Introduction to Noncommutative Geometry Chapter 9: Quantized Calculus on Manifolds. Noncommutative Residue, and Lower Dimensional Volumes

Sichuan University, Summer 2025

Main References

- Slides of my lectures on the noncommutative residue for my 2022 online course on NCG.
- Ponge, R.: *Noncommutative geometry and lower dimensional volumes in Riemannian geometry*. Lett. Math. Phys. **83** (2008), 19–32.

Schwartz Kernels of ΨDOs

Theorem (Schwartz's Kernel Theorem)

Let $P : C_c^\infty(U) \to \mathcal{D}'(U)$ be a linear operator, where $U \subseteq \mathbb{R}^n$ is an open set. TFAE:

(i) *P is continuous.*

(ii) *There is $k_P(x, y) \in \mathcal{D}'(U \times V)$ such that*

$$\langle Pu, v\rangle = \langle k_P(x, y), u(x)v(y)\rangle \qquad \forall u, v \in C_c^\infty(U).$$

Remark

If P is continuous, then $k_P(x, y)$ is unique and is called the Schwartz kernel of P.

Remark

The above result continues to hold if U is a manifold.

Schwartz Kernels of ΨDOs

Lemma

Let $\sigma(x,\xi) \in S^m(U \times \mathbb{R}^n)$, $m \in \mathbb{R}$, *and set* $P = \sigma(x, D)$. *Then the the Schwartz kernel of* P *is*

$$k_P(x,y) = \check{\sigma}_{\xi\to y}(x, x-y).$$

Namely,

$$\langle k_P(x,y), u(x)v(y)\rangle = \langle \check{\sigma}_{\xi\to y}(x,y), u(x)v(x-y)\rangle .$$

Remark

- If $m < -n$, then

$$\check{\sigma}_{\xi \to y}(x,y) = \int_{\mathbb{R}^n} e^{ix\cdot y} \sigma(x,\xi) đ\xi \in C(U \times \mathbb{R}^n).$$

- In particular, on the diagonal $x = y$ we have

$$k_P(x,x) = \check{\sigma}_{\xi \to y}(x,0) = \int_{\mathbb{R}^n} \sigma(x,\xi) đ\xi \in C^\infty(U).$$

- In general, $\check{\sigma}_{\xi \to y}(x,y)$ makes sense as a distribution,

$$\langle \check{\sigma}_{\xi \to y}(x,y), v(y) \rangle := \langle \sigma(x,\xi), \check{v}(\xi) \rangle = \int_{\mathbb{R}^n} \sigma(x,\xi) \check{v}(\xi) d\xi.$$

Schwartz Kernels of ΨDOs

Lemma

Let $\sigma(x,\xi) \in S^m(U \times \mathbb{R}^n)$, $m \in \mathbb{R}$. *Then* $\check{\sigma}_{\xi\to y}(x,y)$ *is* C^∞ *on* $U \times (\mathbb{R}^n \backslash 0)$.

Notation

$\Gamma = \{(x,x);\ x \in U\}$ (diagonal of $U \times U$).

Proposition

Let $P \in \Psi^m(U)$, $m \in \mathbb{C}$, *have Schwartz kernel* $k_P(x,y)$.

1. $k_P(x,y)$ *is* C^∞ *on* $(U \times U)\backslash\Gamma$.
2. *If* $m < -n$, *then*

$$k_P(x,y) \in C(U \times U) \quad \text{and} \quad k_P(x,x) \in C^\infty(U).$$

Schwartz Kernels of ΨDOs

Setup

- M = smooth manifold of dimension n.
- $\Gamma = \{(x,x);\ x \in M\}$.

Proposition

Let $P \in \Psi^m(M)$, $m \in \mathbb{R}$, have Schwartz kernel $k_P(x,x)$.

1. *$k_P(x,y)$ is C^∞ on $(M \times M) \setminus \Gamma$.*
2. *If $m < -n$, then $k_P(x,y)$ is continuous on $M \times M$, and $k_P(x,x)$ is a smooth density on M.*

Remark

Let $\phi : M' \rightarrow M$ be a diffeomorphism.

