Introduction to Noncommutative Geometry
Chapter 7:
Connes' Trace Theorem on Euclidean Spaces
Part 1:
Pseudodifferential Operators

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Additional References

- Taylor, M.E.: *Pseudodifferential operators*. Princeton University Press, Princeton, NJ, 1981.
- Slides of my 2022 online course.

Notation

- If $\alpha \in \mathbb{N}_0^n$, then $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
- $D_{x_j} = \frac{1}{i}\partial_{x_j}$, $j = 1, \ldots, n$.
- If $\alpha \in \mathbb{N}_0^n$, then $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$.

Differential Operators

Setup

$U \subset \mathbb{R}^n$ is an open set.

Definition

A differential operator $P : C^\infty(U) \to C^\infty(U)$ of order m is of the form,

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha, \qquad a_\alpha(x) \in C^\infty(U).$$

Example

Laplace operator $\Delta := -(\partial_{x_1}^2 + \cdots + \partial_{x_n}^2) = D_{x_1}^2 + \cdots + D_{x_n}^2$.

Differential Operators on U

Definition

Let $P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$ be a differential operator.

- Its symbol is

$$\sigma(x, \xi) := \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha, \qquad (x, \xi) \in U \times \mathbb{R}^n.$$

- The principal part is the m-th degree part,

$$\sigma(x, \xi) := \sum_{|\alpha| = m} a_\alpha \xi^\alpha, \qquad (x, \xi) \in U \times \mathbb{R}^n.$$

Example

For the Laplace operator $\Delta = D_{x_1}^2 + \cdots + D_{x_n}^2$, we have

$$\sigma(x, \xi) = \sigma_2(x, \xi) = \xi_1^2 + \cdots + \xi_n^2 = |\xi|^2.$$

Differential Operators on U

Notation

- If $u \in L^1(\mathbb{R}^n)$, then its Fourier transform is
$$\hat{u}(\xi) = \int e^{-ix\cdot\xi} u(x)dx, \qquad x \in \mathbb{R}^n.$$
- Its inverse Fourier transform is
$$\check{u}(\xi) = \int e^{ix\cdot\xi} u(\xi) đ\xi, \qquad đ\xi := (2\pi)^{-n} d\xi.$$

Remark

If u is in the Schwartz's class $\mathcal{S}(\mathbb{R}^n)$, then
$$(D_x^\alpha u)^\alpha = \xi^\alpha \hat{u}.$$

Differential Operators on U

Fact

Let $P = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha$ be a differential operator on U. If $\sigma(x,\xi)$ is the symbol of P, then

$$Pu(x) = \int e^{ix\cdot\xi}\sigma(x,\xi)\hat{u}(\xi)đ\xi \qquad \forall u \in C_c^\infty(U).$$

Proof.

- As $(D_x^\alpha u)^\wedge = \xi^\alpha \hat{u}$, we have

$$D_x^\alpha u = \left((D_x^\alpha u)^\wedge\right)^\vee = (\xi^\alpha \hat{u})^\vee = \int e^{ix\cdot\xi}\xi^\alpha \hat{u}(\xi)đ\xi.$$

- Thus,

$$\begin{aligned} Pu = \sum a_\alpha(x) D_x^\alpha u &= \sum a_\alpha(x) \int e^{ix\cdot\xi}\xi^\alpha \hat{u}(\xi)đ\xi \\ &= \int e^{ix\cdot\xi}\left(\sum a_\alpha(x)\xi^\alpha\right)\hat{u}(\xi)đ\xi \\ &= \int e^{ix\cdot\xi}\sigma(x,\xi)\hat{u}(\xi)đ\xi. \end{aligned}$$

□

Symbols on $U \times \mathbb{R}^n$

Definition (Classical Symbols)

$S^m(U \times \mathbb{R}^n)$, $m \in \mathbb{R}$, consists $\sigma(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$ that admit an expansion,

$$\sigma(x,\xi) \sim \sum_{j \geq 0} \sigma_{m-j}(x,\xi), \quad \sigma_{m-j} \in C^\infty(U \times (\mathbb{R}^n \setminus 0))$$

$$\sigma_{m-j}(x,\lambda\xi) = \lambda^{m-j}\sigma_{m-j}(x,\xi) \quad \forall \lambda > 0.$$

