

Introduction to Noncommutative Geometry

Chapter 3: Operators on Hilbert Space

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Operators on a Hilbert Space

Overview

This chapter is a review of basic results regarding operators on Hilbert space.

References for this Chapter

- Reed, M.; Simon, B.: *Methods of modern mathematical physics. I. Functional analysis*. Second edition. Academic Press, Inc., New York, 1980.

Notation

- $\mathcal{H}$ is a separable Hilbert space.
- $\mathcal{L}(\mathcal{H})$ is the C^* -algebra of bounded operators on $\mathcal{H}$.

Positive Operators

Definition

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be **positive** if can be written in the form $T = S^*S$ with $S \in \mathcal{L}(\mathcal{H})$.

Remark

We denote by $\mathcal{L}(\mathcal{H})_+$ the set of positive elements of $\mathcal{L}(\mathcal{H})$.

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$. TFAE:

- (i) T is positive.
- (ii) There exists $S \in \mathcal{L}(\mathcal{H})$ selfadjoint such that $T = S^2$.
- (iii) T is selfadjoint and $\text{Sp } T \subseteq [0, \infty)$.
- (iv) $\langle T\xi | \xi \rangle \geq 0$ for all $\xi \in \mathcal{H}$.

Corollary

The set of positive operators $\mathcal{L}(\mathcal{H})_+$ is a positive cone of $\mathcal{L}(\mathcal{H})$, i.e.,

$$\lambda_1 T_1 + \lambda_2 T_2 \in \mathcal{L}(\mathcal{H})_+ \quad \forall T_j \in \mathcal{L}(\mathcal{H})_+ \quad \forall \lambda_j \geq 0.$$

Corollary

Let $T \in \mathcal{L}(\mathcal{H})$ be normal. If $f \in C(\operatorname{Sp} T)$ is non-negative, then $f(T)$ is a positive operator.

Facts

- If $T \in \mathcal{L}(\mathcal{H})$, then T^*T is positive.
- Thus, it is selfadjoint and $\text{Sp}(T^*T) \subseteq [0, \infty)$.
- By continuous functional calculus we then can define its square root $\sqrt{T^*T}$.
- We get a positive operator.

Definition

The operator $\sqrt{T^*T}$ is denoted $|T|$ and is called the **absolute value** of T .

Lemma

Let $T \in \mathcal{L}(\mathcal{H})$. The following hold:

(i) $|T|$ is the unique element of $\mathcal{L}(\mathcal{H})_+$ whose square is T^*T .

(ii) We have

$$\|T\xi\| = \||T|\xi\| \quad \forall \xi \in \mathcal{H},$$

and hence $\ker |T| = \ker T$.

Proposition (Polar Decomposition)

Let $T \in \mathcal{L}(\mathcal{H})$. Then there exists a unique $U \in \mathcal{L}(\mathcal{H})$ such that

- (i) $T = U|T|$.
- (ii) $\ker U = \ker T$.

Definition

- ① The operator U is called the **phase** of U .
- ② The decomposition $T = U|T|$ is called the **polar decomposition** of T .

Polar Decomposition

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$ have polar decomposition $T = U|T|$ and denote by $\Pi_0(T)$ (resp. $\Pi_0(T^*)$) the orthogonal projection onto $\ker T$ (resp. $\ker T^*$).

① The range of U is $\overline{\text{ran } T}$.

② We have

$$U^*U = 1 - \Pi_0(T) \quad \text{and} \quad UU^* = 1 - \Pi_0(T^*)$$

Thus, U is a partial isometry and has norm 1 unless $T = 0$.

③ If T is injective and has dense range, then U is unitary.

④ We have

$$|T| = U^*T, \quad T^* = U^*TU^*, \quad |T^*| = TU^* = U|T|U^*.$$

⑤ The phase of T^* is U^* .

The Spectral Theorem

Setup

- T = normal operator in $\mathcal{L}(\mathcal{H})$ (i.e., $T^*T = TT^*$).
- We set $S = \text{Sp}(T)$.

Theorem (Spectral Theorem)

There exist a finite measure space (X, μ) , a unitary operator $U : \mathcal{H} \rightarrow L^2_\mu(X)$, and a function $f \in L^\infty_\mu(X)$ such that

$$UTU^*\xi = f\xi \quad \forall \xi \in L^2_\mu(X).$$

Definition

For $f \in L^\infty_\mu(X)$ the multiplication operator $M_f : L^2_\mu(X) \rightarrow L^2_\mu(X)$ is given by

$$M_f \xi = f \xi, \quad \xi \in L^2_\mu(X).$$

Remark

The Spectral Theorem asserts that $UTU^* = M_f$.

