Introduction to Noncommutative Geometry
Chapter 8:
Pseudodifferential Operators on Manifolds

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Pseudodifferential Operators

Additional References

- Taylor, M.E.: *Pseudodifferential operators*. Princeton University Press, Princeton, NJ, 1981.
- Slides of my 2022 online course.

Smooth Measures/Densities

Definition

A smooth measure on M is a Radon measure μ such that, for every chart $\kappa : U \to V$, there is $\mu_\kappa(x) \in C^\infty(V)$, such thatt

$$\int_M f(x) d\mu(x) = \int_V f \circ \kappa^{-1}(x) \mu_\kappa(x) dx \qquad \forall f \in C_c(U).$$

That is, $\kappa_*(\mu_{|U}) = \mu_\kappa(x) dx$.

Example

Assume M is oriented and $\omega \in \Omega^n(M)$ is an orientation form. Then $f \to \int_M f\omega$ is a smooth measure.

Smooth Measures/Densities

Fact

Let μ be a smooth measure. If $\kappa : U \to V$ and $\kappa_1 : U \to V_1$ are two charts and $\phi := \kappa_1 \circ \kappa^{-1} : V \to V_1$ is the transition map, then

$$\mu_\kappa(x) = \det(\phi'(x))\mu_{\kappa_1}(\phi(x)) \qquad \forall x \in V.$$

Definition

A smooth density on M is a data $\rho = (\rho_\kappa)$, parametrized by charts $\kappa : U \to V$, for functions $\rho_\kappa \in C^\infty(V)$ such that

$$\rho_\kappa(x) = \det(\phi'(x))\rho_{\kappa_1}(\phi(x)), \qquad \phi := \kappa_1 \circ \kappa^{-1}.$$

Remark

An intrinsic definition of a smooth density is as a smooth section of some line bundle, called density bundle $|\Lambda|(M)|$ (see 2022 slides).

Smooth Measures/Densities

Fact

Every smooth density ρ defines a smooth measure:

- Let (φ_i) be a partition of unity subordinate to an open cover (U_i) by domains of charts $\kappa_i : U_i \to V_i$.
- For $f \in C_c(M)$, set

$$\int_M f\rho := \sum_i \int_{V_i} (\varphi_i f) \circ \kappa_i^{-1}(x)dx.$$

- This does not depend on the choice of the partition of unity.
- This yields a smooth measure $f \to \int_M f\rho$.

Consequence

We have a one-to-one correspondance,

$$\{\text{smooth measures}\} \longleftrightarrow \{\text{smooth densities}\}.$$

Example (Riemannian Density)

Let g be a Riemannian metric on M.

- If $\kappa : U \to V$ is a chart, then in local coordinates $g = g^{\kappa}_{ij}(x)dx^i \otimes dx^j$ on U.
- There is a unique smooth density $\nu(g)$ such that
$$\nu(g)_\kappa(x) = \sqrt{\det(g_{ij}(\kappa^{-1}(x)))}, \qquad x \in V.$$
- This density is called the Riemannian density.
- The corresponding measure is called the Riemannian measure.

Remark

If M is oriented, then $\nu(g)$ agrees with the measure defined by the volume form.

Smooth Measures/Densities

Definition (Riemannian Volume)

The volume of (M, g) is

$$\mathrm{Vol}_g(M) := \int_M \nu(g),$$

where $\nu(g)$ is the Riemannian density.

Remark

If M is oriented and compact, then this agrees with the usual definition of the volume as the integral of the volume form.

Regularity of Distributions

Remarks

1. The previous example shows there always a positive smooth density/measure on M.
2. If μ is a positive smooth density/measure, then every other smooth density/measure is of the form $f\mu$, $f \in C^\infty(M)$.

Facts

Let μ be a (positive) smooth measure/density on M.

