Introduction to Noncommutative Geometry
Chapter 12:
Quantized Calculus and Semiclassical Analysis on Quantum Tori

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Noncommutative Torus

Setup

- $\theta = (\theta_{jk})$ real anti-symmetric $n \times n$-matrix.

Definition

The noncommutative torus $\mathbb{T}^n_\theta$ is the NC space whose C^*-algebra $C(\mathbb{T}^n_\theta)$ is generated by unitaries $U_1, \ldots, U_n$ such that

$$U_k U_j = e^{2i\pi\theta_{jk}} U_j U_k.$$

Remarks

1. For $\theta = 0$ we get the C^*-algebra $C(\mathbb{T}^n)$, where $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ is the ordinary torus.
2. A dense basis of $C(\mathbb{T}^n_\theta)$ is given by the monomials,

$$U^m := U_1^{m_1} \cdots U_n^{m_n}, \qquad m = (m_1, \ldots, m_n) \in \mathbb{Z}^n.$$

Definition

$\tau : C(\mathbb{T}^n_\theta) \to \mathbb{C}$ is the faithful positive trace defined by

$$\tau(1) = 1, \qquad \tau\left(U^m\right) = 0, \quad m \neq 0.$$

Definition

$L^2(\mathbb{T}^n_\theta)$ is the Hilbert space completion with respect to the pre-inner product,

$$\langle u|v\rangle := \tau(v^* u), \qquad u, v \in C(\mathbb{T}^n_\theta).$$

Proposition (GNS Representation)

The action of $C(\mathbb{T}^n_\theta)$ on itself by left-multiplication extends to a $*$-representation in $L^2(\mathbb{T}^n_\theta)$.

Smooth Structure of $\mathbb{T}^n_\theta$

Definition

1. The canonical derivations $\partial_1, \ldots, \partial_n$ are given by

$$\partial_j(U_j) = iU_j, \qquad \partial_j(U_k) = 0, \quad k \neq j.$$

2. The smooth noncommutative torus is

$$C^\infty(\mathbb{T}^n_\theta) := \Big\{ u = \sum_{m \in \mathbb{Z}^n} u_m U^m, \ (u_m) \in \mathcal{S}(\mathbb{Z}^n) \Big\}.$$

Definition

The Laplacian $\Delta : C^\infty(\mathbb{T}^n_\theta) \to C^\infty(\mathbb{T}^n_\theta)$ is given by

$$\Delta := -(\partial_1^2 + \cdots + \partial_n^2).$$

Semiclassical Weyl's Law on $\mathbb{T}^n_\theta$ – Flat Case

Conjecture (McDonald+RP JMP '22)

Given any $q > 0$, *for all* $V = V^* \in C^\infty(\mathbb{T}^n_\theta)$,

$$\lim_{h \to 0^+} h^n N^-\big(h^{2q}\Delta^q + V\big) = \tau\Big[(V_-)^{\frac{n}{2q}}\Big].$$

Remark

- The conjecture is proved for $q = 1$ and $n \geq 3$ by McDonald-Sukochev-Zanin as a consequence of their semiclassical Weyl's laws for spectral triples.
- Their approach does not to allow us to get a semiclassical Weyl's law for quantum 2-tori.

Semiclassical Weyl's Law on $\mathbb{T}^n_\theta$ – Flat Case

Proposition

$\left(C^\infty(\mathbb{T}^n_\theta), L^2(\mathbb{T}^n_\theta), \sqrt{\Delta}\right)$ *is an* n*-summable spectral triple.*

Lemma

Let $a \in C^\infty(\mathbb{T}^n_\theta)$*. As* $t \to 0^+$*, we have*

$$\mathrm{Tr}\left[ae^{-t\Delta}\right] = \pi^{\frac{n}{2}}\tau[a]t^{-\frac{n}{2}} + \mathrm{O}\left(t^{\frac{-(n-1)}{2}}e^{-\frac{\pi^2}{t}}\right).$$

As a consequence the conjecture with Ed McDonald is true:

Theorem (Semiclassical Weyl's Law; RP '25)

Given any $q > 0$*, for all* $V = V^* \in C(\mathbb{T}^n_\theta)$*,*

$$\lim_{h \to 0^+} h^n N^-\left(h^{2q}\Delta^q + V\right) = \tau\left[(V_-)^{\frac{n}{2q}}\right].$$

We also recover the version of Connes's trace theorem for quantum tori:

