

$$\Pi^N = (R_{2\pi}^N)^N$$

Orthonormal Basis

Inner Product  $\langle f|g \rangle = \frac{1}{(2\pi)^N} \int_{\Pi^N} f(x) \overline{g(x)} dx$   
 $e_\xi(x) = e^{i\xi \cdot x}, \xi \in \mathbb{Z}^N$

Fourier series decomp.

$u \in L^2(\Pi^N) \hookrightarrow \sum_{\xi \in \mathbb{Z}^N} \hat{u}(\xi) e_\xi, \quad \hat{u}(\xi) = \langle u | e_\xi \rangle = \frac{1}{(2\pi)^N} \int_{\Pi^N} u(x) e^{-i\xi \cdot x} dx$   
 $u \in L^2(\Pi^N) \iff \hat{u}(\xi) \in \ell^2(\mathbb{Z}^N), \text{ i.e., } \sum_{\xi \in \mathbb{Z}^N} |\hat{u}(\xi)|^2 < \infty.$

$u \in C^\infty(\Pi^N) \iff u = \sum_{\xi \in \mathbb{Z}^N} \hat{u}(\xi) e_\xi \text{ w/ } (\hat{u}(\xi)) \in \mathcal{S}(\mathbb{Z}^N)$

$\ell^p$  rapid decay requ.

$(a_\xi) \in \mathcal{S}(\mathbb{Z}^N) \iff \forall N, |a_\xi| = O((1+|\xi|)^{-N}) \quad |\xi| = |\xi_1|^2 + \dots + |\xi_N|^2$

Laplacian:  $\Delta = (\partial_{x_1}^2 + \dots + \partial_{x_N}^2), \Delta: C^\infty(\Pi^N) \rightarrow C^\infty(\Pi^N) \subset L^2(\Pi^N)$

$\partial_{x_j} e_\xi = \partial_{x_j} (e^{i\xi \cdot x}) = i\xi_j e^{i\xi \cdot x} = i\xi_j e_\xi$   
 $\partial_{x_j}^2 e_\xi = (i\xi_j)^2 e_\xi = -\xi_j^2 e_\xi$

$\Delta e_\xi = -(\partial_{x_1}^2(e_\xi) + \dots + \partial_{x_N}^2(e_\xi)) = (\xi_1^2 + \dots + \xi_N^2) e_\xi = |\xi|^2 e_\xi$

Prop. 1.  $\Delta$  is essentially self-adjt.

2. The domain of its closure is

$W^{2,2}(\Pi^N) = \{u \in L^2(\Pi^N), \sum (1+|\xi|^2) |\hat{u}(\xi)|^2 < \infty\}$

3. Its spectrum is  $\{|\xi|^2, \xi \in \mathbb{Z}^N\}$  with  $\Delta e_\xi = |\xi|^2 e_\xi$ .

4. We have  $\lim_{n \rightarrow \infty} \Delta = \Delta_{\text{cl}} = \Delta_0 = 0 \cdot 1$ .

If  $f: [0, \infty) \rightarrow \mathbb{C}$  is bounded, we have

$u(n+1)$

$f(\Delta) e_\xi = f(|\xi|^2) e_\xi$

In particular, we define  $\Delta^{-q}, q > 0$ , e.

$\Delta^{-q} e_\xi = \begin{cases} |\xi|^{-2q} e_\xi & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0. \end{cases}$

Alternatively, we arrange the eigenvalues of  $\Delta$  as a non-decreasing sequence,

$0 = \lambda_0(\Delta) \leq \lambda_1(\Delta) \leq \lambda_2(\Delta) \leq \dots$

where each eigenvalue is repeated  $\infty$  times, e.g.

$\lambda_0(\Delta) = 0, \lambda_1(\Delta) = \dots = \lambda_n(\Delta) = 1, \lambda_{n+1}(\Delta) = \dots = \lambda_{n+\frac{1}{2}n(n+1)}(\Delta) = 2, \dots$



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We re-arrange the family of  $(e_j)_{j \in \mathbb{N}}$  into a family  $(\varepsilon_j)_{j \geq 0}$  s.t.  
 $\Delta \varepsilon_j = \lambda_j(\Delta) \varepsilon_j, \quad \langle \varepsilon_j | \varepsilon_l \rangle = \delta_{jl}$

We then have

$$\Delta^{-q} \varepsilon_j = \begin{cases} \lambda_j(\Delta)^{-q} \varepsilon_j & \text{if } j \geq 1, \\ 0 & \text{if } j = 0. \end{cases}$$

Fact:  $\Delta^{-q}$  is a positive compact operator with

$$\lambda_j(\Delta^{-q}) = \mu_j(\Delta^{-q}) = \lambda_{j+1}(\Delta)^{-q} \quad \forall j \geq 0.$$

$$\ker \Delta^{-q} = \mathbb{C} \cdot 1.$$

volume unit ball  $B^n$

Lemma (Weyl's Law) As  $j \rightarrow \infty$ , we have

$$\lambda_j(\Delta) \sim \left(\frac{j}{c}\right)^{\frac{2n}{n-2}}$$

$$c = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$$

$$= |B^n| = c(n) |\pi^n|, \quad \text{volume of } \pi^n, \quad c(n) = (2\pi)^n |B^n|$$

Proof See Lecture Notes & Prop. 5.

