Introduction to Noncommutative Geometry
Chapter 2:
C^*-Algebras

Sichuan University, July 2025

References

References for this Chapter

- Arveson, W.: *A Short Course on Spectral Theory*. Graduate Texts in Mathematics, Springer, 2002.
- Slides to be posted on the mini-course website.

Algebras

Definition

An algebra (over $\mathbb{C}$) is a vector space A together with a bilinear map $A \times A \ni (x, y) \to xy \in A$ which defines an associative multiplication on A, i.e.,

$$(x + y)z = xz + yz, \quad x(y + z) = xy + xz,$$
$$(\lambda x)y = x(\lambda y) = \lambda xy,$$
$$x(yz) = (xy)z.$$

Definition

A unit of an algebra A is an element $1 \in A$ such that

$$1x = x1 = x \qquad \forall x \in A.$$

Remark

If a unit exists, then it is unique.

Remarks

- All the vector spaces and algebras are vector spaces or algebras over $\mathbb{C}$.
- Unless otherwise mentioned all the topological spaces are Hausdorff.

C^*-algebras

Definition

A Banach algebra is an algebra A endowed with a Banach norm $\|.\|$ such that

$$\|xy\| \leq \|x\|\|y\| \qquad \forall x, y \in A.$$

Definition

A C^*-algebra is a Banach algebra A together with an anti-linear involution $x \to x^*$ such that

$$(xy)^* = y^* x^* \qquad \forall x, y \in A,$$

$$\|x^*\| = \|x\| \quad \text{and} \quad \|x^* x\| = \|x\|^2 \qquad \forall x \in A.$$

$*$-Homomorphisms

Definition

Let A and B be C^*-algebras.

1. A $*$-homomorphism $\phi : A \to B$ is a continuous homomorphism of algebras such that $\phi(x^*) = \phi(x)^*$ for all $x \in A$.
2. A $*$-isomorphism $\phi : A \to B$ is $*$-homomorphism which is bijective.

Remarks

- By the open mapping theorem any bijective continuous linear map between Banach spaces has a continuous inverse. Therefore, the inverse of any $*$-isomorphism is continuous.
- It can be shown that any $*$-isomorphism between C^*-algebras is isometric.

Examples

Example

Let $M_n(\mathbb{C})$ be the algebra of $n \times n$-matrices with complex entries.

- Its involution is $A \to A^*$, where A^* is the adjoint of A.
- Its C^*-algebra norm is

$$\|A\| := \sup\{\|Ax\|;\ x \in \mathbb{C}^n,\ \|x\| = 1\}, \quad A \in M_n(\mathbb{C}).$$

Examples

Example

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}(\mathcal{H})$ the algebra of continuous linear operators $T : \mathcal{H} \to \mathcal{H}$.

- The involution of $\mathcal{L}(\mathcal{H})$ is $T \to T^*$, where T^* is the adjoint of T, i.e., the unique linear operator on $\mathcal{H}$ such that
$$\langle T^*\xi, \eta\rangle = \langle \xi, T\eta\rangle \quad \forall \xi, \eta \in \mathcal{H}.$$
- The norm of $\mathcal{L}(\mathcal{H})$ is
$$\|T\| := \sup_{\|\xi\| \leq 1} \|T\xi\|$$
- More generally, any closed $*$-subalgebra of $\mathcal{L}(\mathcal{H})$ is a C^*-algebra.

$*$-Representations

Definition

A $*$-representation of a C^*-algebra A in a Hilbert space $\mathcal{H}$ is a $*$-homomorphism from A to $\mathcal{L}(\mathcal{H})$.

Theorem (Gel'fand-Naimark)

Any C^*-algebra A admits an isometric $*$-representation π into some Hilbert space $\mathcal{H}$.

Remark

Any isometric linear map between Banach spaces has closed range and is an isomorphism onto its range (Exercise!).

Consequence

Any C^*-algebra A can be $*$-represented as a closed $*$-subalgebra of some $\mathcal{L}(\mathcal{H})$.

Further Examples

Example

Let X be a compact (Hausdorff) space and $C(X)$ its algebra of continuous complex-valued functions.

- The involution of $C(X)$ is $f \to \overline{f}$, where $\overline{f}$ is the complex conjugate of f.
- The norm of $C(X)$ is

$$\|f\|_{C(X)} := \sup_{x \in X} |f(x)|, \qquad f \in C(X).$$

- The constant function 1 is a unit for $C(X)$, and so $C(X)$ is a unital commutative C^*-algebra.

Further Examples

Example

Let X be a locally compact topological space and $C_0(X)$ its algebra of continuous functions "vanishing at infinity".

- Recall that $f \in C(X)$ vanishes at infinity iff, for all $\epsilon > 0$, there $K \subseteq X$ compact s.t. $|f(x)| \leq \epsilon$ on $X \setminus K$.
- The involution of $C_0(X)$ is $f \to \overline{f}$.
- The norm of $C_0(X)$ is

$$\|f\|_{C_0(X)} := \sup_{x \in X} |f(x)|, \qquad f \in C_0(X).$$

- The C^*-algebra $C_0(X)$ is commutative, but it is not unital, since $1 \notin C_0(X)$.