Theorem (McDonald-Sukochev-Zanin '19, RP '20)

For every $x \in C(\mathbb{T}^n_\theta)$, the operator $x\Delta^{-n/2}$ is strongly measurable, and

$$⨍ x\Delta^{-\frac{n}{2}} = c'(n)\tau[x], \qquad c'(n) := |\mathbb{B}^n|.$$

Definition

$L_p(\mathbb{T}^n_\theta)$, $p \in [1, \infty)$, is the completion of $C(\mathbb{T}^n_\theta)$ with the respect to the norm,

$$\|x\|_p := \left(\tau\left[|x|^p\right]\right)^{\frac{1}{p}}$$

Definition

$L_\infty(\mathbb{T}^n_\theta)$ is the von Neumann algebra generated by the unitaries $U^1, \cdots U^n$, i.e., the weak closure of $C(\mathbb{T}^n_\theta)$ in $\mathscr{L}(L^2(\mathbb{T}^n_\theta))$.

Remark

- The spaces $L_p(\mathbb{T}^n_\theta)$ are the NC L_p-spaces associated to $(L_\infty(\mathbb{T}^n_\theta), \tau)$.
- We have (continuous) inclusions,

$$L_1(\mathbb{T}^n_\theta) \supseteq L_p(\mathbb{T}^n_\theta) \supseteq L_\infty(\mathbb{T}^n_\theta) \supseteq C(\mathbb{T}^n_\theta) \supseteq C^\infty(\mathbb{T}^n_\theta).$$

Theorem (McDonald-RP '22)

Let $q > 0$ *and set* $p = n(2q)^{-1}$.

1. *If* $q < n/2$ *and* $x \in L_p(\mathbb{T}^n_\theta)$, *then* $\Delta^{-q/2} x \Delta^{-q/2} \in \mathcal{L}^{p,\infty}$, *and*
$$\|\Delta^{-q/2} x \Delta^{-q/2}\|_{p,\infty} \leq C_+(n,q)\|x\|_p.$$
2. *If* $q > n/2$ *and* $x \in L_1(\mathbb{T}^n)$, *then* $\Delta^{-q/2} x \Delta^{-q/2} \in \mathcal{L}^{p,\infty}$, *and*
$$\|\Delta^{-q/2} x \Delta^{-q/2}\|_{p,\infty} \leq C_-(n,q)\|x\|_1.$$
3. *If* $x \in L_r(\mathbb{T}^n)$, $r > 1$, *then* $\Delta^{-n/4} x \Delta^{-n/4} \in \mathcal{L}^{1,\infty}$, *and*
$$\|\Delta^{-n/4} x \Delta^{-n/4}\|_{1,\infty} \leq C_0(n,r)\|x\|_r.$$

The constants $C_\pm(n,q)$ *and* $C_0(n,r)$ *depend only on* n, q, r *(and not on* x*).*

Extension to L_p-Potentials

The Cwikel-type estimates allow us to extend the semiclassical Weyl's law to L_p-potentials:

Theorem (McDonald-RP '22, RP '25)

Let $q > 0$ *and* $V = V^* \in L_1(\mathbb{T}^n_\theta)$. *We have*

$$\lim_{h \to 0^+} h^n N^-\big(h^{2q}\Delta^q + V\big) = \tau\Big[(V_-)^{\frac{n}{2q}}\Big].$$

provided that one of the following holds:

- $q < n/2$ *and* $V \in L_{\frac{n}{2q}}(\mathbb{T}^n_\theta)$.
- $q > n/2$ *and* $V \in L_1(\mathbb{T}^n_\theta)$.
- $q = n/2$ *and* $V \in L_p(\mathbb{T}^n_\theta)$ *with* $p > 1$.

L_p-Integration Formula

The Cwikel-type estimates allow us to extend the integration formula to the L_p-setting:

Theorem (McDonald-RP '22)

For every $x \in L_p(\mathbb{T}^n_\theta)$, $p > 1$, the operator $\Delta^{-n/4} x \Delta^{-n/4}$ is strongly measurable, and

$$\fint \Delta^{-\frac{n}{4}} x \Delta^{-\frac{n}{4}} = c'(n)\tau[x].$$

Riemannian Metrics on Quantum Tori

Definition (Vector Fields on $\mathbb{T}_\theta^n$)

$\mathcal{X}_\theta$ is the free left $C^\infty(\mathbb{T}_\theta^n)$-module generated by the canonical derivations $\partial_1, \ldots, \partial_n$.