Here $\sim$ means that, for all $N \geq 0$, compact $K \subset U$, and $\alpha, \beta \in \mathbb{N}_0^n$, there is $C_{NK\alpha\beta} > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta \big(\sigma(x,\xi) - \sum_{j<N} \sigma_{m-j}(x,\xi)\big)\right| \leq C_{NK\alpha\beta}|\xi|^{m-N-|\beta|},$$

for all $x \in K$ and all $\xi \in \mathbb{R}^n$, $|\xi| \geq 1$.

Remark

For $N = 0$, we get the estimates,

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \leq C_{K\alpha\beta}(1+|\xi|)^{m-|\beta|} \qquad \forall (x,\xi) \in K \times \mathbb{R}^n.$$

Remark

- The homogeneous symbol $\sigma_m(x,\xi)$ is called the principal symbol of $\sigma(x,\xi)$.
- We have

$$\sigma_m(x,\xi) = \lim_{\lambda\to\infty} \lambda^{-m}\sigma(x,\lambda\xi) \qquad \forall \xi \neq 0.$$

Symbols on $U \times \mathbb{R}^n$

Example

Let $P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$ be a differential operator of order m.

- Its symbol is
$$\sigma(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha.$$
- We have
$$\sigma(x, \xi) = \sum_{0 \leq j \leq m} \sigma_{m-j}(x, \xi), \quad \sigma_{m-j}(x, \xi) := \sum_{|\alpha| = m-j} a_\alpha(x) \xi^\alpha.$$
- Here $\sigma_{m-j}(x, \lambda\xi) = \lambda^{m-j} \sigma_{m-j}(x, \xi)$
- It then follows that $\sigma(x, \xi) \in S^m(U \times \mathbb{R}^n)$.

Example

- Set $\langle\xi\rangle := (1 + |\xi|^2)^{\frac{1}{2}}$, $\xi \in \mathbb{R}^n$ (Russian bracket).
- For any $s \in \mathbb{R}$, the binomial formula implies that

$$\langle\xi\rangle^s = |\xi|^s(|\xi|^{-2} + 1)^{\frac{s}{2}} \sim \sum_{j\geq 0} \binom{\frac{s}{2}}{j} |\xi|^{s-2j}.$$

- It follows that $\langle\xi\rangle^s$ is a symbol of order s whose principal symbol is $|\xi|^s$.

Pseudodifferential Operators on U

Definition

If $\sigma \in S^m(U \times \mathbb{R}^n)$, $m \in \mathbb{R}$, then $\sigma(x, D) : C_c^\infty(U) \to C^\infty(U)$ is the linear operator defined by

$$\sigma(x, D)u(x) = \int e^{ix\cdot\xi}\sigma(x, \xi)\hat{u}(\xi)đ\xi, \quad u \in C_c^\infty(U).$$

Example (Differential Operators)

If $\sigma(x, \xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$, then

$$\sigma(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x)D_x^\alpha.$$

Example

Assume $U = \mathbb{R}^n$, and let $\Delta = D_{x_1}^2 + \cdots + D_{x_n}^2$ be its Laplacian.

- We have $\Delta = \sigma(x, D)$, with $\sigma(x, \xi) = |\xi|^2$.
- That is,

$$(\Delta u)(x) = \int e^{ix\cdot\xi} |\xi|^2 \hat{u}(\xi) đ\xi = \left(|\xi|^2 \hat{u}\right)^{\vee}(x).$$

- Define $V : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ by $Vu = (2\pi)^{-n}\hat{u}$.
- This is a unitary operator, with $V^{-1}u = V^*u = (2\pi)^n \check{u}$.
- We then have

$$\Delta = V^* M_{|\xi|^2} V,$$

  where $M_{|\xi|^2}$ is the operator of multiplication by $|\xi|^2$.
- This is precisely the spectral theorem for Δ.