- We have
$$k_{\phi^* P}(x,y) = |\phi'(y)| k_P\left(\phi(x), \phi(y)\right), \qquad x, y \in M'.$$
- In particular, on the diagonal $y = x$ we get
$$k_{\phi^* P}(x,x) = |\phi'(x)| k_P\left(\phi(x), \phi(x)\right), \qquad x \in M'.$$

  This is the transformation law for densities.

Trace Formula

Proposition

Assume M is compact. Every operator $P \in \Psi^m(M)$, $m < -n$, is trace-class, and

$$\mathrm{Tr}[P] = \int_M k_P(x,x).$$

Setup

- $U \subseteq \mathbb{R}^n$ open set.
- $P \in \Psi^m(U)$, $m \in \mathbb{Z}$, with symbol $\sigma(x,\xi) \sim \sum \sigma_{m-j}(x,\xi)$.
- $k_P(x,y) =$ Schwartz kernel of P.

Reminder

1. $k_P(x,y)$ is C^∞ for $x \neq y$.
2. If $m < -n$, then $k_P(x,y)$ is continuous on $U \times U$.

Question

If $m \geq -n$, then what is the behaviour of $k_P(x,y)$ near $x = y$?

Logarithmic Singularity of ΨDO Kernels

Proposition

Let $P \in \Psi^m(U)$, $m \in \mathbb{Z}$, $m \geq -n$. *Near* $x = y$ *its Schwartz kernel* $k_P(x,y)$ *has a behaviour of the form,*

$$k_P(x,y) = \sum_{1 \leq k \leq m+n} a_k\Big(x, \frac{x-y}{|x-y|}\Big)|x-y|^{-k} - c_P(x)\log|x-y| + O(1).$$

Here $a_k(x,\theta) \in C^\infty(U \times \mathbb{S}^{n-1})$, *and* $c_P(x) \in C^\infty(U)$ *is given by*

$$c_P(x) = (2\pi)^{-n} \int_{\mathbb{S}^{n-1}} \sigma_{-n}(x,\xi)d\xi,$$

where $\sigma_{-n}(x,\xi)$ *is the symbol of degree* $-n$ *of* P.

Remark

$a_k(x,\theta)$ only depends on the homogeneous symbol $\sigma_{m-k}(x,\xi)$ of degree $m-k$ of P.

Setup

- $U_1 \subseteq \mathbb{R}^n$ open set.
- $\phi : U_1 \to U$ is a C^∞-diffeomorphism.

Fact

$\phi^* P := (\phi^{-1})_* \in \Psi^m(U_1)$.

Proposition

We have

$$c_{\phi^* P}(x) = |\phi'(x)| c_P\left(\phi(x)\right) \qquad \forall x \in U_1.$$

Remark

In other words the logarithmic singularities $c_P(x)$ satisfy the transformation law of densities.

Setup

M is a smooth manifold of dimension n.

Reminder

If $P \in \Psi^m(M)$, then, $\kappa_*(P_{|U}) \in \Psi^m(V)$ for every chart $\kappa : U \rightarrow V$.

Proposition

Let $P \in \Psi^m(M)$, $m \in \mathbb{Z}$, $m \geq -n$. There is a unique smooth density $c_P(x)$ on M such that, for every chart $\kappa : U \rightarrow V$, we have

$$(c_P)_\kappa(x) = c_{\kappa_*(P_{|U})}(x) = (2\pi)^{-n} \int_{\mathbb{S}^{n-1}} \sigma^\kappa_{-n}(x,\xi) d\xi,$$

where $\sigma^\kappa_{-n}(x,\xi)$ is the symbol of degree $-n$ of $\kappa_*(P_{|U})$.

Remark

If P is a differential operator on M, then $c_P(x) = 0$ everywhere, since P has no symbols of negative degree.