Definition (Rosenberg)

A Riemannian metric on $\mathbb{T}_\theta^n$ is given by a positive invertible matrix $g = (g_{ij}) \in M_n(C^\infty(\mathbb{T}_\theta^n))$ whose entries g_{ij} are self-adjoint.

Proposition

Any Riemannian metric $g = (g_{ij})$ defines a Hermitian metric on $\mathcal{X}_\theta$,

$$(X, Y)_g = \sum X_i g_{ij} Y_j^*, \quad X = \sum X_i \partial_i, \ Y = \sum Y_j \partial_j,$$

$$(\partial_i, \partial_j)_g = g_{ij} \text{ is selfadjoint.}$$

Examples

1. Flat metric: $g_{ij} = \delta_{ij}$.
2. Conformally Flat Metric (Connes-Tretkoff, Connes-Moscovici, Fathizadeh-Khalkhali):
$$g_{ij} = k^2 \delta_{ij}, \qquad k \in C^\infty(\mathbb{T}^n_\theta),\ k > 0.$$
3. Asymmetric Metrics (Dabrowski-Sitarz):
$$g = \begin{pmatrix} I_\ell & 0 \\ 0 & k^2 I_{n-\ell} \end{pmatrix}, \qquad k_i \in C^\infty(\mathbb{T}^n_\theta),\ k > 0.$$
4. Diagonal Metrics (Connes-Fathizadeh):
$$g = \begin{pmatrix} k_1^2 I_\ell & 0 \\ 0 & k_2^2 I_{n-\ell} \end{pmatrix}, \qquad k_i \in C^\infty(\mathbb{T}^n_\theta),\ k_i > 0.$$

Riemannian Volume

Definition

Let g be a Riemannian metric on $C^\infty(\mathbb{T}^n_\theta)$.

1. The Riemannian density of g is
$$\nu(g) := \sqrt{\det(g)} = \exp\big(\frac{1}{2}\,\mathrm{Tr}[\log(g)]\big).$$

2. The Riemannian weight of g is
$$\varphi_g(u) = (2\pi)^n\tau[u\nu(g)], \qquad u \in C(\mathbb{T}^n_\theta).$$

3. The Riemannian volume with respect to g is
$$\mathrm{Vol}_g(\mathbb{T}^n_\theta) := \varphi_g(1) = (2\pi)^n\tau\left[\nu(g)\right].$$

Definition

$L^2_g(\mathbb{T}^n_\theta)$ is the completion of $C(\mathbb{T}^n_\theta)$ w.r.t. the inner product,

$$\langle u|v\rangle_g := \tau\left[uv^*\nu(g)\right].$$

Remarks

1. $L^2_g(\mathbb{T}^n_\theta)$ is the GNS space of the opposite algebra $C(\mathbb{T}^n_\theta)^{\mathrm{op}}$ for the Riemannian weight φ_g.
2. $C(\mathbb{T}^n_\theta)$ is $*$-represented in $L^2_g(\mathbb{T}^n_\theta)$ by

$$a \longrightarrow a^\circ := \nu(g)^{-\frac{1}{2}} a\nu(g)^{\frac{1}{2}}.$$

Here $\boldsymbol{\Delta}(u) = \nu(g)^{-1}u\nu(g)$ is the modular operator.

Laplace-Beltrami Operator

Definition (Ha+RP JGP '20)

Let $g = (g_{ij})$ be a Riemannian metric with inverse $g^{-1} = (g^{ij})$.
The Laplace-Beltrami operator $\Delta_g : C^\infty(\mathbb{T}^n_\theta) \to C^\infty(\mathbb{T}^n_\theta)$ is

$$\Delta_g u = \frac{-1}{\nu(g)} \sum \partial_i \Big(\nu(g)^{\frac{1}{2}} g^{ij} \nu(g)^{\frac{1}{2}} \partial_j(u) \Big).$$

Remarks

1. On (M^n, g) we have $\Delta_g u = \frac{-1}{\sqrt{\det(g)}} \sum \partial_{x_i}\big(\sqrt{\det(g)} g^{ij} \partial_{x_j} u\big)$.
2. $\Delta_g = d^* d$, where d is the (analogue of) de Rham differential.