Consequence: Let  $q > 0$ , then

$$\lambda_j(\Delta^{-q}) = \lambda_{j+1}(\Delta)^{-q} \sim \left(\frac{c}{j}\right)^{\frac{2q}{n}}.$$

In particular:

$$\mu_j(\Delta^q) = \lambda_j(\Delta^q) = O\left(j^{-\frac{2q}{n}}\right).$$

Therefore, we get:

Prop.  $\Delta^{-q} \in \mathcal{L}^{\frac{n}{2q}, \infty}$

In fact, for  $q = \frac{n}{2}$  we get:

$$\lambda_j(\Delta^{\frac{n}{2}}) \sim \frac{c}{j}$$

Proof, see get:

Prop.  $\Delta^{-\frac{n}{2}}$  is self-adjoint and we have

$$f \Delta^{-\frac{n}{2}} = c = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$$



Theorem (Carleson's Inequality) For all  $f \in C(\mathbb{T}^n)$ , the operator  $f\Delta^{n/2}$  is strongly measurable, and we have

$$\|f\Delta^{n/2}\|_{HS}^2 = c(n) \int_{\mathbb{T}^n} |f(x)|^2 dx.$$

In particular, the RK integral enables to recapture the Haar measure of  $\mathbb{T}^n$ .

Proof. Let  $\varphi$  be a positive trace on  $\mathcal{L}^{1, \infty}$ . We know it is continuous, i.e.,  $\exists C > 0$  s.t.

$$|\varphi(T)| \leq C \|T\|_{1, \infty} \quad \forall T \in \mathcal{L}^{1, \infty}.$$

Define  $\Lambda: C(\mathbb{T}^n) \rightarrow \mathbb{R}$  by

$$\Lambda(f) = \varphi(f\Delta^{n/2}), \quad f \in C(\mathbb{T}^n).$$

We see

$$|\Lambda(f)| \leq C \|f\Delta^{n/2}\|_{1, \infty} \leq C \|f\|_{\infty} \|\varphi\|_{1, \infty}.$$

Thus, we see that  $\varphi$  is a continuous linear form on  $C(\mathbb{T}^n)$ .

Let  $f \in C(\mathbb{T}^n)$ ,  $f \geq 0$ , and set  $g = \sqrt{f} \in C(\mathbb{T}^n)$ . As  $\varphi$  is a trace we get

$$\Lambda(f) = \Lambda(g^2) = \varphi(g^2 \Delta^{n/2}) = \varphi(g \Delta^{n/2} g).$$

Here  $g \Delta^{n/2} g \geq 0$ , and  $\langle g \Delta^{n/2} g | \xi \rangle = \langle \Delta^{n/2} g | g \xi \rangle \geq 0$ .

[The fact that  $\Delta^{n/2}$  is self-adj. and has non-neg. spectrum  $\Rightarrow \Delta^{n/2} \geq 0$ , and  $\langle \Delta^{n/2} \xi | \xi \rangle \geq 0$ ]

As  $\varphi$  is a positive trace, we get

$$\Lambda(f) = \varphi(g \Delta^{n/2} g) \geq 0$$

This shows that  $\Lambda$  is a positive continuous linear functional on  $C(\mathbb{T}^n)$ , and thus  $\Lambda$  is a positive Radon measure on  $\mathbb{T}^n$ .

Let  $x_0 \in \mathbb{T}^n$ . Let  $\tau_{x_0}: \mathbb{T}^n \rightarrow \mathbb{T}^n$  be the translation by  $x_0$ , i.e.,  $\tau_{x_0}(x) = x + x_0$ . Define

$$U: L^2(\mathbb{T}^n) \rightarrow L^2(\mathbb{T}^n) \text{ by } U\xi =$$

$$U_{x_0} \xi = \xi \circ \tau_{x_0}^{-1} = \xi \circ \tau_{-x_0}, \quad \xi \in L^2(\mathbb{T}^n) \quad [ (U_{x_0} \xi)(x) = \xi(x+x_0) ]$$