Example (Continued)

Let $s \in \mathbb{R}$.

- The Borel functional calculus for Δ gives:

$$(1+\Delta)^{\frac{s}{2}} = V^* M_{(1+|\xi|^2)^{\frac{s}{2}}} V.$$

- For $s > 0$ this is a selfadjoint unbounded operator whose domain is the Sobolev space,

$$W^{2,s}(\mathbb{R}^n) := \left\{u \in L^2(\mathbb{R}^n);\ (1+|\xi|^2)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R}^n)\right\}.$$

- In terms of the Fourier transform, we have

$$(1+\Delta)^{\frac{s}{2}} u(x) = \int e^{ix\cdot\xi}(1+|\xi|^2)^{\frac{s}{2}} \hat{u}(\xi) đ\xi.$$

- We saw that $(1+|\xi|^2)^{s/2} \in S^s(\mathbb{R}^n \times \mathbb{R}^n)$. Thus,

$$(1+\Delta)^{\frac{s}{2}} = \sigma^{(s)}(x, D), \quad \text{with } \sigma^{(s)}(x,\xi) = (1+|\xi|^2)^{\frac{s}{2}}.$$

Smoothing Operators

Definition (Smoothing Operators)

- An operator $R : C_c^\infty(U) \to C^\infty(U)$ is called smoothing if it is given by a kernel $k_R(x, y) \in C^\infty(U \times U)$, i.e.,

$$Ru(x) = \int_U k_R(x, y)u(y)dy, \qquad u \in C_c^\infty(U).$$

- The space of smoothing operators is denoted $\Psi^{-\infty}(U)$.

Proposition

Let $R : C_c^\infty(U) \to C^\infty(U)$ be a continuous linear operator. TFAE:

(i) *R is smoothing.*

(ii) *It uniquely extends to a continuous operator $\mathcal{E}'(U) \to C^\infty(U)$.*

Definition

$S^{-\infty}(U \times \mathbb{R}^n) := \bigcap_{m\in\mathbb{R}} S^m(U \times \mathbb{R}^n)$.

Remark

If $\sigma(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$, then $\sigma(x,\xi) \in S^{-\infty}(U \times \mathbb{R}^n)$ if and only if, for every $N \geq 0$, we have the estimates,

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \leq C_{NK\alpha\beta}(1+|\xi|)^{-N} \qquad \forall (x,\xi) \in K \times \mathbb{R}^n.$$

Example

If $\sigma(x,\xi) \in S^m(U \times \mathbb{R}^n)$, then

$$\sigma(x,D) \in \Psi^{-\infty}(U) \Longleftrightarrow \sigma(x,\xi) \in S^{-\infty}(U \times \mathbb{R}^n).$$

Pseudodifferential Operators on U

Definition (Pseudodifferential Operators (ΨDOs))

$\Psi^m(U)$, $m \in \mathbb{R}$, consists of linear operators $P : C_c^\infty(U) \to C^\infty(U)$ of the form,

$$P = \sigma(x, D) + R,$$

with $\sigma(x, \xi) \in S^m(U \times \mathbb{R}^n)$, $\sigma(x, \xi) \sim \sum \sigma_{m-j}(x, \xi)$, and $R \in \Psi^{-\infty}(U)$.

Remark

- The symbol $\sigma_m(x, \xi)$ is called the principal symbol of P.
- The homogeneous symbols $\sigma_{m-j}(x, \xi)$ depends only on P.

Example

Let $P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$ be a differential operator.

- If $\sigma(x, \xi) = \sum a_\alpha(x) \xi^\alpha$, then $\sigma(x, \xi) \in S^m(U \times \mathbb{R}^n)$.
- We then have
$$P = \sigma(x, D) \in \Psi^m(U).$$

Example

Assume $U = \mathbb{R}$, and let $s \in \mathbb{R}$.

