Introduction to Noncommutative Geometry
Spectral Triples and Dirac Operators

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Spectral Triples

Definition (Baaj-Julg, Connes)

A spectral triple $(\mathscr{A}, \mathscr{H}, D)$ consists of

- A Hilbert space $\mathscr{H}$.
- A unital $*$-subalgebra $\mathscr{A} \subseteq \mathscr{L}(\mathscr{H})$.
- A selfadjoint operator D on $\mathcal{H}$ such that:
  - (i) It has compact resolvent, i.e., $(D+i)^{-1}$ is a compact operator.
  - (ii) $a(\mathrm{dom}(D)) \subseteq \mathrm{dom}(D)$ and $[D, a] \in \mathcal{L}(\mathcal{H})$ for all $a \in \mathcal{A}$.

Remark

- The compactness of the resolvent of D and its selfadjointness ensure that the spectrum of D consists of isolated real eigenvalues with finite multiplicity.
- In particular, $\dim \ker D < \infty$, and so D is a Fredholm operator.

Spectral Triples

Definition

A spectral triple $(\mathscr{A}, \mathscr{H}, D)$ is p-summable with $p > 0$ if $|D|^{-1} \in \mathcal{L}^{p,\infty}$, i.e., $\lambda_j(|D|^{-1}) = \mathrm{O}(j^{-1/p})$.

Remark

- p-summability implies that $\mathrm{Tr}[|D|^{-q}] < \infty$ for all $q > p$.
- Degree of summability $\simeq$ dimension of $(\mathscr{A}, \mathscr{H}, D)$.

Spectral Triples – Dirichlet/Neumann Examples

Setup

$\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, bounded domain with smooth boundary.

Example

$$\left(C_c^\infty(\Omega) \oplus \mathbb{C}, L^2(\Omega), \sqrt{\Delta} \right),$$

where Δ is the (positive) Dirichlet/Neumann Laplacian.

Reminder

- The Dirichlet problem is the boundary value problem,

$$\Delta u = v \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

- The Neumann problem is the boundary value problem,

$$\Delta u = v \quad \text{on } \Omega, \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega,$$

  where ∂_ν is the normal derivative.

Spectral Triples – Riemannian Examples

Setup

(M^n, g) closed Riemannian manifold.

Example

$$\left(C^\infty(M), L^2(M), \sqrt{\Delta_g}\right),$$

where $\Delta_g = d^*d$ is the Laplacian of (M, g)

Example (Pseudodifferential Operators)

$$\left(\Psi^0(M), L^2(M), \sqrt{\Delta_g}\right),$$

where $\Psi^0(M) =$ algebra of pseudodifferential operators of order 0.

Remark

These spectral triples are n-summable.

Spectral Triples – Dirichlet-to-Neumann

Example (Dirichlet to Neumann)

If $M = \partial X$ and $\Lambda_g =$ Dirichlet to Neumann operator, then

$$\left(C^\infty(M), L^2(M), \Lambda_g\right)$$

is a spectral triple.

Remark

If $u \in C^\infty(M)$, then by definition

$$\Lambda_g u = \partial_\nu \tilde{u} \qquad \text{on } M = \partial X,$$

where $\tilde{u}$ is the unique harmonic extension of u to X.

Even Spectral Triples

Definition

A spectral $(\mathcal{A}, \mathcal{H}, D)$ is called an even spectral triple if there is a $\mathbb{Z}_2$-grading operator $\gamma \in \mathcal{L}(\mathcal{H})$ such that

$$\gamma^2 = -1, \qquad \gamma^* = \gamma, \qquad D\gamma = -\gamma D, \qquad a\gamma = \gamma a \quad \forall a \in \mathcal{A}.$$

Remark

- Equivalently, there is an orthogonal splitting $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ s.t.

$$\gamma_{|\mathcal{H}^\pm} = \mathrm{id}_{\mathcal{H}^\pm}, \quad D\big(\mathrm{dom}(D) \cap \mathcal{H}^\pm\big) \subseteq \mathcal{H}^\mp, \quad a(\mathcal{H}^\pm) \subseteq \mathcal{H}^\pm.$$

- With respect to this splitting D takes the form,

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \qquad D^\pm : \mathrm{dom}(D) \cap \mathcal{H}^\pm \to \mathcal{H}^\mp.$$

- The selfadjointness of D then means that $(D^\pm)^* = D^\mp$.

