Introduction to Noncommutative Geometry
Chapter 6:
Connes' Trace Theorem on Tori (Part 2)

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Pseudodifferential Operators

Additional References

- Ruzhansky, M.; Turunen, V.: *Pseudo-differential operators and symmetries*. Birkhäuser, Basel, 2010.

Notation

- If $\alpha \in \mathbb{N}_0^n$, then $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
- $D_{x_j} = \frac{1}{i}\partial_{x_j}$, $j = 1, \ldots, n$.
- If $\alpha \in \mathbb{N}_0^n$, then $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$.

Differential Operators on $\mathbb{T}^n$

Definition

If $P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$ is a differential operator on $\mathbb{T}^n$, then its symbol is

$$\sigma(x,\xi) := \sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha, \qquad (x,\xi) \in \mathbb{T}^n \times \mathbb{R}^n.$$

Remark

$\sigma(x,\xi)$ is a function in $C^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ which is polynomial in ξ.

Fact

Let P be a differential operator on $\mathbb{T}^n$ with symbol $\sigma(x,\xi)$. For all $u = \sum \hat{u}(k) e_k$ in $C^\infty(\mathbb{T}^n)$, we have

$$Pu(x) = \sum_{k \in \mathbb{Z}^n} \sigma(x,k)\hat{u}(k)e_k(x), \qquad x \in \mathbb{T}^n.$$

Example

Let $\Delta = -(\partial_{x_1}^2 + \cdots + \partial_{x_n}^2)$ be the Laplacian on $\mathbb{T}^n$.

- As $\Delta = D_{x_1}^2 + \cdots + D_{x_n}^2$, its symbol is
$$\sigma(x,\xi) = \xi_1^2 + \cdots + \xi_n^2 = |\xi|^2.$$
- As $\Delta e_k = |k|^2 e_k$, for all $u = \sum \hat{u}(k) e_k \in C^\infty(\mathbb{T}^n)$, we have
$$\Delta u = \sum_{k\in\mathbb{Z}^n} \hat{u}(k)\Delta e_k = \sum_{k\in\mathbb{Z}^n} |k|^2 \hat{u}(k) e_k.$$

Symbols on $\mathbb{T}^n$

Definition

$S^m(\mathbb{T}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, consists of all $\sigma(x,\xi) \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ that admit an expansion,

$$\sigma(x,\xi) \sim \sum_{j\geq 0} \sigma_{m-j}(x,\xi), \quad \sigma_{m-j} \in C^\infty(\mathbb{T}^n \times (\mathbb{R}^n \setminus 0))$$

$$\sigma_{m-j}(x,\lambda\xi) = \lambda^{m-j}\sigma_{m-j}(x,\xi) \quad \forall \lambda > 0.$$

Here $\sim$ means that, for all $N \geq 0$ and $\alpha, \beta \in \mathbb{N}_0^n$, there is $C_{N\alpha\beta} > 0$ such that

$$\Big|\partial_x^\alpha \partial_\xi^\beta \Big(\sigma(x,\xi) - \sum_{j<N} \sigma_{m-j}(x,\xi)\Big)\Big| \leq C_{N\alpha\beta}|\xi|^{m-N-|\beta|},$$

for all $x \in \mathbb{T}^n$ and all $\xi \in \mathbb{R}^n$, $|\xi| \geq 1$.

Remark

For $N = 0$, we get the estimates,

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\beta|} \qquad \forall (x,\xi) \in \mathbb{T}^n \times \mathbb{R}^n.$$

Remark

- The homogeneous symbol $\sigma_m(x,\xi)$ is called the principal symbol of $\sigma(x,\xi)$.
- We have

$$\sigma_m(x,\xi) = \lim_{\lambda\to\infty} \lambda^{-m}\sigma(x,\lambda\xi) \qquad \forall \xi \neq 0.$$

Pseudodifferential Operators on $\mathbb{T}^n$

Definition

If $\sigma \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, then $P_\sigma : C^\infty(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n)$ is the linear operator defined by

$$(P_\sigma u)(x) = \sum_{k \in \mathbb{Z}^n} \sigma(x,k)\hat{u}(k)e_k(x), \quad u = \sum \hat{u}(k)e_k \in C^\infty(\mathbb{T}^n).$$

Remark

We have

$$(P_\sigma e_k)(x) = \sigma(x,k)e_k(x) \qquad \forall k \in \mathbb{Z}^n.$$

Pseudodifferential Operators on $\mathbb{T}^n$

Definition

$\Psi^m(\mathbb{T}^n)$, $m \in \mathbb{R}$, consists of all operators $P : C^\infty(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n)$ of the form,

$$P = P_\sigma \qquad \text{for some } \sigma \in S^m(\mathbb{T}^n \times \mathbb{R}^n).$$

Remark

- The symbol $\sigma(x, \xi)$ is not unique.
- However, if $\sigma(x, \xi) \sim \sum \sigma_{m-j}(x, \xi)$, then each homogenous symbol $\sigma_{m-j}(x, \xi)$ is uniquely determined by P.
- We call $\sigma_m(x, \xi)$ the principal symbol of P.