Definition

The **essential range** of $f \in L^\infty_\mu(X)$, denoted $\text{ess-ran}(f)$, consists of all $\lambda \in \mathbb{C}$ such that $\mu(|f - \lambda| < \epsilon) > 0$ for all $\epsilon > 0$.

Fact

$$\text{Sp}(M_f) = \text{ess-ran}(f).$$

Fact

$L_\mu^\infty(X)$ is a unital commutative C^* -algebra with respect to the involution $f \rightarrow \bar{f}$ and the norm,

$$\begin{aligned}\|f\|_{L^\infty} &= \inf\{C; |f(x)| \leq C \text{ a.e.}\} \\ &= \{|\lambda|; \lambda \in \text{ess-ran}(|f|)\}, \quad f \in L_\mu^\infty(X).\end{aligned}$$

Proposition

The map $L_\mu^\infty(X) \ni f \rightarrow M_f \in \mathcal{L}(L_\mu^2(X))$ is an isometric $$ -representation.*

Reminder

By the Spectral Theorem $UTU^* = M_f$, i.e, $T = U^*M_fU$.

Borel Functional Calculus

Lemma

- (i) $\text{ess-ran}(f) = S$. (Here $S = \text{Sp}(T)$.)
- (ii) $g(T) = U^* T_{g \circ f} U$ for all $g \in C(S)$.

Proof.

- $\text{ess-ran}(f) = \text{Sp}(M_f) = \text{Sp}(U^* T U) = \text{Sp}(T) = S$.
- As $T = U^* M_f U$, if $g = z^m \bar{z}^n$, then
$$g(T) = T^m (T^*)^n = (U^* M_f U)^m (U^* M_f^* U)^n = U^* M_f^m (M_f^*)^n U.$$
- Here $M_f^m (M_f^*)^n = M_{f^m} M_{\bar{f}^n} = M_{f^m \bar{f}^n} = M_{g \circ f}$. Thus,
$$g(T) = U^* M_{g \circ f} U.$$
- By linearity this holds for any polynomial $g(z) = \sum c_{m,n} z^m \bar{z}^n$.
- By density of those polynomials and continuity of the continuous functional calculus, this holds for all $g \in C(S)$. $\square$

Definition

If $g \in L^\infty(S)$, then the operator $g(T) \in \mathcal{L}(\mathcal{H})$ is given by

$$g(T) := U^* M_{g \circ f} U.$$

Theorem (Borel Functional Calculus)

- ① The map $g \rightarrow g(T)$ is a $*$ -homomorphism from $L^\infty(S)$ to $\mathcal{L}(\mathcal{H})$ such that

$$\|g(T)\| \leq \|g\|_{L^\infty(S)} \quad \forall g \in L^\infty(S).$$

- ② If $(g_n)_{n \geq 0}$ is a bounded sequence in $L^\infty(S)$ such that $g_n \rightarrow g$ a.e., then $g_n(T) \rightarrow g(T)$ strongly (i.e., $g_n(T)\xi \rightarrow g(T)\xi$ for all $\xi \in \mathcal{H}$).
- ③ If $g \in L^\infty(S)$ is real-valued (resp., non-negative), then $g(T)$ is selfadjoint (resp., positive).

Unbounded Operators

Definition

- An **(unbounded) operator** on $\mathcal{H}$ is a linear operator $T : D(T) \rightarrow \mathcal{H}$, where the domain $D(T)$ is a subspace of $\mathcal{H}$.
- It is called **densly defined** if $D(T)$ is a dense subspace of $\mathcal{H}$.

Definition

- The **graph** of an operator T is

$$G(T) = \{(\xi, \eta) \in \mathcal{H} \oplus \mathcal{H}; \eta = T\xi\}.$$

- We say that T is **closed** if $G(T)$ is closed.
- We say that T is **closable** if there is a (closed) operator $\overline{T}$ such that $\overline{G(T)} = G(\overline{T})$.
- The operator $\overline{T}$ is called the **closure** of T .

Unbounded Operators

Remark

- An operator S is called an **extension** of T , and we write $T \subseteq S$ if $G(T) \subseteq G(S)$.
- Equivalently, $D(T) \subseteq D(S)$ and $S = T$ on $D(T)$.