Remark

It can be shown that $C(X)$ and $C_0(X)$ are essentially the only examples of commutative C^*-algebras.

Adding a Unit

Definition

Let A be a (possibly non-unital) Banach algebra. Define

$$A^+ = A \oplus \mathbb{C}.$$

We endow A^+ with the product and norm given by

$$(x_1, \lambda_1).(x_2, \lambda_2) := (x_1x_2 + \lambda_1 x_2 + \lambda_2 x_1, \lambda_1\lambda_2), \quad (x_j, \lambda_j) \in A^+,$$
$$\|(x, \lambda)\|_{A^+} := \sup\{\|xy + \lambda y\|_A;\ y \in A,\ \|y\|_A = 1\}, \quad (x, \lambda) \in A^+.$$

Proposition

1. *A^+ is a Banach algebra with unit $1_{A^+} := (0, 1)$.*
2. *The map $x \to (x, 0)$ is an isometric embedding of A into A^+.*

Definition

A^+ is called the unitalization of A.

Adding a Unit

Remark

- The embedding of $x \to (x, 0)$ identifies A with the closed ideal $A \oplus \{0\}$ of A^+.
- This allows us to write any element of A^+ as

$$(x, \lambda) = x + \lambda 1_{A^+}, \qquad x \in A, \quad \lambda \in \mathbb{C}.$$

Proposition

Assume A is a C^*-algebra, and equip A^+ with the involution

$$(x + \lambda 1_{A^+})^* = x^* + \overline{\lambda} 1_{A^+}, \qquad x \in A, \quad \lambda \in \mathbb{C}.$$

Then A is a C^*-algebra.

Adding a Unit

Example

Let $A = C_0(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with boundary $\partial\Omega$.

- Here
$$C_0(\Omega) = \left\{ f \in C(\overline{\Omega});\ f = 0 \text{ on } \partial\Omega \right\}.$$
- The unitalization of A is
$$A^+ = \left\{ f \in C(\overline{\Omega});\ f_{|\partial\Omega} \text{ is constant} \right\}.$$

Spectrum

Setup

- A is a Banach algebra with a unit 1_A such that $\|1_A\| = 1$.
- This is always satisfied if A is a C^*-algebra.
- A^{-1} is the group of invertible elements of A. This is an open subset of A.

Definition

Let $x \in A$. The spectrum of x is

$$\mathrm{Sp}_A(x) := \{\lambda \in \mathbb{C};\ x - \lambda \notin A^{-1}\}.$$

The complement of $\mathrm{Sp}_A(x)$ is called the resolvent set of x.

Spectrum

Proposition

1. *$\mathrm{Sp}_A(x)$ is a non-empty compact subset of $\mathbb{C}$ contained in the disk $\overline{D(0, \|x\|)}$.*
2. *The resolvent $\lambda \to (x-\lambda)^{-1}$ is an analytic map from $\mathbb{C} \setminus \mathrm{Sp}_A(x)$ to A.*

Partial Proof.

- By definition $\lambda \in \mathbb{C} \setminus \mathrm{Sp}_A(x) \Leftrightarrow x - \lambda \in A^{-1}$.
- Here $\mathbb{C} \ni \lambda \to x - \lambda \in A$ is continuous and A^{-1} is an open set of A.
- Thus, $\mathbb{C} \setminus \mathrm{Sp}_A(x)$ is an open set, and hence $\mathrm{Sp}_A(x)$ is closed.

□

Spectrum

Partial Proof (Continued).

- If $\|x\| < 1$, then $\sum_{\geq 0} x^n = (1 - x)^{-1}$.
- If $\lambda > \|x\|$, then $\lambda^{-1}\|x\| < 1$, and so

$$\sum \lambda^{-n-1} x^n = \lambda^{-1} \sum (\lambda^{-1} x)^n = \lambda^{-1}(1 - \lambda^{-1} x)^{-1} = (\lambda - x)^{-1}.$$

  In particular, $x - \lambda \in A^{-1}$, i.e., $\lambda \in \mathbb{C} \setminus \mathrm{Sp}_A(x)$.
- Thus, $\mathbb{C} \setminus \overline{D}(0, \|x\|) \subseteq \mathbb{C} \setminus \mathrm{Sp}_A(x)$, i.e., $\mathrm{Sp}_A(x) \subseteq \overline{D}(0, \|x\|)$.
- Therefore, $\mathrm{Sp}_A(x)$ is a bounded closed set, and so this is a compact set. □

Spectrum. Non-unital Case

Remarks

- If A is not unital, we define the spectrum of $x \in A$ to be its spectrum in A^+, i.e.,

$$\mathrm{Sp}_A(x) := \mathrm{Sp}_{A^+}(x).$$

- 0 is always contained in $\mathrm{Sp}_{A^+}(x)$, since the proper ideal A cannot contained invertible elements of A^+.
- If A is unital, then $\mathrm{Sp}_{A^+}(x) = \mathrm{Sp}_A(x) \cup \{0\}$.