Remark

- In case $P \in \Psi^{-n}(M)$, i.e., P has order $m = -n$, then its symbol of degree $-n$ agrees with its principal symbol.
- It thus makes sense as a function $\sigma_{-n}(x, \xi) \in C^\infty(T^*M \setminus 0)$.
- It then can be shown that, if $u \in C_c(M)$, then

$$\int_M u(x)c_P(x) = (2\pi)^{-n} \iint_{S^*M} f(x)\sigma_{-n}(x, \xi)dxd\xi,$$

where $S^*M = (T^*M \setminus 0)/\mathbb{R}_+^*$ is the cosphere bundle and $dxd\xi$ is its Liouville measure.

Noncommutative Residue

Setup

M is a compact manifold of dimension n.

Definition (Noncommutative Residue)

If $P \in \Psi^m(M)$, $m \in \mathbb{Z}$, then its noncommutative residue is

$$\operatorname{Res}(P) := \int_M c_P(x).$$

Remarks

1. $\operatorname{Res}(P) = 0$ if P is a differential operator or if P has order $< -n$.
2. If P has order $-n$, then

$$\operatorname{Res}(P) = (2\pi)^{-n} \iint_{S^*M} \sigma_{-n}(x,\xi)\, dx d\xi,$$

where $\sigma_{-n}(x,\xi)$ is the principal symbol of P.

Noncommutative Residue

Notation

$\Psi^{\mathbb{Z}}(M) := \bigcup_{m\in\mathbb{Z}} \Psi^m(M)$.

Remark

As M is compact, $\Psi^{\mathbb{Z}}(M)$ is an algebra:

- If $P_1 \in \Psi^{m_1}(M)$ and $P_2 \in \Psi^{m_2}(M)$ with $m_1, m_2 \in \mathbb{Z}$, then $m_1 + m_2 \in \mathbb{Z}$, and so
$$P_1 P_2 \in \Psi^{m_1+m_2}(M) \subseteq \Psi^{\mathbb{Z}}(M).$$
- If $m_2 \geq m_1$, then $\Psi^{m_1}(M) \subseteq \Psi^{m_2}(M)$, and so we have
$$P_1 + P_2 \in \Psi^{m_1}(M) + \Psi^{m_2}(M) \subseteq \Psi^{m_2}(M) \subseteq \Psi^{\mathbb{Z}}(M).$$

Remark

We then can regard the noncommutative residue as a linear functional $\mathrm{Res} : \Psi^{\mathbb{Z}}(M) \to \mathbb{C}$.

Proposition

1. *If* $P_1 \in \Psi^{m_1}(M)$ *and* $P_2 \in \Psi^{m_2}(M)$ *are such that* $m_1 + m_2 \in \mathbb{Z}$*, then*

$$\operatorname{Res}(P_1 P_2) = \operatorname{Res}(P_2 P_1).$$

2. *In particular, the noncommutative residue* $\operatorname{Res} : \Psi^{\mathbb{Z}}(M) \to \mathbb{C}$ *is a linear trace on the algebra* $\Psi^{\mathbb{Z}}(M)$*.*

Theorem (Wodzicki, Guillemin)

Every linear trace on $\Psi^{\mathbb{Z}}(M)$ *is a constant multiple of the noncommutative residue.*

Example

Δ_g = Laplacian on M associated with some Riemannian metric g.

- The principal symbol of Δ_g is $\sigma_2(x,\xi) = |\xi|_g^2$, where $|\xi|_g^2 = \sum g^{ij}\xi_i\xi_j$ is the Riemannian metric on T^*M.
- $\Delta_g^{-n/2}$ is in $\Psi^{-n}(M)$, and its principal symbol is $\sigma_{-n}(x,\xi) = |\xi|_g^{-n}$.
- It then can be shown that
$$c_{\Delta_g^{-n/2}}(x) = (2\pi)^{-n}\int_{S_x^*M} |\xi|_g^{-n} d\xi = (2\pi)^{-n}|\mathbb{S}^{n-1}|\nu(g)(x),$$
where $\nu(g)$ is the Riemannian density.
- Thus, if we set $c(n) := (2\pi)^{-n}|\mathbb{B}^{n-1}| = n^{-1}(2\pi)^{-n}|\mathbb{S}^{n-1}|$, then
$$\frac{1}{n}\operatorname{Res}\left(\Delta_g^{-\frac{n}{2}}\right) = c(n)\int_M \nu(g) = c(n)\operatorname{Vol}_g(M).$$

Meromorphic Extension of the Trace

Setup

- M = compact manifold of dimension n.
- μ = positive smooth measure on M.
- $P \in \Psi^m(M)$, $m > 0$, is positive-elliptic.
- $\sigma_m(x,\xi)$ = principal symbol of P.