Definition

If T is densely defined, then its **adjoint** is the operator T^* given by the graph

$$G(T^*) = \{(\xi, \eta) \in \mathcal{H} \oplus \mathcal{H}; \langle T\zeta | \xi \rangle = \langle \zeta | \eta \rangle \ \forall \zeta \in D(T)\}.$$

Remark

As $G(T^*)$ is a closed subspace, we see that T^* is always a closed operator.

Unbounded Operators

Definition

Let T be densely defined. We say that:

- T is **selfadjoint** if $T^* = T$.
- T is **symmetric** if $T \subseteq T^*$.
- T is **essentially selfadjoint** if T is symmetric and closable.

Remarks

- T is symmetric if and only if $\langle T\xi|\eta\rangle = \langle \xi|T\eta\rangle \quad \forall \xi, \eta \in D(T)$.
- T is selfadjoint iff T is symmetric and $D(T) = D(T^*)$.
- If T is selfadjoint, then it is closed.
- If T is essentially selfadjoint, then T^* is the closure of T .
- A symmetric operator may have several different selfadjoint extensions.

Proposition

Let T be symmetric. TFAE:

- ① T is selfadjoint.
- ② T is closed and $\ker(T^* \pm i) = \{0\}$.
- ③ $\operatorname{ran}(T \pm i) = \mathcal{H}$.

Corollary

Let T be symmetric. TFAE:

- ① T is essentially selfadjoint.
- ② $\ker(T^* \pm i) = \{0\}$.
- ③ $\operatorname{ran}(T \pm i)$ is dense.

Unbounded Operators

Definition

Suppose that T is closed.

- The **resolvent set** of T consists of all $\lambda \in \mathbb{C}$ such that
 - (i) $T - \lambda : D(T) \rightarrow \mathcal{H}$ is a bijection.
 - (ii) The inverse $(T - \lambda)^{-1} : \mathcal{H} \rightarrow D(T)$ is bounded.
- The **spectrum** of T , denoted $\text{Sp}(T)$, is the complement of the resolvent set.

Proposition

- ① $\text{Sp}(T)$ is (possibly empty) closed subset of $\mathbb{C}$.
- ② The resolvent map $\mathbb{C} \setminus \text{Sp}(T) \rightarrow (T - \lambda)^{-1} \in \mathcal{L}(\mathcal{H})$ is analytic.

Remark

If T is selfadjoint, then $\text{Sp}(T) \subseteq \mathbb{R}$.

Unbounded Operators

Setup

T = selfadjoint operator on $\mathcal{H}$.

Theorem (Spectral Theorem;)

There exist a measure space (X, μ) with $\mu(X) < \infty$, a unitary operator $U : \mathcal{H} \rightarrow L^2_\mu(X)$, and a measurable real-valued function f on X such that

$$U(D(T)) = \{\xi \in L^2_\mu(X); f\xi \in L^2_\mu(X)\},$$
$$UTU^*\xi = f\xi \quad \forall \xi \in U(D(T)).$$

Definition

If g is any bounded Borel function on $\mathbb{R}$, then $g(T)$ is the bounded operator on $\mathcal{H}$ defined by

$$g(T) := U^* M_{g \circ f} U.$$

Remark

If g is Borel and bounded, then $g \circ f \in L^\infty_\mu(X)$, and so $M_{g \circ f}$ is a bounded operator.

Theorem (Borel Functional Calculus)

- ① The map $g \rightarrow g(T)$ is a $*$ -homomorphism from $L^\infty(\mathbb{R})$ to $\mathcal{L}(\mathcal{H})$ such that

$$\|g(T)\| \leq \|g\|_{L^\infty(\mathbb{R})} \quad \forall g \in L^\infty(\mathbb{R}).$$

- ② If $(g_n)_{n \geq 0}$ is a bounded sequence in $L^\infty(\mathbb{R})$ such that $g_n \rightarrow g$ a.e., then $g_n(T) \rightarrow g(T)$ strongly.
- ③ If $g \in L^\infty(\mathbb{R})$ is real-valued (resp., non-negative), then $g(T)$ is selfadjoint (resp., positive).

Remark

If g is a possibly unbounded Borel function on $\mathbb{R}$, then $g(T)$ makes sense as an unbounded operator as follows.

