

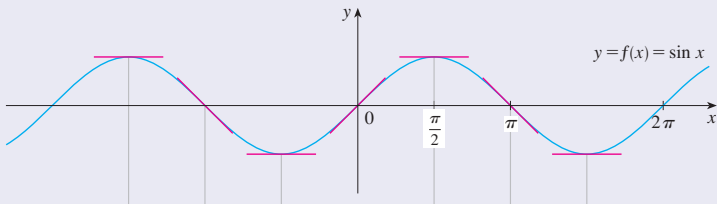
College Mathematics II
§2.4 Derivatives of Trigonometric Functions

Sichuan University, Spring 2026

The Derivative of $y = \sin x$ at $x = 0$

Facts

- If $g(x) = \sin x$, then $g'(0)$ is the slope of the tangent line of $y = g(x)$ at $(0, g(0)) = (0, 0)$.



- The tangent line has equation $y = x$.
- Its slope is equal to 1, and so $g'(0) = 1$.
- Thus,

$$1 = g'(0) = \lim_{h \rightarrow 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \rightarrow 0} \frac{\sin(h)}{h}.$$

The Derivative of $y = \sin x$ at $x = 0$

We have thus got the following limit.

Theorem

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$

Using this limit we also get the following limits.

Theorem

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2} = \frac{1}{2} \quad \text{and} \quad \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0.$$

The Derivative of $y = \sin x$

Reminder (see Appendix D)

We have

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y.$$

$$\frac{d}{dx}(\sin x) = \cos x.$$

Example

Differentiate $y = x^2 \sin x$.

The Derivative of $y = \cos x$

Reminder (see Appendix D)

We have

$$\cos(x + y) = \cos x \cos y - \sin x \sin y,$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y.$$

$$\frac{d}{dx}(\cos x) = -\sin x.$$

The Derivative of $y = \sec x$

$$\frac{d}{dx}(\sec x) = \sec x \tan x.$$

The Derivative of $y = \sec x$

Proof.

- By the Inverse Rule, we have

$$\frac{d}{dx}(\sec x) = \frac{d}{dx} \left(\frac{1}{\cos x} \right) = -\frac{1}{\cos^2 x} \frac{d}{dx}(\cos x).$$

- We know that

$$\frac{d}{dx}(\cos x) = -\sin x.$$

- Thus,

$$\frac{d}{dx}(\sec x) = -\frac{1}{\cos^2 x}(-\sin x) = \frac{1}{\cos x} \frac{\sin x}{\cos x} = \sec x \tan x.$$



The Derivative of $y = \tan x$

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

The Derivative of $y = \tan x$

Proof.

- By the Quotient Rule, we have

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{1}{\cos^2 x} \left(\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x) \right).$$

- We know that

$$\frac{d}{dx}(\sin x) = \cos x \quad \text{and} \quad \frac{d}{dx}(\cos x) = -\sin x.$$

- Thus,

$$\begin{aligned} \frac{d}{dx}(\tan x) &= \frac{1}{\cos^2 x} \left(\cos x(\cos x) - \sin x(-\sin x) \right) \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x. \end{aligned}$$



The Derivative of $y = \tan x$

Remark

- As $1 = \cos^2 x + \sin^2 x$, we get

$$\sec^2 x = \frac{1}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x.$$

- Therefore, we also have

$$\frac{d}{dx}(\tan x) = 1 + \tan^2 x.$$

Example

Differentiate $y = \frac{\sec x}{1 + \tan x}$.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x,$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x,$$

$$\frac{d}{dx}(\tan x) = \sec^2 x,$$

$$\frac{d}{dx}(\cos x) = -\sin x,$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x,$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x.$$

We have

$$\frac{d^2}{dx^2}(\sin x) = \frac{d}{dx} \left(\frac{d}{dx} \sin x \right) = \frac{d}{dx}(\cos x) = -\sin x,$$

$$\frac{d^3}{dx^3}(\sin x) = \frac{d}{dx} \left(\frac{d^2}{dx^2} \sin x \right) = \frac{d}{dx}(-\sin x) = -\cos x,$$

$$\frac{d^4}{dx^4}(\sin x) = \frac{d}{dx} \left(\frac{d^3}{dx^3} \sin x \right) = \frac{d}{dx}(-\cos x) = -(-\sin x) = \sin x.$$

Theorem

For every non-negative integer k , we have

$$\frac{d^{2k}}{dx^{2k}}(\sin x) = (-1)^k \sin x, \quad \frac{d^{2k+1}}{dx^{2k+1}}(\sin x) = (-1)^k \cos x.$$

Higher Order Derivatives

Theorem

For every non-negative integer k , we have

$$\frac{d^{2k}}{dx^{2k}}(\cos x) = (-1)^k \cos x, \quad \frac{d^{2k+1}}{dx^{2k+1}}(\cos x) = (-1)^{k+1} \sin x.$$

Proof.

We have

$$\frac{d^{2k}}{dx^{2k}}(\cos x) = \frac{d^{2k}}{dx^{2k}} \left(\frac{d}{dx} \sin x \right) = \frac{d^{2k+1}}{dx^{2k+1}}(\sin x) = (-1)^k \cos x.$$

$$\frac{d^{2k+1}}{dx^{2k+1}}(\cos x) = \frac{d^{2k+1}}{dx^{2k+1}} \left(\frac{d}{dx} \sin x \right) = \frac{d^{2(k+1)}}{dx^{2(k+1)}}(\sin x) = (-1)^{k+1} \sin x.$$

□

Higher Order Derivatives

Example

Find $\frac{d^{27}}{dx^{27}}(\sin x)$ and $\frac{d^{28}}{dx^{28}}(\cos x)$.

Example

Find $\lim_{x \rightarrow 0} \frac{\sin 7x}{4x}$.

Example

Find $\lim_{x \rightarrow 0} x \cot x$.