Fredholm Index

Remark

Let $(\mathcal{A}, \mathcal{H}, D)$ be an even spectral triple.

- With respect to the splitting $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ the operator D takes the form,

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \qquad D^- = (D^+)^*.$$

- We then have

$$\ker D = \ker D^+ \oplus \ker D^-.$$

Definition

The (Fredholm) index of D is

$$\operatorname{ind}(D) := \dim \ker D^+ - \dim \ker D^-.$$

Setup

- M^n is a compact oriented Riemannian manifold (n even).
- $d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$ is the de Rham differential with adjoint d^*.

Remark

$$\Lambda^* T^*M = \Lambda^{\mathrm{ev}} T^*M \oplus \Lambda^{\mathrm{odd}} T^*M.$$

Proposition

The following is an even spectral triple,

$$\left(C^\infty(M), L^2(M, \Lambda^* T^*M), d + d^*\right),$$

with $L^2(M, \Lambda^* T^*M) = L^2(M, \Lambda^{\mathrm{ev}} T^*M) \oplus L^2(M, \Lambda^{\mathrm{odd}} T^*M)$.

Remark

We have

$$\operatorname{ind}(d+d^*) := \dim\ker\left[(d+d^*)_{|\Lambda^{\mathrm{ev}}}\right] - \dim\ker\left[(d+d^*)_{|\Lambda^{\mathrm{odd}}}\right].$$

Definition (Euler Characteristic $\chi(M)$)

$$\chi(M) := \sum_{k=0}^{n} (-1)^k \dim H^k(M),$$

where $H^k(M)$ is the de Rham cohomology of M.

Chern-Gauss-Bonnet Theorem

Theorem (Chern-Gauss-Bonnet)

We have

$$\chi(M) = \operatorname{ind}(d + d^*) = (2i\pi)^{-\frac{n}{2}} \int_M \operatorname{Pf}\left(R^M\right),$$

where $\operatorname{Pf}(R^M)$ *is the* *Pfaffian* *of the curvature* R^M *of* M*.*

Remark

In particular, for $n = 2$ we recover the Gauss-Bonnet theorem,

$$\chi(M) = \frac{1}{2\pi} \int_M \kappa_g(x) dv_g(x),$$

where $\kappa_g(x)$ is the scalar curvature of M.

Signature Spectral Triple

Setup

- (M^n, g) compact oriented Riemannian manifold (n even).

Definition (Hodge Operator)

The operator $\star : \Lambda^k T^*M \to \Lambda^{n-k} T^*M$ is defined by

$$\star\alpha \wedge \beta = \langle \alpha, \beta \rangle \, \mathrm{Vol}_g(x) \quad \forall \alpha, \beta \in \Lambda^k T^*_x M,$$

where $\mathrm{Vol}_g(x)$ is the volume form of M.

Remark

As $\star^2 = 1$, there is a splitting

$$\Lambda^* T^* M = \Lambda^+ \oplus \Lambda^-, \quad \text{with } \Lambda^\pm := \{\alpha;\ \star\alpha = \pm\alpha\}.$$

Signature Spectral Triple

Proposition

The following is an even spectral triple,

$$\left(C^\infty(M), L^2(M, \Lambda^* T^* M), d - \star d \star\right),$$

with $L^2(M, \Lambda^* T^* M) = L^2(M, \Lambda^+) \oplus L^2(M, \Lambda^-)$.