- For $f \in C(M)$ and $u \in C_c^\infty(M)$, define

$$\langle f, u \rangle_\mu := \langle f\mu, u \rangle = \int_M u(x) f(x) d\mu(x).$$

- The map $f \to \langle f, \cdot \rangle_\mu$ yields an embedding $C(M) \hookrightarrow \mathcal{D}'(M)$.
- We also get an embedding $C^\infty(M) \hookrightarrow \mathcal{D}'(M)$.

Regularity of Distributions

Remark

This enables to speak about continuity/smoothness for distributions.

- More precisely, a distribution $K \in \mathcal{D}'(M)$ is continuous on M if, given any positive smooth measure μ on M, there is $K_\mu(x) \in C(M)$ such that

$$\langle K, u \rangle = \int_M K_\mu(x) u(x) d\mu(x) \qquad \forall u \in C_c^\infty(M).$$

- We say that K is smooth, if $K_\mu(x) \in C^\infty(M)$.

Remark

Let $V \subset U \subset \mathbb{R}^n$ be open sets and $P : C_c^\infty(U) \to C^\infty(U)$ a linear operator.

- The restriction $P_{|V} : C_c^\infty(V) \to C^\infty(V)$ is defined by
$$P_{|V}u = (Pu)_{|V}, \qquad u \in C_c^\infty(V).$$
- If $P \in \Psi^m(U)$, then $P_{|V} \in \Psi^m(V)$.

Remark

If $P : C_c^\infty(M) \to C^\infty(M)$ is a linear operator and $\kappa : U \to V$ is a chart, then $\kappa_*(P_{|U}) : C_c^\infty(V) \to C^\infty(V)$ is defined by

$$\kappa_*(P_{|U})u = [P(u \circ \kappa)] \circ \kappa^{-1}, \qquad u \in C_c^\infty(V).$$

Definition

$\Psi^m(U)$, $m \in \mathbb{R}$, consists of continuous linear operators $P : C_c^\infty(M) \to C^\infty(M)$ such that

$$\kappa_*(P_{|U}) \in \Psi^m(V) \quad \text{for every chart } \kappa : U \to V.$$

Remark

We can take m to be any complex number in the definitions of $\Psi^m(V)$ and $\Psi^m(M)$.

Remark (Smoothing Operators)

- $\Psi^{-\infty}(M)$ consists of operators $R : C_c^\infty(M) \rightarrow C^\infty(M)$ that are smoothing, i.e., they have a smooth Schwartz kernel.
- This means that, given any smooth measure μ, there is $K_\mu(x,y) \in C^\infty(M \times M)$ such that

$$Ru(x) = \int_M K_\mu(x,y)u(y)d\mu(y), \qquad u \in C_c^\infty(M).$$

Principal Symbol

Proposition

If $P \in \Psi^m(M)$, $m \in \mathbb{R}$, *then there is a unique function* $\sigma_m(x,\xi)$ *in* $C^\infty(T^*M \setminus 0)$ *such that:*

- $\sigma_m(x,\lambda\xi) = \lambda^m \sigma_m(x,\xi)$ *for all* $\lambda > 0$ *and* $(x,\xi) \in T^*M \setminus 0$.
- *For every chart* $\kappa : U \to V$, *we have*

$$\sigma_m(x,\xi) = \sigma_m^\kappa\left(\kappa(x), (\kappa'(x)^{-1})^t \xi\right) \quad \forall (x,\xi) \in T^*U \setminus 0.$$

Definition

$\sigma_m(x,\xi)$ is called the principal symbol of P.

ΨDOs on Manifolds

Assumption

From now on we assume M is compact.

Proposition

Let $P_1 \in \Psi^{m_1}(M)$ and $P_2 \in \Psi^{m_2}(M)$ have respective principal symbols $\sigma_{m_1}(x,\xi)$ and $\sigma_{m_2}(x,\xi)$.