Examples

Example

Let $A = C(X)$, where X is a compact (Hausdorff) space.

- If $f \in C(X)$ and $\lambda \in \mathbb{C}$, then
  $$f - \lambda \text{ invertible} \iff (f(x) - \lambda \neq 0 \ \forall x \in X) \iff \lambda \notin f(X).$$
- Thus, $\mathbb{C} \setminus \mathrm{Sp}_{C(X)}(f) = \mathbb{C} \setminus f(X)$, i.e.,
  $$\mathrm{Sp}_{C(X)}(f) = f(X).$$

Examples

Example

Let $A = C_0(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is a bounded open set.

- Here $A^+ = \{f \in C(\overline{\Omega});\, f_{|\partial\Omega} \text{ is constant }\}$.
- Thus, if $f \in C_0(\Omega)$, then, by spectral permanence,

$$\mathrm{Sp}_{C_0(\Omega)}(f) = \mathrm{Sp}_{A^+}(f) = \mathrm{Sp}_{C(\overline{\Omega})}(f) = f(\overline{\Omega}) = f(\Omega) \cup \{0\}.$$

- More generally, if X is a locally compact Hausdorff space, then

$$\mathrm{Sp}_{C_0(X)}(f) = f(X) \cup \{0\} \qquad \forall f \in C_0(X).$$

Spectral Radius

Definition

Let $x \in A$. The spectral radius of x is

$$\rho(x) = \sup\{|\lambda|;\ \lambda \in \mathrm{Sp}_A(x)\}.$$

Remark

We always have $\rho(x) \leq \|x\|$.

Proposition (Gel'fand-Mazur)

For all $x \in A$, we have

$$\rho(x) = \lim_{n\to\infty} \sqrt[n]{\|x^n\|}.$$

Proposition

Let A be a C^*-algebra. If $x \in A$ is normal (i.e., $x^*x = xx^*$), then

$$\rho(x) = \|x\|.$$

Proof.

- As A is a C^*-algebra and $x^*x = xx^*$, we have

$$\|x^2\| = \|(x^2)^* x^2\|^{\frac{1}{2}} = \|(x^*x)^*(x^*x)\|^{\frac{1}{2}} = \|x^*x\| = \|x\|^2.$$

- An induction shows that $\|x^{2^n}\| = \|x\|^{2^n}\ \forall n \in \mathbb{N}$.
- By Gel'fand-Mazur's result,

$$\rho(x) = \lim_{n\to\infty} \|x^{2^n}\|^{\frac{1}{2^n}} = \|x\|.$$

The proof is complete. □

Spectral Radius

Remark

If A is a Banach algebra with an antilinear involution $x \to x^*$, we call C^*-norm any norm such that

$$\|x\|^2 = \|x^*\|^2 = \|x^* x\| \qquad \forall x \in A.$$

Proposition

If A is a C^*-algebra, then its norm is its unique C^*-norm.

Proof.

- If $x \in A$, for any C^*-norm we have
$$\|x\| = \sqrt{\|x^* x\|} = \sqrt{\rho(x^* x)}.$$
- The spectral radius $\sqrt{\rho(x^* x)}$ does not depend on the norm, so this uniquely defines the C^*-norm. □

Corollary

Every $*$-isomorphism between C^*-algebras is isometric.

Proof.

- Let $\phi : A_1 \to A_2$ be a $*$-isomorphism between C^*-algebras A_1 and A_2.
- In this case $x \to \|\phi(x)\|_{A_2}$ is a C^*-norm on A_1.
- Therefore, it agrees with the original C^*-norm of A_1.
- That is, ϕ is isometric. □

Holomorphic Functional Calculus

Setup

- A is a Banach algebra with unit 1 such that $\|1\| = 1$.
- $x \in A$ and we set $S = \mathrm{Sp}_A(x)$ (this a compact subset of $\mathbb{C}$).
- $\Omega \subseteq \mathbb{C}$ is an open containing S.
- $\mathrm{Hol}(\Omega)$ is the algebra of holomorphic functions on Ω, equipped with the topology of uniform convergence on compact sets.

Holomorphic Functional Calculus

Remarks

- By Cauchy's formula, if $f \in \mathrm{Hol}(\Omega)$, then

$$f(z) = \frac{1}{2i\pi} \int_\Gamma f(\lambda)(\lambda - z)^{-1} d\lambda \qquad \forall z \in S,$$

  Here Γ is any oriented contour in Ω whose interior contains S.
- The map $\lambda \to (x - \lambda)^{-1}$ is analytic on $\mathbb{C} \setminus S$.