Remark

We have

$$\mathrm{ind}(d - \star d \star) := \dim \ker \left[(d - \star d \star)_{|\Lambda^+} \right] - \dim \ker \left[(d - \star d \star)_{|\Lambda^-} \right].$$

Definition (Signature $\sigma(M)$)

If $n = 4p$, then $\sigma(M)$ of M is the signature of the bilinear form,

$$H^{\frac{n}{2}}(M) \times H^{\frac{n}{2}}(M) \ni (\alpha, \beta) \to \int_M \alpha \wedge \beta.$$

Hirzebruch Signature Theorem

Theorem (Hirzebruch)

We have

$$\begin{aligned} \sigma(M) &= \operatorname{ind}(d - \star d \star) \quad \textit{if } n = 4p, \\ &= (i\pi)^{-\frac{n}{2}} \int_M L\left(R^M\right), \end{aligned}$$

where $L\left(R^M\right) := \det^{\frac{1}{2}}\left[\frac{R^M/2}{\tanh\left(R^M/2\right)}\right]$ *is the* L*-form of the curvature* R^M.

Dolbeault Spectral Triple

Setup

- M^n compact Kälher manifold ($n =$ complex dimension).
- $\Lambda^{0,q}T^*M := \text{Span}\left\{d\overline{z_{k_1}} \wedge \cdots \wedge d\overline{z_{k_q}}\right\}$ is the bundle of anti-holomorphic q-forms.
- $\overline{\partial} : C^\infty(M, \Lambda^{0,q}T^*M) \to C^\infty(M, \Lambda^{0,q+1}T^*M)$ is the Dolbeault differential with adjoint $\overline{\partial}^*$.

Remark

$$\Lambda^{0,*}T^*M = \Lambda^{0,\text{ev}}T^*M \oplus \Lambda^{0,\text{odd}}T^*M.$$

Dolbeault Spectral Triple

Proposition

The following is an even spectral triple,

$$\left(C^\infty(M), L^2\left(M, \Lambda^{0,*}T^*M\right), \overline{\partial} + \overline{\partial}^*\right),$$

with $L^2\left(M, \Lambda^{0,*}T^*M\right) = L^2\left(M, \Lambda^{0,\mathrm{ev}}T^*M\right) \oplus L^2\left(M, \Lambda^{0,\mathrm{odd}}T^*M\right)$.

Remark

We have

$$\mathrm{ind}(\overline{\partial} + \overline{\partial}^*) := \dim\ker\left[(\overline{\partial} + \overline{\partial}^*)_{|\Lambda^{0,\mathrm{ev}}}\right] - \dim\ker\left[(\overline{\partial} + \overline{\partial}^*)_{|\Lambda^{0,\mathrm{odd}}}\right].$$

Definition (Holomorphic Euler Characteristic)

$$\chi(M) := \sum_{q=0}^{n} (-1)^q \dim H^{0,q}(M),$$

where $H^{0,q}(M)$ is the Dolbeault cohomology of M.

Hirzebruch-Riemann-Roch Theorem

Theorem (Hirzebruch-Riemann-Roch)

We have

$$\chi(M) = \text{ind}\left(\overline{\partial} + \overline{\partial}^*\right) = (2i\pi)^{-\frac{n}{2}} \int_M \text{Td}\left(R^{1,0}\right),$$

where $\text{Td}\left(R^{1,0}\right) := \det\left[\frac{R^{1,0}}{e^{R^{1,0}}-1}\right]$ *is the Todd form of the holomorphic curvature* $R^{1,0}$ *of* M.

The Dirac Operator

Fact

On $\mathbb{R}^n$ the square root $\sqrt{\Delta}$ is a ΨDO, but not a differential operator.

Dirac's Idea

Seek for a square root of Δ as a differential operator with matrix coefficients,

$$\not{D} = \sum c^j \partial_j.$$

Dirac Operator on $\mathbb{T}^n$

Setup

- n is even and $N = 2^{[n/2]}$.
- $\gamma_1, \ldots, \gamma_n$ are skew-adjoint matrices in $M_N(\mathbb{C})$ satisfying the Clifford relations,

$$\gamma_j\gamma_k + \gamma_k\gamma_j = -2\delta_{jk}, \qquad j, k = 1, \ldots, n.$$

- $L^2(\mathbb{T}^n; \mathbb{C}^N)$ is a Hilbert space with inner-product,

$$\langle u|v\rangle = (2\pi)^{-n} \int_{\mathbb{T}^n} (u(x)|v(x))\, dx,$$

where $(\cdot|\cdot)$ is the standard Hermitian inner-product of $\mathbb{C}^N$.