- The domain of $g(T)$ is

$$D(g(T)) = \{ \xi \mathcal{H}; (g \circ f)(U\xi) \in L^2_\mu(X) \},$$

- On $D(T)$ we have

$$g(T)\xi := U^* ((g \circ f)U\xi), \quad \xi \in D(g(T)).$$

- For instance, if $g(t) = t$, then $g(T) = T$.
- If (g_n) is a sequence of Borel functions on $\mathbb{R}$ such that $g_n \rightarrow g$ and $|g_n| \leq |g|$, then

$$g_n(T)\xi \longrightarrow g(T)\xi \quad \forall \xi \in D(g(T)).$$

Unbounded Operators

Example

Let $\Delta = -(\partial_{x_1}^2 + \dots + \partial_{x_n}^2)$ be the (positive) Laplacian on $\mathbb{R}^n$.

- This is a selfadjoint operator on $\mathbb{R}^n$ with domain

$$D(\Delta) = \{u \in L^2(\mathbb{R}^n); (1 + |\xi|^2)\hat{u}(\xi) \in L^2(\mathbb{R}^n)\}.$$

Here $\hat{u}$ is the Fourier transform.

- The Fourier transform $U : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is a unitary operator (with the suitable normalization).
- As $(\Delta u)^\wedge = |\xi|^2 \hat{u}$ we see that

$$\Delta = U^* M_{|\xi|^2} U.$$

- If $g \in L^\infty(\mathbb{R})$, then $g(T) = U^* M_{g(|\xi|^2)} U$. That is,

$$g(T)u = (g(|\xi|^2)\hat{u})^\vee \quad \forall u \in L^2(\mathbb{R}^n).$$

Here $v \rightarrow \check{v}$ is the inverse Fourier transform.

Compact Operators

Definition

- An operator $T \in \mathcal{L}(\mathcal{H})$ is **compact** if the image by T of the unit ball $B(0,1)$ is precompact.
- The set of compact operators is denoted $\mathcal{K}$.

Example

Every finite rank operator is compact.

Proof.

Let $T \in \mathcal{L}(\mathcal{H})$ have finite rank, i.e., $\dim \operatorname{ran}(T) < \infty$.

- By Riesz Theorem any bounded subset of $\operatorname{ran}(T)$ is precompact.
- Here $T(B(0,1))$ is a bounded subset of $\operatorname{ran}(T)$.
- Therefore, it is precompact, and hence T is compact. □

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$. TFAE:

- ① T is a compact operator.
- ② For any bounded sequence $(\xi_n)_{n \geq 0} \subseteq \mathcal{H}$ there is a subsequence $(\xi_{n_k})_{k \geq 0}$ such that the sequence $(T\xi_{n_k})_{k \geq 0}$ converges in norm.
- ③ For any sequence $(\xi_n)_{n \geq 0} \subseteq \mathcal{H}$ converging weakly to 0 the sequence $(T\xi_n)_{n \geq 0}$ converges to 0 in norm.
- ④ There is an orthonormal basis $(\xi_n)_{n \geq 0}$ of $\mathcal{H}$ such that

$$\lim_{N \rightarrow \infty} \|T|_{E_N^\perp}\| = 0, \quad E_N := \text{Span}\{\xi_0, \dots, \xi_{N-1}\}.$$

- ⑤ For any orthonormal basis $(\xi_n)_{n \geq 0}$ of $\mathcal{H}$, we have

$$\lim_{N \rightarrow \infty} \|T|_{E_N^\perp}\| = 0, \quad E_N := \text{Span}\{\xi_0, \dots, \xi_{N-1}\}.$$

Proposition

$\mathcal{K}$ is a closed two-sided ideal of $\mathcal{L}(\mathcal{H})$.

Corollary

Let $T \in \mathcal{L}(\mathcal{H})$. TFAE:

- 1 T is compact.
- 2 T is the norm limit of finite-rank operators.

Proof.

- Any finite-rank operator is compact and $\mathcal{K}$ is closed.
- Thus, any norm-limit of finite-rank operators is compact.
- If T is compact and $(\xi_n)_{n \geq 0}$ is any orthonormal basis of $\mathcal{H}$, then

$$\lim_{N \rightarrow \infty} \|T|_{E_N^\perp}\| = 0, \quad E_N := \text{Span}\{\xi_0, \dots, \xi_{N-1}\}.$$

- Let Π_N be the orthogonal projection onto E_N .
- Then $T_N := T\Pi_N$ has finite rank, and

$$\|T_N - T\| = \|T(1 - \Pi_N)\| = \|T|_{E_N^\perp}\| \rightarrow 0.$$

- Thus, T is the norm-limit of finite-rank operators. □

Compact Operators

Corollary

Let $T \in \mathcal{L}(\mathcal{H})$. Then

$$T \in \mathcal{K} \iff T^* \in \mathcal{K} \iff |T| \in \mathcal{K}.$$

Proof.