Definition

If $f \in \mathrm{Hol}(\Omega)$, then we define

$$f(x) := \frac{1}{2i\pi} \int_\Gamma f(\lambda)(\lambda - x)^{-1} d\lambda \in A,$$

where the integral is meant as a Riemann integral.

Remark

The integral does not depend on Γ.

Holomorphic Functional Calculus

Theorem (Holomorphic Functional Calculus)

1. *The map* $f \to f(x)$ *is a continuous unital homomorphism of algebras from* $\mathrm{Hol}(\Omega)$ *to* A.
2. *If* f *and* g *are elements of* $\mathrm{Hol}(\Omega)$ *that agree on* S, *then* $f(x) = g(x)$.
3. *For all* $f \in \mathrm{Hol}(\Omega)$, *we have*

$$\mathrm{Sp}_A f(x) = f(S).$$

Example

Example

Let $f(z) = \sum_{n\geq 0} a_n z^n$ have convergence radius $R > \|x\|$.

- Let $\Gamma = \{|\lambda| = r\}$ with $\|x\| < r < R$. By definition,

$$f(x) = \frac{1}{2i\pi} \int_\Gamma f(\lambda)(\lambda - x)^{-1} d\lambda.$$

- If $|\lambda| > \|x\|$, then $\|\lambda^{-1}x\| < 1$, and so

$$(\lambda - x)^{-1} = \lambda^{-1}(1 - \lambda^{-1}x)^{-1} = \sum_{n\geq 0} \lambda^{-(n+1)} x^n.$$

- Thus,

$$f(x) = \sum_{n\geq 0} \left(\frac{1}{2i\pi} \int_\Gamma \lambda^{-(n+1)} f(\lambda) d\lambda \right) x^n = \sum_{n\geq 0} a_n x^n.$$

Reminder

- If S is a subset of $\mathbb{C}$, then its boundary is $\partial S = \overline{S} \cap \overline{\mathbb{C} \setminus S}$.
- If S is closed, then ∂S consists of all points $z \in S$ for which there is sequence $(z_n)_{n \geq 0}$ such that $z_n \to z$ and $z_n \notin S$.

Proposition

Let B be a closed subalgebra of A containing the unit 1_A.

1. *If $x \in B$ is invertible in A, and $0 \in \partial(\mathrm{Sp}_B(x))$, then x^{-1} belongs to B.*
2. *For all $x \in B$, we have*

$$\partial\left(\mathrm{Sp}_B(x)\right) \subseteq \mathrm{Sp}_A(x) \subseteq \mathrm{Sp}_B(x).$$

In particular, if $\mathrm{Sp}_B(x)$ has empty interior, then

$$\mathrm{Sp}_B(x) = \mathrm{Sp}_A(x).$$

Proof of (1).

Let $x \in B \cap A^{-1}$, and assume that $0 \in \partial(\mathrm{Sp}_B(x))$.

- As $0 \in \partial(\mathrm{Sp}_B(x))$, there is $(z_n) \subseteq \mathbb{C} \setminus \mathrm{Sp}_B(x)$ s.t. $z_n \to 0$.
- In particular, $x - z_n \in B^{-1}$, and hence $(x - z_n)^{-1} \in B$.
- The map $x \to x^{-1}$ is continuous on A^{-1}.
- Here $x \in A^{-1}$, and $x - z_n \to x$ in A.
- Thus, $(x - z_n)^{-1} \to x^{-1}$ in A.
- As $(x - z_n)^{-1} \in B$, it follows that $x^{-1} \in \overline{B} = B$.

□

Proof of (2).

Let $x \in B$.

- As $B^{-1} \subseteq A^{-1}$, we always have $\mathrm{Sp}_A(x) \subseteq \mathrm{Sp}_B(x)$.
- Let $\lambda \in \partial(\mathrm{Sp}_B(x))$. Then $\lambda \in \mathrm{Sp}_B(x)$, i.e., $x - \lambda \notin B^{-1}$,
- Assume that $\lambda \notin \mathrm{Sp}_A(x)$. Then $x - \lambda \in A^{-1}$.
- As $\lambda \in \partial(\mathrm{Sp}_B(x))$, we have that $0 \in \partial(\mathrm{Sp}_B(x - \lambda))$.
- Part (1) then ensures that $x - \lambda \in B^{-1}$ (contradiction!).
- Thus, $\lambda \in \mathrm{Sp}_A(x)$, and hence $\partial(\mathrm{Sp}_B(x)) \subseteq \mathrm{Sp}_A(x)$.

The proof is complete. □

Spectral Permanence for C^*-Algebras

Proposition

Assume that A is a unital C^*-algebra.

1. *If $x \in A$ is unitary (i.e., $x^*x = xx^* = 1$), then $\mathrm{Sp}_A(x) \subseteq S^1$.*
2. *If $x \in A$ is selfadjoint (i.e., $x^* = x$), then $\mathrm{Sp}_A(x) \subseteq \mathbb{R}$.*

Proof of (1).

Let $x \in A$ be unitary.