Example

- Let $q \in \mathbb{R}$. By definition,

$$\Delta^q e_k = \begin{cases} |k|^{2q} e_k & \text{if } k \neq 0, \\ 0 & \text{if } k = 0. \end{cases}$$

- Thus,

$$\Delta^q u = \sum_{k \in \mathbb{Z}^n \setminus 0} |k|^{2q} \hat{u}(k) e_k = \sum_{k \in \mathbb{Z}^n} \sigma(x,k) \hat{u}(k) e_k,$$

where $\sigma(x,\xi)$ is any function in $C^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ such that

$$\sigma(x,\xi) = \begin{cases} |\xi|^{2q} e_k & \text{for } |\xi| \geq 1, \\ 0 & \text{near } \xi = 0. \end{cases}$$

- In particular $\sigma \in S^{2q}(\mathbb{T}^n \times \mathbb{R}^n)$ with $\sigma(x,\xi) \sim |\xi|^{2q}$.
- It follows that

$$\Delta^q = P_\sigma \in \Psi^{2q}(\mathbb{T}^n).$$

Proposition

Let $P_1 \in \Psi^{m_1}(\mathbb{T}^n)$ *and* $P_2 \in \Psi^{m_2}(\mathbb{T}^n)$ *have respective principal symbols* $\sigma_{m_1}(x, \xi)$ *and* $\sigma_{m_2}(x, \xi)$.

1. $P_1 P_2 \in \Psi^{m_1+m_2}(\mathbb{T}^n)$.
2. *Its principal symbol is* $\sigma_{m_1}(x, \xi)\sigma_{m_2}(x, \xi)$.

Weak Schatten Class Properties

Proposition

If $P \in \Psi^m(\mathbb{T}^n)$ with $m \leq 0$, then P uniquely extends to a continuous linear operator,

$$P : L^2(\mathbb{T}^n) \longrightarrow L^2(\mathbb{T}^n).$$

Proposition

Every $P \in \Psi^{-m}(\mathbb{T}^n)$, $m > 0$, is in the weak Schatten class $\mathcal{L}^{\frac{n}{m},\infty}$.

Proof.

- We know that $\Delta^{-m/2} \in \mathcal{L}^{\frac{n}{m},\infty}$ and $\Delta^{m/2} \in \Psi^m(\mathbb{T}^n)$.
- We have $\Delta^{m/2}\Delta^{-m/2} = 1 - |e_0\rangle\langle e_0|$.
- Thus,

$$P = P\big(\Delta^{\frac{m}{2}}\Delta^{-\frac{m}{2}} + |e_0\rangle\langle e_0|\big) = \big(P\Delta^{\frac{m}{2}}\big)\Delta^{-\frac{m}{2}} + |Pe_0\rangle\langle e_0|.$$

- Here $|Pe_0\rangle\langle e_0|$ has rank 1, and hence $|Pe_0\rangle\langle e_0| \in \mathcal{L}^{\frac{n}{m},\infty}$.
- Here $P\Delta^{m/2} \in \Psi^0(\mathbb{T}^n) \subseteq \mathcal{L}(L^2(\mathbb{T}^n))$
- As $\mathcal{L}^{\frac{n}{m},\infty}$ is an ideal, we see that $(P\Delta^{m/2})\Delta^{-m/2} \in \mathcal{L}^{\frac{n}{m},\infty}$.
- It follows that $P \in \mathcal{L}^{\frac{n}{m},\infty}$.

The proof is complete. □

Trace Formula

Proposition

Let $P \in \Psi^m(\mathbb{T}^n)$, $m < -n$. *Then:*

1. P *is trace-class.*
2. *For any* $\sigma \in S^{-n}(\mathbb{T}^n \times \mathbb{R}^n)$ *such that* $P = P_\sigma$, *we have*

$$\mathrm{Tr}[P] = (2\pi)^{-n} \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{T}^n} \sigma(x,k)dx.$$

Proof.

- P is trace-class, because as $m < -n$, we have $n|m|^{-1} < 1$, and hence
$$\Psi^m(\mathbb{T}^n) = \Psi^{-|m|}(\mathbb{T}^n) \subseteq \mathcal{L}^{\frac{n}{|m|},\infty} \subseteq \mathcal{L}^1.$$
- By definition,
$$\mathrm{Tr}[P] = \sum_{k \in \mathbb{Z}^n} \langle P e_k | e_k \rangle \,.$$
- If $P = P_\sigma$, then $P e_k = P_\sigma e_k = \sigma(x,k) e_k$.
- Thus,
$$\begin{aligned}\langle P e_k | e_k \rangle &= \langle \sigma(x,k) e_k | e_k \rangle \\ &= (2\pi)^{-n} \int \sigma(x,k) e_k(x) \overline{e_k(x)} dx \\ &= (2\pi)^{-n} \int_{\mathbb{T}^n} \sigma(x,k) dx.\end{aligned}$$
- This gives the trace formula.

Remark

- If $P \in \Psi^{-m}(\mathbb{T}^n)$ with $m > 0$, then $P \in \mathcal{L}^{\frac{n}{m},\infty}$.
- In particular, for $m = n$, we get that $P \in \mathcal{L}^{1,\infty}$.

Theorem (Connes' Trace Theorem on $\mathbb{T}^n$)

Every operator $P \in \Psi^{-n}(\mathbb{T}^n)$ is strongly measurable, and

$$
⨍ P = \frac{1}{n}(2\pi)^{-n} \iint_{\mathbb{T}^n \times \mathbb{S}^{n-1}} \sigma_{-n}(x,\xi)dxd\xi,
$$

where $\sigma_{-n}(x,\xi)$ is the principal symbol of P.

Remarks

1. A proof is given in the handwritten notes.
2. For $P = f\Delta^{-n/2}$ we recover Connes' integration formula.