Remark

The positive-elliptic assumption means that

- $\sigma_m(x,\xi) > 0$ for all $(x,\xi) \in T^*M \setminus 0$.
- $P^* = P$ and $\mathrm{Sp}(P) \subseteq [0,\infty)$.

Meromorphic Extension of the Trace

Reminder

- Each operator P^{-z}, $z \in \mathbb{C}$, is in $\Psi^{-mz}(M)$.
- Its principal symbol is $\sigma_m m(x,\xi)^{-z}$.

Facts

Let $A \in \Psi^{a}(M)$, $a \in \mathbb{R}$.

- The operator AP^{-z} is in $\Psi^{a-mz}(M)$.
- In particular AP^{-z} is trace-class for $a - m\Re z > -n$, i.e., $\Re z > m^{-1}(n+a)$.
- Thus, $\mathrm{Tr}[AP^{-z}]$ is well defined for $\Re z > m^{-1}(n+a)$.

Meromorphic Extension of the Trace

Proposition (Wodzkicki, Guillemin)

Let $A \in \Psi^a(M)$, $a \in \mathbb{R}$, *and set* $\Sigma := \{m^{-1}(n+a-j);\ j \in \mathbb{N}_0\}$.

1. *The function* $z \to \operatorname{Tr}[AP^{-z}]$ *has a meromorphic extension to* $\mathbb{C}$ *with at worst simple pole singularities on* Σ.
2. *If* $\sigma \in \Sigma$, *then*
$$m \operatorname{Res}_{z=\sigma} \operatorname{Tr}\left[AP^{-z}\right] = \operatorname{Res}\left[AP^{-\sigma}\right].$$
3. *In particular, if* $a \in \mathbb{Z}$, $a \geq -n$, *then for* $\sigma = 0$ *we get*
$$m \operatorname{Res}_{z=0} \operatorname{Tr}\left[AP^{-z}\right] = \operatorname{Res}(A).$$

Remark

$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.

Schatten Properties of ΨDOs

Setup

- M = compact manifold with positive smooth density μ.
- $P \in \Psi^m(M)$, $m > 0$, is positive-elliptic.
- $\sigma_m(x, \xi)$ = principal symbol of P.

Reminder

The positive-ellipticity of P ensures the following:

1. Its spectrum of P can be arranged as a sequence,

$$0 \leq \lambda_0(P) \leq \lambda_1(P) \leq \lambda_2(P) \leq \cdots,$$

   where each eigenvalue is repeated according to multiplicity.

2. For every $z \in \mathbb{C}$, the power P^z is in $\Psi^{mz}(M)$ and has $\sigma_m(x, \xi)^z$ as principal symbol.

Schatten Properties of ΨDOs

Reminder (Weyl's Law)

- By Weyl's law, as $j \to \infty$ we have

$$\lambda_j(P) \sim \left(\frac{j}{\gamma(P)}\right)^{\frac{m}{n}},$$

where
$$\gamma(P) = \frac{(2\pi)^{-n}}{n} \iint_{S^*M} \sigma_m(x,\xi)^{-\frac{n}{m}} dx d\xi.$$

- Note that $P^{-m/m}$ is an operator in $\Psi^{-n}(M)$ whose principal symbol is $\sigma_m(x,\xi)^{-n/m}$.
- Thus,

$$\operatorname{Res}\left[P^{-\frac{n}{m}}\right] = (2\pi)^{-n} \iint_{S^*M} \sigma_m(x,\xi)^{-\frac{n}{m}} dx d\xi.$$

- It then follows that

$$\gamma(P) = \frac{1}{n} \operatorname{Res}\left[P^{-\frac{n}{m}}\right].$$

Schatten Properties of ΨDOs

Facts

Assume that $\ker P = \{0\}$.