- As $\|x\|^2 = \|x^*x\| = \|1_A\| = 1$, we have $\mathrm{Sp}_A(x) \subseteq \overline{D(0,1)}$.
- Since $x^{-1} = x^*$ is unitary, we also have $\mathrm{Sp}_A(x^{-1}) \subseteq \overline{D(0,1)}$.
- If $f(z) = z^{-1}$, then

$$\mathrm{Sp}_A(x) = \mathrm{Sp}_A(f(x^{-1})) = f(\mathrm{Sp}_A(x^{-1})) \subseteq \mathbb{C} \setminus D(0,1).$$

- Thus,

$$\mathrm{Sp}_A(x) \subseteq \overline{D(0,1)} \cap [\mathbb{C} \setminus D(0,1)] = \mathbb{S}^1.$$

□

Proof of (2).

Let $x \in A$ be selfadjoint.

- Set $u = \exp(ix) = \sum \frac{1}{n!}(ix)^n$. We have
$$u^* = \sum \frac{1}{n!}\left((ix)^n\right)^* = \sum \frac{1}{n!}(-ix^*)^n = \exp(-ix).$$
- As $f \to f(x)$ is an algebra homomorphism and $\exp(-iz)\exp(iz) = \exp(iz)\exp(-iz) = 1$, we have
$$uu^* = u^*u = \exp(-ix)\exp(ix) = 1.$$
- Thus, u is unitary, and hence $\mathrm{Sp}_A(u) \subseteq \mathbb{S}^1$.
- By property of the holomorphic functional calculus, we have
$$\mathrm{Sp}_A(u) = \mathrm{Sp}_A(\exp(ix)) = \exp(i\,\mathrm{Sp}_A(x)) \subseteq \mathbb{S}^1$$
- It follows that $\mathrm{Sp}_A(x) \subseteq \mathbb{R}$.

□

Spectral Permanence for C^*-Algebras

Proposition

Assume that A is a unital C^*-algebra, and B is a closed $*$-subalgebra containing 1_A.

1. *If $x \in B$ is invertible in A, then it is invertible in B.*
2. *For all $x \in B$, we have*

$$\mathrm{Sp}_B(x) = \mathrm{Sp}_A(x).$$

Proof.

- If $x \in B$ is invertible in A, then x^*x is in B and is invertible in A.
- As x^*x is selfadjoint, $\text{Sp}_B(x^*x) \subseteq \mathbb{R}$.
- In particular, $\text{Sp}_B(x^*x)$ has empty interior.
- Thus, x^*x is invertible in B, and hence $(x^*x)^{-1} \in B$.
- It then follows that $x^{-1} = (x^*x)^{-1}x^* \in B$.
- Using this it follows that, given any $x \in B$, we have

$$\lambda \notin \text{Sp}_A(x) \Longleftrightarrow x - \lambda \in A^{-1} \Longleftrightarrow x - \lambda \in B^{-1} \Longleftrightarrow \lambda \notin \text{Sp}_B(x).$$

- That is, $\mathbb{C} \setminus \text{Sp}_A(x) = \mathbb{C} \setminus \text{Sp}_B(x)$, i.e., $\text{Sp}_B(x) = \text{Sp}_A(x)$.

The proof is complete. □

Gel'fand Spectrum

Setup

A is a unital commutative C^*-algebra.

Definition

- A character of A is a linear map $\chi : A \to \mathbb{C}$ such that
$$\chi(xy) = \chi(x)\chi(y) \qquad \forall x, y \in A,$$
$$\chi(1_A) = 1.$$
- The set of characters of A is called the Gel'fand spectrum of A and is denoted $\operatorname{Sp} A$.

Remark

It can be shown that characters are in one-to-one correspondence with maximal ideals of A.

Gel'fand Spectrum

Example

Let $A = C(X)$, where X is a compact space.

- Every $x \in X$ defines a character of $C(X)$ by
$$\chi_x(f) = f(x), \qquad f \in C(X).$$
- The map $X \ni x \to \chi_x \in \mathrm{Sp}(A)$ is one-to-one.
- It can be shown to be onto.
- Therefore, the characters of $C(X)$ are in one-to-one correspondence with the points of X.

Gel'fand Spectrum

The relationship between Gel'fand spectrum and the spectra of the points of A is provided by the following result.

Proposition

For all $x \in A$, we have

$$\mathrm{Sp}_A(x) = \{\chi(x);\ \chi \in \mathrm{Sp}(A)\}.$$

Proof.

Set $S = \{\chi(x);\ \chi \in \mathrm{Sp}(A)\}$.