Remark

The Clifford relations mean that

$$\gamma_j^2 = -1, \qquad \gamma_j\gamma_k = -\gamma_k\gamma_j, \quad j \neq k.$$

Dirac Operator on $\mathbb{T}^n$

Definition

The Dirac operator $\not{D} : C^\infty(\mathbb{T}^n; \mathbb{C}^N) \to C^\infty(\mathbb{T}^n; \mathbb{C}^N)$ is defined by

$$\not{D} := \gamma_1 \partial_{x_1} + \cdots + \gamma_n \partial_{x_n}.$$

Proposition

1. *We have*
$$\not{D}^2 = \Delta,$$
where $\Delta := -(\partial_{x_1}^2 + \cdots + \partial_{x_n}^2)$ *is the (positive) Laplacian.*
2. *The Dirac operator* $\not{D}$ *is a (formally) selfadjoint first order elliptic differential operator.*

Dirac Operator on $\mathbb{T}^n$

Definition

Assume n is even. The chirality operator is

$$\gamma := i^{\frac{n(n-1)}{2}} \gamma_1 \cdots \gamma_n.$$

Lemma

Assume n is even. We have

$$\gamma^2 = 1, \qquad \gamma^* = \gamma, \qquad D\!\!\!/\,\gamma = -\gamma D\!\!\!/.$$

Remark

- We then get splitting $\mathbb{C}^N = \$^+ \oplus \$^-$, where
$$\$^\pm := \{\xi \in \mathbb{C}^N;\ \gamma\xi = \pm\xi\}.$$
- This yields an orthogonal splitting,
$$L^2(\mathbb{T}^n; \mathbb{C}^N) = L^2(\mathbb{T}^n; \$^+) \oplus L^2(\mathbb{T}^n; \$^-).$$

Dirac Spectral Triple for $\mathbb{T}^n$

Proposition

1. *The following is a spectral triple,*
$$\left(C^\infty(\mathbb{T}^n), L^2(\mathbb{T}^n;\mathbb{C}^N), \not{D}\right),$$
2. *If n is even, then this is an even spectral triple, with*
$$L^2(\mathbb{T}^n;\mathbb{C}^N) = L^2(\mathbb{T}^n;\not{S}^+) \oplus L^2(\mathbb{T}^n;\not{S}^-).$$

Remark

- $\ker \not{D} = (\ker \Delta) \otimes \mathbb{C}^N$ consists of constant maps $u : \mathbb{T}^n \to \mathbb{C}^N$.
- If n is even, then $\ker \not{D}^\pm \simeq \not{S}^\pm$.
- As $\dim \not{S}^- = \dim \not{S}^+$, we then get
$$\operatorname{ind} \not{D} = \dim \ker \not{D}^+ - \dim \ker \not{D}^- = \dim \not{S}^+ - \dim \not{S}^- = 0.$$

Definition

The Clifford algebra of $\mathbb{R}^n$ is the $\mathbb{C}$-algebra $\mathrm{Cl}(\mathbb{R}^n)$ generated by the canonical basis vectors $e^1, \ldots, e^n$ of $\mathbb{R}^n$ with relations,

$$e^i e^j + e^j e^i = -2\delta^{ij}.$$

Remark

Any Euclidean space $(V, \langle \cdot, \cdot \rangle)$ defines a Clifford algebra.

Quantization Map

Denote by $\Lambda^\bullet_{\mathbb{C}}\mathbb{R}^n$ the complexified exterior algebra of $\mathbb{R}^n$.