Examples

1. Flat Metric $g_{ij} = \delta_{ij}$:

$$\Delta_g = \Delta, \qquad \Delta := -(\partial_1^2 + \cdots + \partial_n^2).$$

2. Conformally Flat Metric $g_{ij} = k^2\delta_{ij}$, $k \in C^\infty(\mathbb{T}_\theta^n)$, $k > 0$:

$$\Delta_g u = k^{-2}\Delta u - \sum k^{-n}\partial_i(k^{n-2})\partial_i(u).$$

In particular, when $n = 2$ we get

$$\Delta_g = k^{-2}\Delta.$$

Elliptic Theory and Spectral Theory

Proposition (Ha + RP JGP '20)

1. Δ_g *is an elliptic differential operator on* $\mathbb{T}^n_\theta$ *with* $\ker \Delta_g = \mathbb{C}$.
2. Δ_g *is essentially selfadjoint and positive on* $L^2_g(\mathbb{T}^n_\theta)$.
3. $\mathrm{Sp}(\Delta_g)$ *is discrete and consists of non-negative eigenvalues with finite multiplicity.*

Pseudodifferential Operators

Definition

A symbol of order $q \in \mathbb{C}$ is a map $\sigma(\xi) \in C^\infty(\mathbb{R}^n; C^\infty(\mathbb{T}^n_\theta))$ with an asymptotic expansion,

$$\sigma(\xi) \simeq \sum_{j \geq 0} \sigma_{q-j}(\xi), \qquad \sigma_{q-j}(\lambda\xi) = \lambda^{q-j}\sigma_{q-j}(\xi).$$

Definition (Connes)

$\Psi^q(\mathbb{T}^n_\theta)$, $q \in \mathbb{C}$, consists of operators $P_\sigma : C^\infty(\mathbb{T}^n_\theta) \to C^\infty(\mathbb{T}^n_\theta)$ s.t.

$$P_\sigma u = \sum u_m \sigma(m) U^m, \quad u = \sum u_m U^m.$$

Example

If $P = \sum a_\alpha \partial^\alpha$ ($a_\alpha \in C^\infty(\mathbb{T}^n_\theta)$) is a differential operator, then $P = P_\sigma$ with $\sigma(\xi) = \sum a_\alpha (i\xi)^\alpha$.

Weyl Laws for Negative Order ΨDOs

Conjecture (McDonald+RP '23)

Let $P = P^* \in \Psi^{-q}(\mathbb{T}^n_\theta)$, $q > 0$, *have principal symbol* $\sigma_{-q}(\xi)$. *Then*

$$\lim_{j\to\infty} j^{\frac{q}{n}} \lambda_j^{\pm}(P) = \left(\int_{\mathbb{S}^{n-1}} \tau\left[\sigma_{-q}(\xi)_{\pm}^{\frac{n}{q}}\right] d\xi \right)^{\frac{q}{n}}.$$

Remarks

1. This is the full version for quantum tori of the Weyl's laws for negative order ΨDOs of Birman-Solomyak.
2. A version of this result for singular values was established by Sukochev-Xiao-Zanin (JFA '23) for $q = 1$ and a smaller class of ΨDOs.

Weyl Laws for Negative Order ΨDOs

Proposition

$(\Psi^0(\mathbb{T}^n), L^2(\mathbb{T}^n_\theta), \sqrt{\Delta})$ *is an* n*-summable spectral triple.*

Proposition (RP)

Let $A \in \Psi^0(\mathbb{T}^n_\theta)$ *have principal symbol* $\sigma_0(\xi)$. *Then*

$$\operatorname{Res}_{z=n} \operatorname{Tr}\left[A\Delta^{-\frac{z}{2}}\right] = \int_{\mathbb{S}^n} \tau\left[\sigma_0(\xi)\right] d\xi \quad \text{(simple pole)}.$$

Weyl Laws for Negative Order ΨDOs

We thus can apply the first part of the main result to get a version of Birman-Solomyak's Weyl's Law for quantum tori:

Theorem (RP '25)

1. *Let* $A = A^* \in \Psi^0(\mathbb{T}^n_\theta)$ *have principal symbol* $\sigma_0(\xi)$. *For all* $q > 0$, *we have*

$$\lim_{j\to\infty} j^{\frac{2q}{n}} \lambda_j^{\pm}(\Delta^{-\frac{q}{2}} A \Delta^{-\frac{q}{2}}) = \left(\int_{\mathbb{S}^{n-1}} \tau\left[\sigma_0(\xi)_{\pm}^{\frac{n}{2q}}\right] d\xi \right)^{\frac{2q}{n}}.$$

