Introduction to Noncommutative Geometry
Chapter 7:
Connes' Trace Theorem on Euclidean Spaces
Part 3:
Integration Formula

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Semi-Compactly Supported ΨDOs

Definition

We say that $P \in \Psi^m(U)$, $m \in \mathbb{R}$, is (left) semi-compactly supported if there is a compact $K \subseteq U$ such that

$$\operatorname{supp} Pu \subseteq K \qquad \forall u \in C_c^\infty(U).$$

Remark

If P is semi-compactly supported, then there is $\varphi \in C_c^\infty(U)$ such that $\varphi P = P$.

Remark

If $U = \mathbb{R}^n$ and $f \in C_c^\infty(\mathbb{R}^n)$, then $f(1+\Delta)^q$, $q \in \mathbb{R}$, is semi-compactly supported.

Semi-Compactly Supported ΨDOs

Proposition (Calderon-Vaillancourt)

If $P \in \Psi^m(U)$, $m \leq 0$, is semi-compactly supported, then it uniquely extends to a continuous linear operator,

$$P : L^2(U) \longrightarrow L^2(U).$$

Proposition

Every $P \in \Psi^{-m}(U)$, $m > 0$, that is semi-compactly supported is in the weak Schatten class $\mathcal{L}^{\frac{n}{m},\infty}$.

Connes' Trace Theorem

Theorem (Connes's Trace Theorem)

If $P \in \Psi^{-n}(U)$ *is semi-compactly supported, then* P *is strongly measureable, and*

$$⨍ P = \frac{1}{n}(2\pi)^{-n} \iint_{U \times \mathbb{S}^{n-1}} \sigma_{-n}(x,\xi) dx d\xi,$$

$\sigma_{-n}(x,\xi)$ *is the principal symbol of* P.

Connes' Integration Formula

Theorem

If $f \in C_c^\infty(\mathbb{R}^n)$, *then* $f(1+\Delta)^{-n/2}$ *is strongly measurable, and*

$$⨍ f(1+\Delta)^{-\frac{n}{2}} = c(n)\int_{\mathbb{R}^n} f(x)dx.$$

This shows that the NC integral recovers Lebesgue measure on $\mathbb{R}^n$.

Remark

- If $f \in C_c^\infty(\mathbb{R}^n)$, then $(1+\Delta)^{-n/4} f(1+\Delta)^{-n/4}$ is weak-trace class and agrees with $f(1+\Delta)^{-n/2}$ modulo $\mathrm{Com}(\mathcal{L}^{1,\infty})$.
- Thus, $(1+\Delta)^{-n/4} f(1+\Delta)^{-n/4}$ is strongly measurable, and

$$⨍ (1+\Delta)^{-\frac{n}{4}} f(1+\Delta)^{-\frac{n}{4}} = c(n)\int_{\mathbb{R}^n} f(x)dx.$$

Zygmund Space

Definition (Zygmund)

$L\log L(\mathbb{R}^n)$ consists of measurable functions $f : M \to \mathbb{C}$ such that

$$\int_M |f(x)| \log(1 + |f(x)|)dx < \infty.$$

Proposition

$L\log L(\mathbb{R}^n)$ is a Banach space with respect to the norm,

$$\|f\|_{L\log L} := \inf\left\{\lambda > 0; \int_M |\lambda^{-1} f(x)| \log(1 + \lambda^{-1}|f(x)|)dx < 1\right\}.$$

Theorem (Solomyak, Sukochev-Zanin)

1. *If* $f \in L\log L(\mathbb{R}^n)$, *then* $(1+\Delta)^{-n/4} f (1+\Delta)^{-n/4} \in \mathcal{L}^{1,\infty}$.
2. *There is* $C > 0$ *such that*

$$\left\|(1+\Delta)^{-\frac{n}{4}} f (1+\Delta)^{-\frac{n}{4}}\right\|_{1,\infty} \leq C \|f\|_{L\log L} \qquad \forall f \in L\log L(\mathbb{R}^n).$$

Theorem

If $f \in L\log L(\mathbb{R}^n)$*, then* $(1+\Delta)^{-n/4} f (1+\Delta)^{-n/4}$ *is strongly measurable, and*

$$⨍ (1+\Delta)^{-\frac{n}{4}} f (1+\Delta)^{-\frac{n}{4}} = c(n) \int_{\mathbb{R}^n} f(x)dx.$$