- If $\lambda \in \mathbb{C} \setminus \mathrm{Sp}_A(x)$ and $\chi \in \mathrm{Sp}\, A$, then
$$\chi\left((x-\lambda)^{-1}\right)(\chi(x) - \lambda) = \chi\left((x-\lambda)^{-1}(x-\lambda)\right) = \chi(1) = 1.$$
- In particular, $\chi(x) \neq \lambda$ for all $\chi \in \mathrm{Sp}(A)$, and so $\lambda \notin S$.
- Thus, $\mathbb{C} \setminus \mathrm{Sp}_A(x) \subseteq \mathbb{C} \setminus S$, i.e., $S \subseteq \mathrm{Sp}_A(x)$.
- It can be shown that $\mathrm{Sp}_A(x) \subseteq S$ (see Arveson, Thm. 1.9.5), and hence $S = \mathrm{Sp}_A(x)$. □

Corollary

If $\chi \in \mathrm{Sp}(A)$, *then*

$$\chi(x^*) = \overline{\chi(x)} \qquad \forall x \in A.$$

Proof.

Let $x \in A$.

- If $x^* = x$, then $\chi(x) \in \mathrm{Sp}_A(x) \subseteq \mathbb{R}$.
- In general, $x = x_1 + ix_2$, with $x_i^* = x_i$. Then

$$\chi(x) = \chi(x_1 + ix_2) = \chi(x_1) + i\chi(x_2),$$
$$\chi(x^*) = \chi(x_1 - ix_2) = \chi(x_1) - i\chi(x_2).$$

- As $\chi(x_1)$ and $\chi(x_2)$ are in $\mathbb{R}$, we see that $\chi(x^*) = \overline{\chi(x)}$. □

Gel'fand Spectrum

Setup

- $A^* =$ topological dual of A. Banach space with norm,

$$\|\varphi\| := \sup_{\|x\|=1} |\langle \varphi, x\rangle|, \qquad \varphi \in A^*.$$

- $\Omega =$ unit sphere A^*.
- By Banach-Alaoglu theorem Ω is compact with respect to the weak-$*$ topology, i.e., the topology of pointwise convergence.

Proposition

$\mathrm{Sp}\, A$ *is a closed subset of* Ω, *and hence is compact with respect to the weak-*$*$ *topology.*

Gel'fand Spectrum

Proof.

- Let $\chi \in A$. If $x \in A$, then $\chi(x) \in \mathrm{Sp}_A(x)$, and hence $|\chi(x)| \leq \|x\|$.
- This shows that $\chi \in A^*$ and $\|\chi\| \leq 1$.
- As $\chi(1) = 1$, we see that $\|\chi\| = 1$, i.e., $\chi \in \Omega$. Thus, $\mathrm{Sp}(A) \subseteq \Omega$.
- By definition $\mathrm{Sp}(A)$ is the intersection of the sets,

$$\{\varphi \in A^*;\ \varphi(1) = 1\}, \quad \{\varphi \in A^*;\ \varphi(x)\varphi(y) = \varphi(xy)\},\ x, y \in A.$$

- There are closed subsets of A^* w.r.t. the weak-$*$ topology.
- Therefore, $\mathrm{Sp}(A)$ is closed with respect to that topology. □

Gel'fand Transform

Definition

The Gel'fand transform of A is the map

$$G_A : A \longrightarrow C\left(\mathrm{Sp}(A)\right), \qquad x \longrightarrow \hat{x},$$

where

$$\hat{x}(\chi) = \chi(x), \qquad x \in A, \quad \chi \in \mathrm{Sp}(A).$$

Remark

G_A is an algebra homomorphism.

Theorem (Gel'fand-Naimark)

G_A is an (isometric) $*$-isomorphism from A onto $C(\mathrm{Sp}\, A)$.

Proof.

- Let $x \in A$. If $\chi \in \mathrm{Sp}(A)$, then
$$\overline{\hat{x}(\chi)} = \overline{\chi(x)} = \chi(x^*) = (x^*)^\wedge.$$
This shows that G_A is a $*$-homomorphism.
- We have
$$\mathrm{Sp}_A(x) = \{\chi(x);\ \chi \in \mathrm{Sp}(A)\} = \hat{x}\left(\mathrm{Sp}(A)\right) = \mathrm{Sp}_{C(\mathrm{Sp}\,A)}(\hat{x}).$$
- In particular, $\rho(x) = \rho(\hat{x})$ for all $x \in A$.
- Thus,
$$\|x\|^2 = \|x^*x\| = \rho(x^*x) = \rho\left(\widehat{x^*x}\right) = \rho\left(\bar{\hat{x}}\hat{x}\right) = \|\bar{\hat{x}}\hat{x}\| = \|\hat{x}\|^2.$$
It follows that G_A is isometric.
- It can be shown that $G_A(A)$ separates the points of $\mathrm{Sp}(A)$, and so $G_A(A) = C(\mathrm{Sp}(A))$ by Stone-Weierstrass theorem. □

Gel'fand Transform

Consequence

The Gel'fand transform provides a one-to-one correspondence,

$$
\begin{array}{ccc}
\{\text{Compact Hausdorff spaces}\} & & \{\text{Unital commutative } C^*\text{-algebras}\} \\
X & \longrightarrow & C(X) \\
\operatorname{Sp} A & \longleftarrow & A
\end{array}
$$

Consequence

We may regard unital C^*-algebras as the noncommutative analogue of compact spaces.