- We then have
$$\lambda_j(P^{-1}) = \lambda_j(P)^{-1} \sim \left(\frac{j}{\gamma(P)}\right)^{-\frac{m}{n}}.$$
- Thus,
$$\mu_j(P^{-1}) = \lambda_j(P^{-1}) = \operatorname{O}\left(j^{-\frac{m}{n}}\right).$$
That is, $P^{-1} \in \mathcal{L}^{m^{-1}n,\infty}$.
- In particular, if g is a Riemannian metric, then
$$(1+\Delta_g)^{-\frac{m}{2}} \in \mathcal{L}^{m^{-1}n,\infty} \qquad \forall m > 0.$$

Proposition

Every $P \in \Psi^{-m}(M)$, $m > 0$, is in the weak Schatten class $\mathcal{L}^{nm^{-1},\infty}$

In particular, for $m = n$ we obtain:

Corollary

Every $P \in \Psi^{-n}(M)$ is in the weak trace-class $\mathcal{L}^{1,\infty}$.

Proof of the Proposition.

Let $m>0$ and $P\in\Psi^{-m}(M)$.

- Pick a Riemannian metric g on M, and write
$$P=P\left(1+\Delta_g\right)^{\frac{m}{2}}\cdot\left(1+\Delta_g\right)^{-\frac{m}{2}}$$
- We already know that $(1+\Delta_g)^{-m/2}\in\mathcal{L}^{nm^{-1},\infty}$.
- As $P\in\Psi^{-m}(M)$ and $(1+\Delta_g)^{m/2}\in\Psi^{m}(M)$, we see that $P(1+\Delta_g)^{m/2}\in\Psi^{0}(M)$.
- This ensures that $P(1+\Delta_g)^{m/2}$ is bounded, i.e., it is contained in $\mathcal{L}(L^2(M))$.
- As $\mathcal{L}^{nm^{-1},\infty}$ is an ideal of $\mathcal{L}(L^2(M))$, it follows that
$$P=P\left(1+\Delta_g\right)^{\frac{m}{2}}\cdot\left(1+\Delta_g\right)^{-\frac{m}{2}}\in\mathcal{L}^{nm^{-1},\infty}.$$

The proof is complete. □

Facts

Let $P \in \Psi^n(M)$ be positive-elliptic with $\ker P = \{0\}$ (e.g., $P = (1 + \Delta_g)^{n/2}$).

- In this case the Weyl's law gives

$$\lambda_j(P^{-1}) = \lambda_j(P)^{-1} \sim \left(\frac{j}{\gamma(P)}\right)^{-1},$$

  where $\gamma(P) = \frac{1}{n}\operatorname{Res}(P^{-1})$.

- It then follows that P^{-1} is strongly measurable, and we have

$$⨍ P^{-1} = \gamma(P) = \frac{1}{n}\operatorname{Res}\left(P^{-1}\right).$$

Connes' Trace Theorem

More generally, we have:

Theorem (Connes' Trace Theorem)

If $P \in \Psi^{-n}(M)$, then P is strongly measurable, and we have

$$⨍ P = \frac{1}{n} \operatorname{Res}(P).$$

Remark

1. Connes (CMP '88) established measurability and derived the trace formula.
2. Lord-Potapov-Sukochev (Adv. Math. '13) obtained strong measurability.

Proof.

Step 1: $P \in \Psi^{-(n+1)}(M)$.

- In this case $\mathrm{Res}(P) = 0$.
- Moreover, P is in the weak Schatten class $\mathcal{L}^{p,\infty}$, with $p = n(n+1)^{-1} < 1$.
- Thus, P is in $\mathcal{L}^1 \subseteq \mathcal{L}_0^{1,\infty}$.
- It follows that P is strongly measurable, and we have

$$\int\!\!\!\!\!- P = 0 = \frac{1}{n}\mathrm{Res}(P).$$

□

Proof.

Step 2: $P = Q^{-1}$, where $Q \in \Psi^n(M)$ is positive-elliptic.