1. *The composition $P_1 P_2$ is an operator in $\Psi^{m_1+m_2}(M)$.*
2. *Its principal symbol is $\sigma_{m_1}(x,\xi)\sigma_{m_2}(x,\xi)$.*

Definition

Let $P \in \Psi^m(M)$ have principal symbol $\sigma_m(x,\xi)$. We say that P is elliptic if

$$\sigma_m(x,\xi) \neq 0 \qquad \forall (x,\xi) \in T^*M \setminus 0.$$

Proposition

Let $P \in \Psi^m(M)$ have principal symbol $\sigma_m(x,\xi)$. TFAE:

(i) *P is elliptic.*

(ii) *It admits a parametrix $Q \in \Psi^{-m}(M)$, i.e.,*

$$PQ = QP = 1 \qquad \text{mod } \Psi^{-\infty}(M).$$

Moreover, if (ii) holds, then the principal symbol of Q is $\sigma_m(x,\xi)^{-1}$.

Adjoints of ΨDOs

Setup

- $\mu =$ smooth measure on M (e.g., Riemannian measure).
- $L^2_\mu(M)$ has inner product,

$$\langle u|v\rangle = \int_M u(x)\overline{v(x)}d\mu(x), \quad u, v \in L^2_\mu(M).$$

Remark

As M is compact the topology of $L^2_\mu(M)$ does not depend on μ (i.e., all the norms are equivalent to each other).

Adjoints of ΨDOs

Definition

A formal adjoint of an operator $P : C_c^\infty(M) \to C^\infty(M)$ is any operator $P^* : C_c^\infty(M) \to C^\infty(M)$ such that

$$\langle Pu|v\rangle = \langle u|P^*v\rangle \qquad \forall u, v \in C^\infty(M).$$

Remark

If a formal adjoint exists, then it is unique.

Proposition

Let $P \in \Psi^m(M)$, $m \in \mathbb{R}$, have principal symbol $p_m(x,\xi)$.

1. *P has a formal adjoint $P^* \in \Psi^m(M)$.*
2. *The principal symbol of P^* is $\overline{p_m(x,\xi)}$.*

Boundedness of ΨDOs

Proposition

Let $P \in \Psi^m(M)$, $m \leq 0$.

1. P *uniquely extends to a bounded operator,*
$$P : L^2_\mu(M) \longrightarrow L^2_\mu(M).$$
2. *This operator is compact if* $m < 0$.

Lemma

Let $P \in \Psi^m(M)$, $m > 0$, be elliptic. TFAE:

1. *P is formally selfadjoint (i.e., it agrees with its formal adjoint).*
2. *P is essentially selfadjoint (i.e., its closure is selfadjoint).*

Remark

In what follows we shall simply say that an elliptic ΨDO is selfadjoint if it is formally selfadjoint.

Spectral Theory of Elliptic ΨDOs

Proposition

Let $P \in \Psi^m(M)$, $m > 0$, *be elliptic and selfadjoint. Then:*

- *Its spectrum consists of an unbounded set of isolated real eigenvalues with finite multiplicity.*
- *Every eigenvector is a function in* $C^\infty(M)$.
- *It admits an orthonormal eigenbasis consisting of* C^∞*-functions.*

Remark

- We say that a principal symbol $\sigma_m(x,\xi)$ is positive (and write $\sigma_m(x,\xi) > 0$) if
$$\sigma_m(x,\xi) > 0 \qquad \forall (x,\xi) \in T^*M \setminus 0.$$
- This implies that P is elliptic.

Proposition

Let $P \in \Psi^m(M)$, $m > 0$, be selfadjoint and have a positive principal symbol.

1. P *is bounded from below, i.e., there is* $c \in \mathbb{R}$ *such that*

$$\langle Pu|u\rangle \geq c\|u\|^2 \qquad \forall u \in C^\infty(M).$$

2. *Its spectrum can be arranged as a(n unbounded) non-decreasing sequence of real eigenvalues,*

$$\lambda_0(P) \leq \lambda_1(P) \leq \lambda_2(P) \leq \cdots,$$

where each eigenvalue is repeated according to multiplicity.