2. *Let* $P = P^* \in \Psi^{-q}(M)$, $q > 0$, *have principal symbol* $\sigma_{-q}(\xi)$. *Then:*

$$\lim_{j\to\infty} j^{\frac{q}{n}} \lambda_j^{\pm}(P) = \left(\int_{\mathbb{S}^{n-1}} \tau\left[\sigma_{-q}(\xi)_{\pm}^{\frac{n}{q}}\right] d\xi \right)^{\frac{q}{n}}.$$

This shows that the 2nd conjecture with Ed McDonald is true.

Semiclassical Weyl's Law on $\mathbb{T}^n_\theta$ – Curved Case

Applying the previous result to $P = \Delta_g^{-q/2} V^\circ \Delta_g^{-q/2}$ yields:

Proposition

For all $V = V^* \in C^\infty(\mathbb{T}^n_\theta)$ *and* $q > 0$, *we have*

$$\lim_{j\to\infty} j^{\frac{2q}{n}} \lambda_j^{\pm}(\Delta_g^{-\frac{q}{2}} V^\circ \Delta_g^{-\frac{q}{2}}) = \left(\int_{\mathbb{S}^{n-1}} \tau \left[\left(|\xi|_g^{-q} V |\xi|_g^{-q} \right)_{\pm}^{\frac{n}{2q}} \right] d\xi \right)^{\frac{2q}{n}}.$$

where $|\xi|_g^2 := \sum g^{ij} \xi_i \xi_j$

By applying the Birman-Schwinger principle we then get a semiclassical Weyl's law for curved quantum tori.

Theorem (Semiclassical Weyl's Law; RP '25)

For all $V = V^* \in C^\infty(\mathbb{T}^n_\theta)$ *and* $q > 0$, *we have*

$$\lim_{h\to 0^+} h^n N^-\left(h^{2q}\Delta_g^{q} + V^\circ\right) = \int_{\mathbb{S}^{n-1}} \tau \left[\left(|\xi|_g^{-q} V |\xi|_g^{-q} \right)_{\pm}^{\frac{n}{q}} \right] d\xi.$$

Example

For $g_{ij} = k^2\delta_{ij}$, $k \in C^\infty(\mathbb{T}_\theta^n)$, $k > 0$, then the semiclassical Weyl's law becomes

$$\lim_{h\to 0^+} h^n N^-\left(h^{2q}\Delta_g^q + V^\circ\right) = c(n)\tau\left[(kVk)_-^{\frac{n}{q}}\right].$$

Definition

1. The spectral Riemannian density is
$$\tilde{\nu}(g) := \frac{1}{|\mathbb{S}^{n-1}|}\int_{|\xi|=1} |\xi|_g^{-n} d\xi, \qquad |\xi|_g^2 = \sum \xi_i g^{ij} \xi_j.$$
2. The spectral volume is
$$\widetilde{\mathrm{Vol}}_g(\mathbb{T}^n_\theta) = (2\pi)^n \tau\left[\tilde{\nu}(g)\right].$$

Remark

If $[g_{ij}, g_{kl}] = 0$, then $\tilde{\nu}(g) = \nu(g) = \sqrt{\det(g)}$.

Theorem (Weyl's Law; Lee-RP)

As $j \to \infty$, we have
$$\lambda_j(\Delta_g) \sim \left(\frac{j}{c'(n)\widetilde{\mathrm{Vol}}_g(\mathbb{T}^n_\theta)}\right)^{\frac{2}{n}}.$$

Theorem (RP '20, McDonald-RP '23)

1. *For every* $x \in C(\mathbb{T}^n_\theta)$*, the operator* $x^\circ \Delta_g^{-n/2}$ *is strongly measurable, and*

$$⨍ x^\circ \Delta_g^{-\frac{n}{2}} = c'(n)\tau\big[x\tilde{\nu}(g)\big].$$

2. *For every* $x \in L_p(\mathbb{T}^n_\theta)$*,* $p > 1$*, the operator* $\Delta_g^{-n/4} x^\circ \Delta_g^{-n/4}$ *is strongly measurable, and*

$$⨍ \Delta_g^{-\frac{n}{4}} x^\circ \Delta_g^{-\frac{n}{4}} = c'(n)\tau\big[x\tilde{\nu}(g)\big].$$