- In this case we saw that P is strongly measurable, and we have
$$⨍ P = \frac{1}{n} \int_{S^*M} \sigma_n(Q)(x,\xi)^{-1} dx đ\xi.$$
- Here $\sigma_n(Q)(x,\xi)^{-1} = \sigma_{-n}(Q^{-1})(x,\xi) = \sigma_{-n}(P)(x,\xi)$.
- Thus,
$$⨍ P = \frac{1}{n} \int_{S^*M} \sigma_{-n}(P)(x,\xi) dx đ\xi = \frac{1}{n} \operatorname{Res}(P).$$

□

Proof.

Step 3: $\sigma_{-n}(P)(x,\xi) > 0$.

- Let $Q \in \Psi^n(M)$ be pos.-ellipt. w/ $\sigma_n(Q) = \sigma_{-n}(P)(x,\xi)^{-1}$.
- Here Q^{-1} is an operator in $\Psi^{-n}(M)$ with

$$\sigma_{-n}\left(Q^{-1}\right)(x,\xi) = \sigma_n(Q)(x,\xi)^{-1} = \sigma_{-n}(P)(x,\xi).$$

- This ensures that $R := P - Q^{-1}$ is in $\Psi^{-(n+1)}(M)$.
- In particular $\operatorname{Res}(P) = \operatorname{Res}(Q^{-1})$.
- By Step 2 Q^{-1} is strongly meas. and $⨍ Q^{-1} = \frac{1}{n}\operatorname{Res}(Q^{-1})$.
- As $R \in \Psi^{-(n+1)}(M)$, by Step 1 it is strongly measurable and $⨍ R = 0$.
- It follows that $P = Q^{-1} + R$ is strongly measurable, and

$$⨍ P = ⨍ Q^{-1} + ⨍ R = \frac{1}{n}\operatorname{Res}\left(Q^{-1}\right) = \frac{1}{n}\operatorname{Res}(P).$$

□

Proof.

Step 4: $P^* = P$.

- Here $\sigma_{-n}(P)(x, \xi)$ is Hermitian for all $(x, \xi) \in T^*M \setminus 0$.
- Bearing in mind that S^*M is compact, set

$$c := \sup \{|\sigma_{-n}(P)(x, \xi); (x, \xi) \in S^*M\} < \infty.$$

- For $(x, \xi) \in T^*M \setminus 0$, we then have

$$\begin{aligned} \sigma_{-n}(P)(x, \xi) &\leq |\sigma_{-n}(P)(x, \xi)| \\ &\leq |\xi|_g^{-n} |\sigma_{-n}(P)(x, |\xi|_g^{-n}\xi)| \\ &\leq c|\xi|_g^{-n}. \end{aligned}$$

□

Proof.

- Let $P_1 \in \Psi^{-n}(M)$ have principal symbol

$$\sigma_{-n}(P_1)(x,\xi) = (c+\epsilon)|\xi|_g \operatorname{id}_{\mathcal{E}_x}, \quad \epsilon > 0.$$

- Set $P_2 = P_1 - P$. Then $P_2 \in \Psi^{-n}(M)$, and

$$\begin{aligned}\sigma_{-n}(P_2)(x,\xi) &= \sigma_{-n}(P_1)(x,\xi) - \sigma_{-n}(P)(x,\xi) \\ &\geq (c+\epsilon)|\xi|_g^{-n} - c|\xi|_g^{-n} \\ &\geq \epsilon|\xi|_g^{-n} > 0.\end{aligned}$$

- As $\sigma_{-n}(P_j)(x,\xi)$, Step 3 ensures that each P_j is strongly measurable and $\fint P_j = \frac{1}{n}\operatorname{Res}(P_j)$.
- It then follows that $P = P_1 - P_2$ is strongly measurable, and

$$\fint P = \fint P_1 - \fint P_2 = \frac{1}{n}\operatorname{Res}(P_1) - \frac{1}{n}\operatorname{Res}(P_2) = \frac{1}{n}\operatorname{Res}(P).$$

□

Proof.

Step 5: General case $P \in \Psi^{-n}(M)$.