Remark

- We have a natural action of $\mathbb{R}_+^*$ on $(T^*M)\setminus 0$ given by

$$\lambda\cdot(x,\xi)=(x,\lambda\xi), \qquad (x,\xi)\in T^*M,\ \lambda>0.$$

- The cosphere bundle S^*M is the sphere-bundle,

$$S^*M=\left[T^*M\setminus 0\right]/\mathbb{R}_+^*.$$

- The Liouville measure $dxd\xi$ of T^*M descends to S^*M.
- If g is any Riemannian metric on M, then

$$S^*M\simeq S_g^*M:=\left\{\xi\in T^*M;\ |\xi|_g=1\right\},$$

where $|\xi|_g^2=\sum\xi_i g^{ij}\xi_j$ is the Riemannian metric on T^*M.

Weyl's Law for Elliptic ΨDOs

Theorem (Weyl's Law)

Let $P \in \Psi^m(M)$, $m > 0$, *be selfadjoint and have principal symbol* $\sigma_m(x,\xi) > 0$. *As* $j \to \infty$, *we have*

$$\lambda_j(P) \sim j^{\frac{m}{n}} \left(\frac{(2\pi)^{-n}}{n} \iint_{S^*M} \sigma_m(x,\xi)^{-\frac{n}{m}} dx d\xi \right)^{-\frac{m}{n}}.$$

The Laplacian on a Riemannian Manifold

Setup

- $(M^n, g) =$ compact Riemannian manifold.
- $\nu(g) =$ Riemannian density.
- $d : C^\infty(M) \to C^\infty(M, T^*M) =$ de Rham differential.

Notation

- If $g(x) = g_{ij}(x)dx^i \otimes dx^j$ and $g(x)^{-1} = (g^{ij}(x))$, then the Riemannian metric defined by g on T^*M is given by

$$(\xi|\eta)_g = \sum \xi_i g^{ij} \eta_j, \quad \xi = \sum \xi_i dx^i, \ \eta = \sum \eta_i dx^i.$$

- The inner product on 1-forms is then given by

$$\langle \xi|\eta \rangle_g = \int_M (\xi(x)|\eta(x))_g \, d\nu_g(x), \quad \xi, \eta \in C^\infty(M, T^*M).$$

The Laplacian on a Riemannian Manifold

Definition

The Laplacian $\Delta_g : C^\infty(M) \to C^\infty(M)$ is defined by

$$\langle \Delta_g u | u \rangle = \langle du | du \rangle \, , \qquad \forall u \in C^\infty(M).$$

Proposition

In local coordinates, we have

$$\Delta_g u = \frac{-1}{\sqrt{\det(g(x))}} \sum \partial_i \left(g^{ij}(x) \sqrt{\det(g(x))} \partial_j u \right) .$$

Remark

- On T_x^*M the norm defined by g is

$$|\xi|_g = \sqrt{(\xi,\xi)_g} = \sqrt{\sum g^{ij}(x)\xi_i\xi_j}, \qquad \xi = \sum \xi_i dx^i.$$

- We often refer to $|\xi|_g^2 = \sum g^{ij}(x)\xi_i\xi_j$ as the Riemannian metric on T^*M.

From the expression of Δ_g in local coordinates we get:

Proposition

1. Δ_g *is a 2nd order differential operator with principal symbol,*

$$\sigma_2(x,\xi) = \sum \xi_i g^{ij}(x)\xi_j = |\xi|_g^2.$$

2. *In particular, this is an elliptic operator with positive principal symbol.*
3. $\ker \Delta_g = H^0(M,\mathbb{C})$ *(de Rham coholomogy group).*

Reminder

The volume of (M, g) is

$$\mathrm{Vol}_g(M) := \int_M \nu_g .$$

Theorem (Weyl's Law)

As $j \to \infty$, we have

$$\lambda_j(\Delta_g) \sim j^{\frac{2}{n}} \left(c(n)\,\mathrm{Vol}_g(M)\right)^{-\frac{2}{n}}, \quad c(n) := (2\pi)^{-n}|\mathbb{B}^n|.$$

Setup

$M^n =$ compact manifold with a smooth (positive) measure μ.