- We saw that $\sigma^{(s)} := (1 + |\xi|^2)^{\frac{s}{2}} \in S^s(\mathbb{R}^n \times \mathbb{R}^n)$.
- We also saw that
$$(1 + \Delta)^{\frac{s}{2}} = \sigma^{(s)}(x, D).$$
- Thus,
$$(1 + \Delta)^{\frac{s}{2}} \in \Psi^s(\mathbb{R}^n) \qquad \forall s \in \mathbb{R}.$$

Remark

Let $\sigma(x,\xi) \in S^m(U \times \mathbb{R}^n)$ with $m < -n$.

- For any compact $K \subset U$,

$$|\sigma(x,\xi)| \leq C_K(1+|\xi|)^m, \quad (x,\xi) \in K \times \mathbb{R}^n.$$

- As $m < -n$, the function $(1+|\xi|)^m$ is in $L^1(\mathbb{R}^n)$, and so we may define

$$\check{\sigma}_{\xi\to y}(x,y) := \int e^{ix\cdot y}\sigma(x,\xi)đ\xi \in C(K \times \mathbb{R}^n).$$

Therefore, we obtain:

Lemma

If $\sigma(x,\xi) \in S^m(U \times \mathbb{R}^n)$, $m < -n$, *then*

$$\check{\sigma}_{\xi\to y}(x,y) := \int e^{ix\cdot y}\sigma(x,\xi)đ\xi \in C(U \times \mathbb{R}^n).$$

Schwartz Kernels of ΨDOs

Lemma

Let $\sigma(x,\xi) \in S^m(U \times \mathbb{R}^n)$ *with* $m < -n$, *and set* $P = \sigma(x, D)$. *We have*

$$Pu(x) = \int_U k_P(x,y)u(y)dy, \quad \text{with } k_P(x,y) := \check{p}_{\xi\to y}(x, x-y).$$

Proof.

If $u \in C_c^\infty(U)$, then

$$\begin{aligned} Pu(x) &= \int e^{ix\cdot y}\sigma(x,\xi)\hat{u}(\xi)đ\xi \\ &= \int e^{ix\cdot y}\sigma(x,\xi)\left(\int e^{-iy\cdot\xi}u(y)dy\right)đ\xi \\ &= \int\left(\int e^{i(x-y)\cdot\xi}\sigma(x,\xi)đ\xi\right)u(y)dy \\ &= \int \check{p}_{\xi\to y}(x,x-y)u(y)dy. \end{aligned}$$

This gives the result. □

Schwartz Kernels of ΨDOs

Remark

- In general, if $\sigma(x,\xi) \in S^m(U \times \mathbb{R}^n)$, $m \geq -n$, then $\check{\sigma}_{\xi\to y}(x,y)$ makes sense as a distribution.
- Namely, if $v \in C_c^\infty(\mathbb{R}^n)$, then

$$\langle \check{\sigma}_{\xi\to y}(x,y), v(y)\rangle := \langle \sigma(x,\xi), \check{v}(\xi)\rangle = \int \sigma(x,\xi)\check{v}(\xi).$$

Lemma

Let $\sigma(x,\xi) \in S^m(U \times \mathbb{R}^n)$, $m \in \mathbb{R}$, *and set* $P = \sigma(x,D)$. *Then*

$$Pu(x) = \langle k_P(x,y), u(y)\rangle\,, \quad k_P(x,y) := \check{\sigma}_{\xi\to y}(x, x-y).$$

More precisely, for all $u \in C_c^\infty(U)$,

$$Pu(x) = \langle k_P(x,y), u(y)\rangle = \langle \check{\sigma}_{\xi\to y}(x,y), u(x-y)\rangle\,.$$

Definition

$k_P(x,y)$ is called the Schwartz kernel of P.

Lemma

Let $\sigma(x,\xi) \in \mathbb{S}^m(U \times \mathbb{R}^n)$, $m \in \mathbb{R}$. *Then* $\check{\sigma}_{\xi \to y}(x,y)$ *is* C^∞ *on* $U \times (\mathbb{R}^n \backslash 0)$.

Notation

$\Gamma = \{(x,x);\ x \in U\}$ (diagonal of $U \times U$).

Proposition

Let $P \in \Psi^m(U)$, $m \in \mathbb{R}$, *have Schwartz kernel* $k_P(x,y)$.