Gel'fand Transform

Remark

- The Gel'fand spectrum and Gel'fand transform make sense for any unital commutative Banach algebra A.
- The Gel'fand transform is an isometric isomorphism if and only if A is a C^*-algebra.
- It is an isometry if and only if $\|x^2\| = \|x\|^2$ for all $x \in A$.
- In general, the Gel'fand transform need not be one-to-one or onto.

Gel'fand Transform. Non-unital Case

Setup

A is a possibly non-unital C^*-algebra.

Definition

- A character of A is any non-zero algebra homomorphism $\chi : A \to \mathbb{C}$.
- The Gel'fand spectrum of A is the set of all characters of A. It is denoted $\mathrm{Sp}(A)$.

Remarks

- If A is unital and $\chi : A \to \mathbb{C}$ is an algebra homomorphism, then $\chi(1) = 1$.
- If A^+ is the unitalization of A, then any character $\chi : A \to \mathbb{C}$ uniquely extends to a character $\tilde{\chi} : A^+ \to \mathbb{C}$ by letting

$$\tilde{\chi}(x + \lambda \cdot 1) = \chi(x) + \lambda, \quad x \in A, \ \lambda \in \mathbb{C}.$$

Gel'fand Transform. Non-unital Case

Proposition

$\mathrm{Sp}(A)$ is contained in the (closed) unit ball B_{A^*} of A^* and is locally compact w.r.t. the weak-$*$ topology.

Proof.

- If $\chi \in A$, then $\|\chi\| \leq \|\tilde{\chi}\| = 1$. Thus $\mathrm{Sp}(A) \subseteq B_{A^*}$.
- We have $\mathrm{Sp}(A) = K \setminus 0$, where
$$K = \bigcap_{x,y\in A} \{\varphi \in B_{A^*}; \varphi(xy) = \varphi(x)\varphi(y)\}.$$
- B_{A^*} is compact w.r.t. the weak-$*$ topology.
- K is weak-$*$ closed, and hence is weak-$*$ compact.
- A basis of the weak $*$-topology of $\mathrm{Sp}(A)$ is given by the compact sets,
$$\{\chi \in K; |\chi(x)| \geq \epsilon\}, \quad x \in A \setminus 0, \ \epsilon > 0.$$
- Thus, $\mathrm{Sp}(A)$ is weak-$*$ locally compact. □

Gel'fand Transform. Non-unital Case

Definition

The Gel'fand transform of A is the map

$$G_A : A \longrightarrow C_0\left(\mathrm{Sp}(A)\right), \qquad x \longrightarrow \hat{x},$$

where

$$\hat{x}(\chi) = \chi(x), \qquad x \in A, \quad \chi \in \mathrm{Sp}(A).$$

Remark

- If $x \in A$, then $\hat{x} \in C_0(\mathrm{Sp}(A))$, since, for any $\epsilon > 0$,

$$\begin{aligned}\{\chi \in \mathrm{Sp}(A);\ |\hat{x}(\chi)| < \epsilon\} &= \{\chi \in \mathrm{Sp}(A);\ |\chi(x)| < \epsilon\} \\ &= \mathrm{Sp}(A) \setminus \{\chi \in \mathrm{Sp}(A);\ |\chi(x)| \geq \epsilon\}.\end{aligned}$$

- Here $\{\chi \in \mathrm{Sp}(A);\ |\chi(x)| \geq \epsilon\}$ is a compact set of $\mathrm{Sp}(A)$ (see previous slide).
- Thus, $|\hat{x}| < \epsilon$ outside some compact set.

Gel'fand Transform. Non-unital Case

Theorem (Gel'fand-Naimark)

G_A is an (isometric) $*$-isomorphism from A onto $C_0(\mathrm{Sp}\, A)$.

Consequence

We have a one-to-one correspondence,

$$\begin{array}{ccc} \{\text{Locally compact Hausdorff spaces}\} & & \{\text{Commutative } C^*\text{-algebras}\} \\ X & \longrightarrow & C_0(X) \\ \mathrm{Sp}\, A & \longleftarrow & A \end{array}$$

Continuous Functional Calculus

Setup

- $A =$ unital C^*-algebra.
- $x \in A$ is normal, i.e., $x^*x = xx^*$, and $S = \mathrm{Sp}_A(x)$.
- $\mathcal{P} = *$-algebra of polynomials $\sum c_{mn} z^m \bar{z}^n$.

Definition

If $f = \sum c_{mn} z^m \bar{z}^n \in \mathcal{P}$, we set

$$f(x) := \sum c_{mn} x^m (x^*)^n.$$

Facts

- $\mathcal{P} \ni f \to f(x) \in A$ is a $*$-homomorphism of algebras.
- Set $\mathcal{B}$ be its range. This is sub-$*$-algebra of A.
- Let B be the closure of $\mathcal{B}$ in A. This is a unital sub-C^*-algebra; this is the (unital) C^*-algebra generated by x.