- Put $P = \Re P + i\Im P$, with

$$P_1 = \Re P = \frac{1}{2}\left(P + P^*\right), \qquad P_2 = \Im P = \frac{1}{2i}\left(P - P^*\right).$$

- Here P_1 and P_2 are selfadjoint operators in $\Psi^{-n}(M)$.
- By Step 4 each operator P_j is is strongly measurable and $⨍ P_j = \frac{1}{n}\operatorname{Res}(P_j)$.
- It then follows that $P = P_1 + iP_2$ is strongly measurable, and

$$⨍ P = ⨍ P_1 + i ⨍ P_2 = \frac{1}{n}\operatorname{Res}(P_1) + \frac{i}{n}\operatorname{Res}(P_2) = \frac{1}{n}\operatorname{Res}(P).$$

□

Connes' Trace Theorem

Consequence

- The NC integral makes sense for ΨDOs of order $\leq -n$
- The NC residue, however, makes sense for all integer order ΨDOs.
- Therefore, the equality between the NC integral and the NC residue enables us to extend the NC integral to all ΨDOs.
- This includes ΨDOs that are not infinitesimals or even bounded!

Definition

For any $P \in \Psi^{\mathbb{Z}}(M)$ we set

$$⨍ P := \frac{1}{n}\operatorname{Res}(P) = \frac{1}{n}\int_M c_P(x).$$

Connes' Integration Formula

Setup

- $(M^n, g) =$ compact Riemannian manifold.
- $\nu(g) =$ Riemannian density.
- $\Delta_g =$ Laplacian associated with g.

Theorem (Connes' Integration Formula)

For every $f \in C^\infty(M)$, the operator $f\Delta_g^{-n/2}$ is strongly measurable, and we have

$$⨍ f\Delta_g^{-\frac{n}{2}} = c(n) \int_M f(x)\nu(g)(x),$$

where we have set $c(n) := (2\pi)^{-n}|\mathbb{B}^n|$.

Proof.

- $\Delta_g^{-n/2}$ is ΨDOs of order $-n$ whose principal symbol is $|\xi|_g^{-n}$.
- As $f \in C^\infty(M)$, this is a ΨDO of order 0.
- Thus $f\Delta_g^{-n/2}$ is in $\Psi^{-n}(M)$ and it has $f(x)|\xi|_g^{-n}$ as principal symbol.
- Thus, by Connes' trace theorem $f\Delta_g^{-n/2}$ is strongly measurable, and we have

$$\begin{aligned} ⨍ f\Delta_g^{-\frac{n}{2}} = \frac{1}{n}\operatorname{Res}\left(f\Delta_g^{-\frac{n}{2}}\right) &= \frac{(2\pi)^{-n}}{n}\iint_{S^*M} f(x)|\xi|_g^{-n} dx d\xi \\ &= \frac{1}{n}(2\pi)^{-n}|\mathbb{S}^{n-1}|\int_M f(x)\nu(g)(x). \end{aligned}$$

This proves the result. □

Connes' Integration Formula

Remark

- The integration formula actually holds for all $f \in C(M)$.
- It can rewritten as

$$⨍ \Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} = c(n) \int_M f(x)\nu(g)(x) \qquad \forall f \in C(M).$$

Definition (Zygmund)

$L\log L(M)$ consists of measurable functions $f : M \to \mathbb{C}$ such that

$$\int_M |f(x)| \log(1 + |f(x)|)\nu(g)(x) < \infty.$$

Proposition

$L\log L(M)$ *is a Banach space with respect to the norm,*

$$\|f\|_{L\log L} := \inf\left\{\lambda > 0; \int_M |\lambda^{-1} f(x)| \log(1+\lambda^{-1}|f(x)|)\nu(g)(x) < 1\right\}.$$

Connes' Integration Formula

Theorem (Solomyak, Sukochev-Zanin)

1. *If* $f \in L\log L(M)$, *then* $(1+\Delta)^{-n/4} f (1+\Delta)^{-n/4} \in \mathcal{L}^{1,\infty}$.
2. *There is* $C > 0$ *such that*

$$\left\|(1+\Delta)^{-\frac{n}{4}} f (1+\Delta)^{-\frac{n}{4}}\right\|_{1,\infty} \leq C\|f\|_{L\log L} \qquad \forall f \in L\log L(M).$$