Proposition

There is a linear isomorphism $c : \Lambda^\bullet_{\mathbb{C}}\mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^n)$ *given by*

$$c(e^{i_1} \wedge \cdots \wedge e^{i_k}) = e^{i_1} \cdots e^{i_k}, \qquad 1 \leq i_1 < \cdots < i_k \leq n.$$

Remark

This is not an isomorphism of algebras, e.g., for all $\xi, \eta \in \mathbb{R}^n$, we have $c^{-1}(c(\xi)c(\eta)) = \xi \wedge \eta - \langle \xi, \eta \rangle$.

Corollary

There is a $\mathbb{Z}_2$-grading,

$$\mathrm{Cl}(\mathbb{R}^n) = \mathrm{Cl}^+(\mathbb{R}^n) \oplus \mathrm{Cl}^-(\mathbb{R}^n), \quad \mathrm{Cl}^{\pm}(\mathbb{R}^n) := c(\Lambda_{\mathbb{C}}^{\mathrm{ev}/odd}\mathbb{R}^n).$$

Remark

$\mathrm{Cl}^+(\mathbb{R}^n)$ is a subalgebra of $\mathrm{Cl}(\mathbb{R}^n)$.

Spinor Representation

Theorem

1. $\mathrm{Cl}(\mathbb{R}^n)$ *has a unique irreducible representation,*

$$\rho : \mathrm{Cl}(\mathbb{R}^n) \to \mathrm{End}(\$_n),$$

where $\$_n$ *is the space of spinors of* $\mathbb{R}^n$*.*

2. *If* n *is even, then this rise to an algebra-isomorphism,*

$$\mathrm{Cl}(\mathbb{R}^n) \simeq \mathrm{End}(\$_n).$$

3. *If* n *is odd, then we get an algebra-isomorphism,*

$$\mathrm{Cl}^+(\mathbb{R}^n) \simeq \mathrm{End}(\$_n).$$

Remark

- If n is even, then, as $\mathrm{Cl}(\mathbb{R}^n) \simeq \mathrm{End}(\$_n)$, we have
$$\dim \mathrm{Cl}(\mathbb{R}^n) = \dim \mathrm{End}(\$_n) = (\dim \$_n)^2 .$$
- As $\dim \mathrm{Cl}(\mathbb{R}^n) = 2^n$, we deduce that
$$\dim \$_n = \big(\dim \mathrm{Cl}(\mathbb{R}^n)\big)^{\frac{1}{2}} = 2^{\frac{n}{2}}.$$
- If n is odd, then, as $\mathrm{End}(\$_n) \simeq \mathrm{Cl}^+(\mathbb{R}^n)$, we have
$$\dim \$_n = \big(\dim \mathrm{Cl}^+(\mathbb{R}^n)\big)^{\frac{1}{2}} = 2^{\frac{n-1}{2}}.$$

Chirality Operator

Definition

Assume n is even. The chirality operator is

$$\gamma := i^{\frac{1}{2}n(n-1)} c(e^1) \cdots c(e^n).$$

Proposition

1. *The chirality operator defines a $\mathbb{Z}_2$-grading,*

$$\slashed{S}_n = \slashed{S}_n^+ \oplus \slashed{S}_n, \qquad \slashed{S}_n := \{\xi \in \slashed{S}_n;\ \gamma\xi = \pm\xi\}.$$

2. *This $\mathbb{Z}_2$-grading is preserved by the action of $\mathrm{Cl}^+(\mathbb{R}^n)$.*
3. *The action of $\mathrm{Cl}^-(\mathbb{R}^n)$ maps $\slashed{S}_n^\pm$ to $\slashed{S}_n^\mp$.*

Spin Group $\mathrm{Spin}(n)$

Definition

The spin group $\mathrm{Spin}(n)$ is the double cover of $\mathrm{SO}(n)$,

$$\{\pm 1\} \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to \{1\}.$$

Remark

The spin group $\mathrm{Spin}(n)$ can be realized as the Lie group of some Lie algebra contained in $\mathrm{Cl}^{+}(\mathbb{R}^{n})$.