Remark

- B is a unital commutative C^*-algebra.
- Therefore, its Gel'fand transform $G_B : B \to C(\mathrm{Sp}(B))$ is a $*$-isomorphism.

Lemma

Set $\xi = G_B(x)$. *Then:*

(i) $f(x) = G_B^{-1}(f \circ \xi)$ *for all* $f \in \mathcal{P}$.

(ii) $\xi(\mathrm{Sp}(B)) = S$.

Proof of (i).

- $G_B : B \to C(\mathrm{Sp}(B))$ is a $*$-isomorphism.
- Therefore, if $f = \sum c_{mn} z^m \bar{z}^n \in \mathcal{P}$, then

$$\begin{aligned} G_B\left(f(x)\right) &= \sum c_{mn} G\left(x^m (x^*)^n\right) \\ &= \sum c_{mn} G(x)^m \overline{G(x)}^n \\ &= \sum c_{mn} \xi^m \bar{\xi}^n = f \circ \xi. \end{aligned}$$

- Thus, $f(x) = G_B^{-1}(f \circ \xi)$.

□

Proof of (ii).

- We have

$$\xi(\mathrm{Sp}(B)) = \mathrm{Sp}_{C(\mathrm{Sp}(B))}(\xi) = \mathrm{Sp}_{C(\mathrm{Sp}(B))}(G_B(x))$$

- As $G_B : B \to C(\mathrm{Sp}(B))$ is an isomorphism, we have

$$\mathrm{Sp}_{C(\mathrm{Sp}(B))}(G(x)) = \mathrm{Sp}_B(x)$$

- By spectral permanence,

$$\mathrm{Sp}_B(x) = \mathrm{Sp}_A(x) = S.$$

- Thus,

$$\xi(\mathrm{Sp}(B)) = S.$$

The proof is complete. □

Continuous Functional Calculus

Definition

If $f \in C(S)$, we define

$$f(x) := G_B^{-1}(f \circ \xi).$$

Remark

If $f \in C(S)$, then $f \circ \xi$ is well defined, since $\xi(\mathrm{Sp}(B)) = S$.

Theorem (Continuous Functional Calculus)

1. *The map $\Phi : x \to f(x)$ is an isometric $*$-homomorphism from $C(S)$ to A, whose image is the C^*-algebra generated by x.*
2. *If $f = \sum c_{mn} z^m \bar{z}^n \in \mathcal{P}$, then*

$$\phi(f) = f(x) = \sum c_{mn} x^m (x^*)^n.$$

3. *For all $f \in C(S)$, we have*

$$\mathrm{Sp}_A f(x) = f(S).$$

Continuous Functional Calculus

Proof of (1)+(2).

- The 2nd part is immediate. In particular, $\mathcal{B} \subseteq \Phi(C(S))$.
- As G_B^{-1} maps onto B, we have $\Phi(C(S)) \subseteq B$.
- The map $\Psi : C(S) \ni f \to f \circ \xi \in C(\mathrm{Sp}(B))$ is a $*$-homomorphism.
- As $\xi(\mathrm{Sp}(B)) = S$, for any $f \in C(S)$, we have
$$\|f \circ \xi\|_{C(\mathrm{Sp}\, B)} = \sup_{\chi \in \mathrm{Sp}\, B} |f(\xi(\chi))| = \sup_{z \in S} |f(z)| = \|f\|_{C(S)}.$$
- Thus, Ψ is an isometric $*$-homomorphism.
- As $\Phi = G_B^{-1} \circ \Psi$, it follows that Φ an isometric $*$-homomorphism.
- In particular, the range of Φ is closed.
- As $\mathcal{B} \subseteq \Phi(C(S)) \subseteq B$ and $\mathcal{B}$ is dense in B, it follows that $\Phi(C(S)) = B$.

□

Proof of (3).

Let $f \in C(S)$. Need to show that $\mathrm{Sp}_A(f(x)) = f(S)$.

- We have $\mathrm{Sp}_{C(S)} f = f(S)$.
- Φ is a $*$-isomorphism from $C(S)$ onto B. Thus,

$$\mathrm{Sp}_B(f(x)) = \mathrm{Sp}_B(\Phi(f)) = \mathrm{Sp}_{C(S)} f = f(S).$$

- By spectral permanence,

$$\mathrm{Sp}_A(f(x)) = \mathrm{Sp}_B(f(x)) = f(S).$$

The proof is complete. □

Remark

- The map $C(S) \ni f \to f(x) \in A$ is a $*$-homorphism.
- Thus, if f is real-valued, then $f(x)$ is selfadjoint.

Remark

- The homomorphism $C(S) \ni f \to f(x) \in A$ is continuous.
- Thus, if $(f_k) \subseteq \mathcal{P}$ converges uniformly on S to f, then

$$f(x) = \lim_{k\to\infty} f_k(x).$$

- This gives an alternative definition of $f(x)$.