are **2** choices for the digit a_4 (and so $n_4 = 2$).
- The total number of choices is

$$n_1 \times n_2 \times n_3 \times n_4 = 3 \times 4 \times 3 \times 2 = 72.$$

Permutations

- We are often interested in sample spaces containing as outcomes all possible orders or arrangements of a group of events.
- Example: number of different arrangements for sitting 6 people around a table.

Definition

A **permutation** is an arrangement of all or part of a set of objects.

Example

- The permutations of the 3 letters a , b , c are

abc , acb , bac , bca , cab , cba .

- 3 possibilities for the 1st letter.
- 2 possibilities for the 2nd letter.
- 1 possibility for the 3rd letter.
- The number of permutations is

$$3 \times 2 \times 1 = 6.$$

Definition

For any positive number n , the number $n!$ (called “ n factorial”) is defined by

$$n! = n(n - 1)(n - 2) \cdots (2)(1).$$

Remark

By convention $0! = 1$.

Theorem

The number of permutations of n object is equal to $n!$.

Example

The number of permutations of the four letters a, b, c, d is equal to

$$4! = 4 \times 3 \times 2 \times 1 = 24.$$

Example

- The permutations of the four letters *a, b, c, d* by taking two at a time are

ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc.

- 4 possibilities for the 1st letter.
- 3 possibilities for the 2nd letter.
- The number of possibilities is

$$4 \times 3 = 12.$$

Remark

Given n objects, we want to take r of them at a time.

- There are n possibilities for the 1st object.
- $n - 1$ possibilities for the 2nd object.
- $n - 2$ possibilities for the 3rd object.
- Etc...
- $n - r + 1$ possibilities for the r th object.

The total number of possibilities then is equal to

$$n(n - 1) \cdots (n - r + 1) = \frac{n!}{(n - r)!}.$$

Definition

If n and r are non-negative integers with $r \leq n$, then we define

$${}_r P_n = \frac{n!}{(n-r)!}$$

Remark

If $1 \leq r \leq n$, then

$${}_r P_n = n(n-1) \cdots (n-r+1).$$

Theorem

The number of permutations of n objects taken r at a time is equal to

$${}_r P_n = \frac{n!}{(n-r)!}$$

Example

In one year, three awards (research, teaching, and service) will be given to a class of 25 graduate students in a statistics department. If each student can receive at most one award, how many possible selections are there?

Circular Permutations

Definition

- **Circular permutations** occur by arranging objects in circle.
- Two circular permutations are equal if we can go from to the other by moving the objects clockwise.

Example

Circular permutations of $\{1, 2, 3\}$.

Circular Permutations

Theorem

The number of circular permutations of n object is equal to $(n - 1)!$.

Example

Number of circular permutations of $\{1, 2, 3, 4\}$.

Permutations of Non-Distinguishable Objects

Example

The words that can be formed from the letters a, a, b, b are

$aabb, abab, abba, baab, baba, bbaa.$

- There are $4! = 24$ permutations of 4 letters.
- We divide by the numbers of permutations of the sets $\{a, a\}$ and $\{b, b\}$.
- The total number of words out of the letters a, a, b, b is

$$\frac{4!}{2!2!} = \frac{24}{2 \cdot 2} = \frac{24}{4} = 6.$$

Permutations of Non-Distinguishable Objects

Theorem

The number of distinct permutations of n objects of n_1 are of one kind, n_2 are of a 2nd kind, ..., n_k of a k th kind is equal to

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Remark

The number of words out of the letters a, a, b, b, b is equal to

$$\frac{5!}{2!3!} = \frac{4 \cdot 5}{2!} = \frac{4 \cdot 5}{2} = 10.$$

Definition

A **partition** of a set S into r **cells** is given by subsets A_1, \dots, A_r such that

- (i) They are mutually disjoint, i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$.
- (ii) Every element of S is in some A_i , i.e., $S = A_1 \cup \dots \cup A_r$.

Example

If A is a subset of S , and A' is its complement in S , then we saw that

$$A \cap A' = \emptyset \quad \text{and} \quad A \cup A' = S.$$

Therefore, $\{A, A'\}$ and $\{A, A'\}$ are partitions of S into two cells.

Example

Partitions of $\{1, 2, 3, 4, 5\}$ into a cell of 2 elements and a cell of 3 elements:

$$\begin{aligned} & \{\{1, 2\}, \{3, 4, 5\}\}, \{\{1, 3\}, \{2, 4, 5\}\}, \{\{1, 4\}, \{2, 3, 5\}\}, \\ & \quad \{\{1, 5\}, \{2, 3, 4\}\}, \\ & \{\{2, 3\}, \{1, 4, 5\}\}, \{\{2, 4\}, \{1, 3, 5\}\}, \{\{2, 5\}, \{1, 3, 4\}\}, \\ & \quad \{\{3, 4\}, \{1, 2, 5\}\}, \{\{3, 5\}, \{1, 2, 4\}\}, \\ & \quad \{\{4, 5\}, \{1, 2, 3\}\}. \end{aligned}$$

- This amounts to finding permutations of $\{1, 2, 3, 4, 5\}$ that take 2 elements and 3 elements at a time.
- Their number is equal to

$$\frac{5!}{2!3!} = 10.$$

Definition

If $n_1 + n_2 + \cdots + n_r = n$, then we define

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}.$$

Theorem

Suppose that $n_1 + n_2 + \cdots + n_r = n$. The number of partitions of a set of n objects into cells of n_1 elements, n_2 elements, ..., n_r elements is equal to

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}.$$

Definition

For $0 \leq r \leq n$ we set

$$\binom{n}{r} = \binom{n}{r, n-r} = \frac{n!}{r!(n-r)!}.$$

Remark

We have

$$\binom{n}{n-r} = \frac{n!}{(n-r)!r!} = \binom{n}{r}.$$

Combinations

Definition

A **combination** of r elements is a selection of r elements of without any regard for the order.

Example

The combinations of 2 elements of $\{1, 2, 3, 4, 5\}$ are

$$\begin{aligned} &\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \\ &\{2, 3\}, \{2, 4\}, \{2, 5\}, \\ &\{3, 4\}, \{3, 5\}, \\ &\{4, 5\}. \end{aligned}$$

- This is equivalent to partitioning $\{1, 2, 3, 4, 5\}$ into a cell of 2 elements and a cell of $5 - 2 = 3$ elements.
- The number of such combinations then is equal to

$$\binom{5}{2, 3} = \binom{5}{2} = \frac{5!}{2!3!} = 10.$$

Theorem

The number of combinations of n distinct objects taken r at a time is equal to

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Example

The number of combinations of 7 objects by taking 3 at time is equal to

$$\binom{7}{3} = \frac{7!}{3!4!} = \frac{5 \cdot 6 \cdot 7}{3!} = \frac{5 \cdot 6 \cdot 7}{6} = 5 \cdot 7 = 35.$$