Proposition

The spinor representation splits into the half-spin representations,

$$\rho_{\pm} : \mathrm{Spin}(n) \longrightarrow \mathrm{End}(\$_{n}^{\pm}).$$

The Dirac Operator

Setup

(M^n, g) is an oriented Riemannian manifold.

Definition

The Clifford bundle of M is the bundle of algebras,

$$\mathrm{Cl}(M) = \bigsqcup_{x \in M} \mathrm{Cl}(T_x^* M),$$

where $\mathrm{Cl}(T_x^* M)$ is the Clifford algebra of $(T_x^* M, g^{-1})$.

The Dirac Operator

Remarks

- There is a quantization map,

$$c : \Lambda^{\bullet}_{\mathbb{C}} T^*M \longrightarrow \mathrm{Cl}(M).$$

- This an isomorphism of vector bundles, but not an isomorphism of algebra bundles.
- There is also a splitting,

$$\mathrm{Cl}(M) = \mathrm{Cl}^{+}(M) \oplus \mathrm{Cl}^{-}(M), \quad \mathrm{Cl}^{\pm}(M) = c\left(\Lambda^{\mathrm{ev/odd}} T^*_{\mathbb{C}} M\right).$$

- Here $\mathrm{Cl}^{+}(M)$ is a sub-bundle of algebras of $\mathrm{Cl}(M)$.

Spin Structure

Definition

A spin structure on M is a reduction of its structure group from $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$.

Remark

This means there is a $\mathrm{Spin}(n)$-principal bundle such that T^*M is isomorphic to the associated bundle $\mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \mathbb{R}^n$.

Proposition

Assume M is oriented. Then M has a spin structure if and only if its 2nd Stieffel-Whitney characteristic class vanishes.

Spin Structure

Theorem

If M has a spin structure, then there is an associated spinor bundle $\not{S} = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \not{S}_n$ (called spinor bundle) such that:

1. *If n is even, then $\mathrm{Cl}(M) \simeq \mathrm{End}\, \not{S}$.*
2. *If n is odd, then $\mathrm{Cl}^+(M) \simeq \mathrm{End}\, \not{S}$.*
3. *The Riemannian metric lifts to a Hermitian metric on $\not{S}$.*
4. *The Levi-Civita connection lifts to a connection $\nabla^{\not{S}}$ on $\not{S}$ (called spin connection) compatible with its Hermitian metric.*

Theorem

Assume n is even. Then:

1. *We have an orthogonal $\mathbb{Z}_2$-grading $\not{S} = \not{S}^+ \oplus \not{S}^-$, with $\not{S}^\pm = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \not{S}_n^\pm$.*
2. *The spin connection $\nabla^{\not{S}}$ preserves this $\mathbb{Z}_2$-grading.*

Setup

(M^n, g) is a spin oriented Riemannian manifold.

Definition (Dirac operator)

The Dirac operator $D\!\!\!/ : C^\infty(M, \$) \to C^\infty(M, \$)$ is the composition,

$$D\!\!\!/ : C^\infty(M,\$) \xrightarrow{\nabla^{\$}} C^\infty(M, \$ \otimes T^*M) \longrightarrow C^\infty(M,\$)$$
$$\sigma \otimes \xi \longrightarrow c(\xi)\sigma,$$

where $c(\xi) \in \mathrm{Cl}_x(M)$ is identified with an element of $\mathrm{End}\, \$_x$.

Remark

- If n is even, then $D\!\!\!/\gamma = -\gamma D\!\!\!/$.
- Thus, with respect to the splitting $\$ = \$^+ \oplus \$^-$, the Dirac operator takes the form,

$$D\!\!\!/ = \begin{pmatrix} 0 & D\!\!\!/^- \\ D\!\!\!/^+ & 0 \end{pmatrix}, \qquad D\!\!\!/^\pm : C^\infty(M, \$^\pm) \to C^\infty(M, \$^\mp).$$

The Dirac Operator

Proposition (Lichnerowicz Formula)

We have

$$\not{D}^2 = \left(\nabla^{\$}\right)^* \nabla^{\$} + \frac{1}{4}\kappa_g,$$

where $\left(\nabla^{\$}\right)^* \nabla^{\$}$ *is the connection Laplacian and* κ_g *is the scalar curvature of* (M, g).