- $\mathcal{K}$ is an ideal, and so $T \in \mathcal{K} \Rightarrow ATB \in \mathcal{K} \forall A, B \in \mathcal{L}(\mathcal{H})$.
- By polar decomposition $T = U|T|$, and so $|T| \in \mathcal{K} \Rightarrow T \in \mathcal{K}$.
- As $|T| = U^*T$, we have $|T| \in \mathcal{K} \Rightarrow T \in \mathcal{K}$.
- We have $T^* = |T|U^* = U^*TU^*$, and hence $T = UT^*U$.
- Thus, $T \in \mathcal{K} \Leftrightarrow T^* \in \mathcal{K}$. □

Remark

The above result is true for any two-sided ideal of $\mathcal{L}(\mathcal{H})$.

Corollary

$\mathcal{K}$ is a closed $*$ -subalgebra of $\mathcal{L}(\mathcal{H})$, and hence is a C^* -algebra.

Proof.

- $\mathcal{K}$ is a closed two-sided ideal of $\mathcal{L}(\mathcal{H})$, and so this is a closed subalgebra.
- $\mathcal{K}$ is a $*$ -subalgebra by the previous corollary.
- Thus, $\mathcal{K}$ is a closed $*$ -subalgebra. □

Theorem (Riesz-Schauder)

Let $T \in \mathcal{K}$. The following hold.

- ① T always contains 0 in its spectrum.
- ② If $\lambda \in \text{Sp } T \setminus \{0\}$, then λ is an eigenvalue with finite multiplicity.
- ③ $\text{Sp } T$ is either finite or consists of a sequence of complex numbers converging to 0 .

Notation

If $\xi, \eta \in \mathcal{H}$, then $|\xi\rangle\langle\eta| \in \mathcal{L}(\mathcal{H})$ is given by

$$(|\xi\rangle\langle\eta|)\zeta := \langle\zeta|\eta\rangle\xi, \quad \zeta \in \mathcal{H}.$$

Remarks

- If $\xi \neq 0$ and $\eta \neq 0$, then $|\xi\rangle\langle\eta|$ has rank 1.
- If $\|\xi\| = 1$, then $|\xi\rangle\langle\xi|$ is the orthogonal projection onto $\mathbb{C}\xi$.
- We have

$$(|\xi\rangle\langle\eta|)^* = |\eta\rangle\langle\xi|,$$

$$|\xi_1\rangle\langle\eta_1| \circ |\xi_2\rangle\langle\eta_2| = \overline{\langle\eta_1|\xi_2\rangle} (|\xi_1\rangle\langle\eta_2|) = |\xi_1\rangle\langle\eta_1|\xi_2\rangle\eta_2|.$$

Theorem (Hilbert-Schmidt)

Let $T \in \mathcal{K}$ be normal.

- 1 T diagonalizes in an orthonormal basis, i.e., there exists an orthonormal basis $(\xi_n)_{n \geq 0}$ of $\mathcal{H}$ and a sequence $(\lambda_n)_{n \geq 0} \subseteq \mathbb{C}$ converging to 0 such that

$$T\xi_n = \lambda_n \xi_n \quad \forall n \geq 0.$$

- 2 T is the sum of its *Schmidt series*,

$$T = \sum \lambda_n |\xi_n\rangle\langle \xi_n|,$$

where the series converges in norm.

Corollary (Borel Functional Calculus for Compact Operators)

Let $T \in \mathcal{K}$ be normal.

- ① For every bounded function f on $\text{Sp}(T)$, we have

$$f(T) = \sum_{n \geq 0} f(\lambda_n) |\xi_n\rangle \langle \xi_n|,$$

where the series converges strongly.

- ② $f(T)$ is a compact operator if and only if

$$\lim_{n \rightarrow \infty} f(\lambda_n) = 0.$$

Moreover, in this case the above series converges in norm.

Example

Suppose that $T \in \mathcal{K}$ is normal, and let $T = U|T|$ be the polar decomposition. Then

$$U = \sum_{\lambda_n \neq 0} |\lambda_n|^{-1} \lambda_n |\xi_n\rangle\langle\xi_n|, \quad |T| = \sum_{n \geq 0} |\lambda_n| |\xi_n\rangle\langle\xi_n|,$$

where the first series converges strongly and the second series converges in norm.