Connes' Integration Formula

Theorem

If $f \in L\log L(M)$*, then* $(1+\Delta)^{-n/4} f (1+\Delta)^{-n/4}$ *is strongly measurable, and*

$$⨍ (1+\Delta)^{-\frac{n}{4}} f (1+\Delta)^{-\frac{n}{4}} = c(n) \int_M f(x)\nu(g)(x).$$

Consequence

- Connes' integration formula shows that the NC integral recaptures the Riemannian volume density. Namely,

$$\int_M f(x)\nu(g)(x) = c(n)^{-1} ⨍ f\Delta_g^{-\frac{n}{2}} \qquad \forall f \in C(M).$$

- We regard $c(n)^{-1}\Delta_g^{-n/2}$ as the NC volume element.
- The volume element is ds^n, where ds is the length element.
- Thus, ds is the n-th root of the volume element.

Lower Dimensional Volumes

Definition

The NC length element of (M^n, g) is the operator,

$$ds := \left(c(n)^{-1}\Delta_g^{-\frac{n}{2}}\right)^{\frac{1}{n}} = c(n)^{-\frac{1}{n}}\Delta_g^{-\frac{1}{2}}.$$

Remark

ds is a ΨDO of order -1.

Lower Dimensional Volumes

Facts

- For $k = 1, \ldots, n$ the k-th dimensional volume is meant to be the integral of ds^k.
- Here ds^k is a ΨDO of order $-k$.
- The NC integral has been extended to all ΨDOs.
- This enables us to define k-dimensional volumes for all $k = 1, \ldots, n-1$.

Lower Dimensional Volumes

Definition

For $k = 1, \dots, n$, the k-th dimensional volume of (M^n, g) is

$$\mathrm{Vol}_g^{(k)}(M) := \fint ds^k = c(n)^{-\frac{k}{n}} \fint \Delta_g^{-\frac{k}{2}}.$$

In particular, the length and area of (M^n, g) are

$$\mathrm{Length}_g(M) := \fint ds = c(n)^{-\frac{1}{n}} \fint \Delta_g^{-\frac{1}{2}},$$

$$\mathrm{Area}_g(M) := \fint ds^2 = c(n)^{-\frac{2}{n}} \fint \Delta_g^{-1}.$$

Lower Dimensional Volumes

Proposition

1. *If* k *and* n *have opposite parities (i.e.,* $n-k$ *is odd), then* $\mathrm{Vol}_g^{(k)}(M)=0$.
2. *If* $k=n-2$*, then*
$$\mathrm{Vol}_g^{(n-2)}(M)=c(n,2)\int_M \kappa_g(x)\nu(g)(x),$$
where $\kappa_g(x)$ *is the scalar curvature of* (M,g).
3. *In general, we have*
$$\mathrm{Vol}_g^{(n-k)}(M)=c(n,k)\int_M I_g^{(k)}(x)\nu(g)(x),$$
where $I_g^{(k)}(x)$ *is a universal polynomial in the curvature tensor and its covariant derivatives.*

Lower Dimensional Volumes

Remark

- The definition of the k-th dimensional volumes involved noncommutative geometry.
- However, the formulas in the previous slide provide purely differential-geometric expressions for the k-th dimensional volumes.

Remark

- The functional $g \to \int_M \kappa_g \nu(g)$ is called the Einstein-Hilbert action.
- It accounts for the contribution of gravity forces in the Standard Model from theoretical physics.
- We have
$$\begin{aligned}\int_M \kappa_g \nu(g) &= c(n,2)^{-1} ⨍ \Delta_g^{-n+2} \\ &= \frac{1}{n} c(n,2)^{-1} \operatorname{Res}\left(\Delta_g^{-n+2}\right) \\ &= \frac{2}{n} c(n,2)^{-1} \operatorname{Res}_{s=\frac{n}{2}-1} \operatorname{Tr}\left[\Delta_g^{-s}\right].\end{aligned}$$
- This yields a spectral theoretic interpretation of the Einstein-Hilbert action.
- This an important ingredient in the spectral action formalism of Connes-Chamseddine-Marcolli.