1. $k_P(x,y)$ *is* C^∞ *on* $(U \times U)\backslash\Gamma$.
2. *If* $\Re m < -n$, *then* $k_P(x,y) \in C(U \times U)$.

Compactly Supported ΨDOs

Setup

- $K \subseteq U$ is compact.

Definition

$\Psi_K^m(U)$, $m \in \mathbb{R}$, consists of all $P \in \Psi^m(U)$ whose Schwartz kernels $k_P(x,y)$ (seen as distributions on $U \times U$) are supported on $K \times K$.

Remark

This means that the following two properties are satisfied:

1. $\operatorname{supp} Pu \subseteq K$ for all $u \in C_c^\infty(U)$.
2. If $\operatorname{supp} u \cap K = \emptyset$, then $Pu = 0$.

Example

If $P \in \Psi^m(U)$ and $\varphi, \psi \in C_K^\infty(U)$, then $\varphi P \psi \in \Psi_K^m(U)$.

Remarks

1. If $P \in \Psi_K(U)$, then it induces a linear operator,
$$P : C_K^\infty(U) \longrightarrow C_K^\infty(U)$$

2. If $V \subseteq \mathbb{R}^n$ is any other open set containing K, then
$$\Psi_K^m(U) = \Psi_K^m(V) = \Psi_K^m(\mathbb{R}^n).$$

3. If $P \in \Psi_K(U)$, then $P = \sigma(x, D)$, with
$$\sigma(x,\xi) = e^{-ix\cdot\xi} P(e_\xi), \quad e_\xi(x) := e^{ix\cdot\xi}.$$

Compactly Supported ΨDOs

Proposition

For $j = 1, 2$ *let* $P_j \in \Psi_K^{m_j}(U)$ *have principal symbol* $\sigma_{m_1}(x, \xi)$.

1. $P_1 P_2 \in \Psi_K^{m_1+m_2}(U)$.
2. *Its principal symbol is* $\sigma_{m_1}(x, \xi)\sigma_{m_2}(x, \xi)$.

Proposition (Calderon-Vaillancourt)

If $P \in \Psi_K^m(U)$, $m \leq 0$, *then* P *uniquely extends to a continuous linear operator,*

$$P : L^2(U) \longrightarrow L^2(U).$$

Weak Schatten Class Properties

Fact

As explained during lecture and in the handwritten notes, the singular values properties of ΨDOs on $\mathbb{T}^n$ extends to compactly supported ΨDOs on U.

In particular, we have:

Proposition

Every $P \in \Psi_K^{-m}(U)$, $m > 0$, is in the weak Schatten class $\mathcal{L}^{\frac{n}{m},\infty}$.

Reminder

Let $P = \sigma(x, D)$ with $\sigma \in S^m(U \times \mathbb{R}^n)$, $m < -n$. Then:

- P has a Schwartz kernel $k_P(x, y) \in C(U \times U)$, i.e.,
$$Pu(x) = \int_U k_P(x, y)u(y)dy, \qquad u \in C_c^\infty(U).$$
- Namely,
$$k_P(x, y) = \sigma_{\xi \to y}(x, x - y) = \int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)}\sigma(x, \xi)đ\xi.$$
- In particular,
$$k_P(x, x) = \int_{\mathbb{R}^n} \sigma(x, \xi)đ\xi.$$
- If in addition $P \in \Psi_K^m(U)$, then
$$k_P(x, y) \in C_{K\times K}(U \times U)$$

Trace Formula

Proposition (Trace Formula)

Let $P \in \Psi_K^m(U)$, $m < -n$. *Then:*

1. P *is trace-class.*
2. *If* $k_P(x, y)$ *is the Schwartz kernel of* P, *then*

$$\mathrm{Tr}[P] = \int_U k_P(x, x)dx.$$

Remark

If $P = \sigma(x, D)$, then

$$k_P(x, x) = \int_{\mathbb{R}^n} \sigma(x, \xi) đ\xi.$$

Thus,

$$\mathrm{Tr}[P] = \iint_{U \times \mathbb{R}^n} \sigma(x, \xi) dx đ\xi.$$