Proposition

The Dirac operator $\not{D}$ *is a (formally) selfadjoint first order elliptic differential operator.*

Remark

If n is even, then $\not{D}^-$ is the (formal) adjoint of $\not{D}^+$.

Dirac Spectral Triple

Proposition

Assume M is a compact spin oriented Riemannian manifold. Then:

1. *The following is a spectral triple,*
$$\left(C^\infty(M), L^2\left(M, \not{S}\right), \not{D}\right).$$
2. *If n is even, then this is an even spectral triple, with*
$$L^2(M, \not{S}) = L^2(M, \not{S}^+) \oplus L^2(M, \not{S}^-).$$

Theorem (Atiyah-Singer)

If n is even, then
$$\operatorname{ind} \not{D} = (2i\pi)^{-\frac{n}{2}} \int_M \hat{A}\left(R^M\right),$$
where $\hat{A}(R^M) := \det^{\frac{1}{2}}\left[\frac{R^M/2}{\sinh(R^M/2)}\right]$ is the $\hat{A}$-form of the curvature R^M.

Non-Unital Spectral Triples

Definition

A non-unital spectral triple $(\mathscr{A}, \mathscr{H}, D)$ consists of

- A Hilbert space $\mathscr{H}$.
- A (possibly non-unital) $*$-subalgebra $\mathscr{A} \subseteq \mathscr{L}(\mathscr{H})$.
- A selfadjoint operator D on $\mathcal{H}$ such that:
  (i) $a(1+D^2)^{-1}$ is a compact operator for all $a \in \mathcal{A}$.
  (ii) $a(\mathrm{dom}(D)) \subseteq \mathrm{dom}(D)$ and $[D, a] \in \mathcal{L}(\mathcal{H})$ for all $a \in \mathcal{A}$.

Remark

If $\mathcal{A}$ is unital, we recover the definition of a (unital) spectral triple.

Definition

A non-unital spectral triple $(\mathscr{A}, \mathscr{H}, D)$ is p-summable if

$$a\left(1+D^2\right)^{-\frac{1}{2}} \in \mathscr{L}^{p,\infty} \qquad \forall a \in \mathcal{A}.$$

Spectral Triples on $\mathbb{R}^n$

Proposition

The following are n-summable non-unital spectral triples:

- $\left(C_c^\infty(\mathbb{R}^n), L^2(\mathbb{R}^n), (1+\Delta)^{1/2}\right)$.
- $\left(C_c^\infty(\mathbb{R}^n), L^2(\mathbb{R}^n), \sqrt{\Delta}\right)$ *if* $n \geq 2$.

Definition

$\Psi_c^0(\mathbb{R}^n)$ is the algebra of compactly supported ΨDOs on $\mathbb{R}^n$ of order 0.

Proposition

The following are n-summable non-unital spectral triples:

- $\left(\Psi_c^0(\mathbb{R}^n), L^2(\mathbb{R}^n), (1+\Delta)^{1/2}\right)$.
- $\left(\Psi_c^0(\mathbb{R}^n), L^2(\mathbb{R}^n), \sqrt{\Delta}\right)$ *if* $n \geq 2$.

Proposition

Let $\not{D} : C^\infty(\mathbb{R}^n; \not{S}_n) \rightarrow C^\infty(\mathbb{R}^n; \not{S}_n)$ *be the Dirac operator on* $\mathbb{R}^n$*.*

1. *The following is an* n*-summable non-unital spectral triple,*

$$\left(C_c^\infty(\mathbb{R}^n), L^2\left(\mathbb{R}^n; \not{S}_n\right), \not{D} \right).$$

2. *If* n *is even, then this is an even spectral triple, with*

$$L^2(\mathbb{R}^n, \not{S}_n) = L^2(\mathbb{R}^n, \not{S}_n^+) \oplus L^2(\mathbb{R}^n, \not{S}_n^-).$$