Definition

If $P \in \Psi^m(M)$, $m > 0$, is called positive-elliptic if

(i) It has a positive principal symbol (and hence P is elliptic).

(ii) P is a selfadjoint and has non-negative spectrum.

Remark

Condition (ii) is equivalent to $\langle Pu|u\rangle \geq 0$ for all $u \in C^\infty(M)$.

Example

If g is a Riemannian metric and $\mu = \nu(g)$, then the Laplacian Δ_g is positive-elliptic.

Setup

- $P \in \Psi^m(M)$, $m > 0$, is positive-elliptic.
- $\sigma_m(x,\xi) =$ principal symbol of P.

Facts

- The spectrum of P can be arranged as a sequence,
$$0 \leq \lambda_0(P) \leq \lambda_1(P) \leq \lambda_2(P) \leq \cdots ,$$
where each eigenvalue is repeated according to multiplicity.
- Each eigenspace $E_\lambda(P) := \ker(P - \lambda)$, $\lambda \in \mathrm{Sp}(P)$, is a finite dimensional subspace of $C^\infty(M)$.
- P admits an orthonormal eigenbasis $(e_j)_{j\geq 0} \subset C^\infty(M)$ with
$$Pe_j = \lambda_j(P)e_j, \qquad j = 0, 1, 2, \ldots .$$

Complex Powers of Elliptic ΨDOs

Definition

- For $z \in \mathbb{C}$, the complex power P^z is defined by using the Borel functional calculus for P for $f(t) = \mathbb{1}_{(0,\infty)} t^z$.
- Equivalently, P^z is the operator on $L^2_\mu(M)$ such that

$$P^z e_j = \begin{cases} \lambda_j(P)^z e_j & \text{if } \lambda_j(P) > 0, \\ 0 & \text{if } \lambda_j(P) = 0. \end{cases}$$

Facts

- For $\Re z \leq 0$ this a bounded operator.
- For $\Re z > 0$ this is a selfadjoint unbounded operator.
- We have

$$P^{z_1} P^{z_2} = P^{z_1 + z_2}, \qquad P^z|_{z=0} = 1 - \Pi_0(P).$$

  where $\Pi_0(P)$ is the orthogonal projection onto $\ker(P)$.
- $\Pi_0(P)$ has finite rank and is a smoothing operator.

Complex Powers of Elliptic ΨDOs

Theorem (Seeley)

P^z, $z \in \mathbb{C}$, is an operator in $\Psi^{mz}(M)$ whose principal symbol is $\sigma_m(x,\xi)^z$.

Corollary

Let $P \in \Psi^m(M)$, $m > 0$, be elliptic. Then:

1. *$|P| := \sqrt{P^*P}$ is an operator in $\Psi^m(M)$ whose principal symbol is $|\sigma_m(x,\xi)|$.*
2. *$|P|^z$, $z \in \mathbb{C}$, is an operator in $\Psi^{mz}(M)$ whose principal symbol is $|\sigma_m(x,\xi)|^z$.*

Complex Powers of Elliptic ΨDOs

Setup

- $g =$ Riemannian metric on M.
- We take μ to be the Riemannian density $\nu(g)$.
- $\Delta_g =$ Laplacian associated with g.

Remark

- Δ_g is positive-elliptic.
- Its principal symbol is $|\xi|_g^2$.

Therefore, Seeley's result gives:

Proposition

The power Δ_g^z, $z \in \mathbb{C}$, is an operator in $\Psi^{2z}(M)$ whose principal symbol is $|\xi|_g^{2z}$.