$U$  is a unitary operator, since  $U_{x_0}^{-1} = U_{-x_0} = U_{x_0}^*$ . Indeed, for all  $\xi, \eta \in L^2(\mathbb{T}^n)$ , we have

$$\langle \xi | U_{x_0} \eta \rangle = \int_{\mathbb{T}^n} \xi(x) \overline{\eta(x+x_0)} dx = \int_{\mathbb{T}^n} \xi(x-x_0) \overline{\eta(x)} dx = \langle U_{-x_0} \xi | \eta \rangle$$

Multiplication operators

Claim. We have

$$U_{x_0} \Delta^{n/2} U_{x_0}^* = \Delta^{n/2} \quad \text{and} \quad U_{x_0} f U_{x_0}^* = f \circ \tau_{x_0} \quad \forall f \in C(\mathbb{T}^n).$$

Proof. We have  $U_{x_0} e_\xi = e_\xi(x+x_0) = e^{i\xi \cdot (x+x_0)} = e^{i\xi \cdot x_0} e^{i\xi \cdot x} = e^{i\xi \cdot x_0} e_\xi$ .

Conversely  $U_{x_0}^* e_\xi = U_{-x_0} e_\xi = e^{-i\xi \cdot x_0} e_\xi$ . Thus,



$$\begin{aligned}
 (U_{x_0} \Delta^{-1/2} U_{x_0}) e_2 &= U_{x_0} \Delta^{-1/2} (e^{-i\ell \cdot x_0} e_2) = e^{-i\ell \cdot x_0} U_{x_0} (\Delta^{-1/2} e_2) \quad (\ell \neq 0) \\
 &= e^{-i\ell \cdot x_0} U_{x_0} (|\ell|^{-1/2} e_2) \\
 &= |\ell|^{-1/2} e^{-i\ell \cdot x_0} U_{x_0} e_2 \\
 &= |\ell|^{-1/2} e_2 \\
 &= \Delta^{1/2} e_2
 \end{aligned}$$

If  $\ell=0$ , then  $U_{x_0} e_0 = e_0$ , and then  
 $U_{x_0} \Delta^{1/2} U_{x_0} e_0 = U_{x_0} \Delta^{1/2} e_0 = 0 = \Delta^{1/2} e_0$ .

As  $(e_2)_{x_0}$  is an arbitrary vector, we see that  $U_{x_0} \Delta^{1/2} U_{x_0} = \Delta^{1/2}$ .

Let  $f \in C(\Pi^u)$  and  $\xi \in L^2(\Pi^u)$ . We have

$$\begin{aligned}
 (U_{x_0} f U_{x_0}^*) \xi &= (U_{x_0} f U_{x_0}) \xi = U_{x_0} (f U_{x_0} \xi) = U_{x_0} (f \xi \circ \tau_{x_0}) \\
 &= (f \circ \tau_{x_0}) (U_{x_0} \xi \circ \tau_{x_0}) \\
 &= (f \circ \tau_{x_0}) \xi
 \end{aligned}$$

This shows that  $U_{x_0} f U_{x_0}^* = f \circ \tau_{x_0}$ . The claim is proved.  $\square$

Now, let  $f \in C(\Pi^u)$  and  $x_0 \in \Pi^u$ . We have

$$\begin{aligned}
 \Lambda(f \circ \tau_{x_0}) &= \varphi((f \circ \tau_{x_0}) \Delta^{1/2}) = \varphi((U_{x_0} f U_{x_0}^*)(U_{x_0} \Delta^{1/2} U_{x_0}^*)) \\
 &= \varphi(U_{x_0} (f \Delta^{1/2}) U_{x_0}^*)
 \end{aligned}$$

As  $\varphi$  is a trace, we have

$$\Lambda(f \circ \tau_{x_0}) = \varphi(U_{x_0} (f \Delta^{1/2}) U_{x_0}^*) = \varphi(f \Delta^{1/2}) = \Lambda(f)$$

This shows that  $\Lambda$  is a translation-invariant positive Radon measure on the compact (Abelian) group. Therefore, it is a constant multiple of its Haar measure. That is,

$$\Lambda(f) = \Lambda(1) \int_{\Pi^u} f \, dx \quad \forall f \in C(\Pi^u)$$

For  $f=1$ , we have that

$$\Lambda(1) = \varphi(\Delta^{1/2}) = c(u) |\Pi^u|, \quad c(u) = (2\pi)^u |B^u|$$

Thus,

$$c(u) |\Pi^u| = \Lambda(1) = \Lambda(1) \int_{\Pi^u} 1 \, dx = \Lambda(1) |\Pi^u| \implies \Lambda(1) = c(u)$$



Thus, if  $f \in C(\mathbb{T}^n)$ , then

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$$\varphi(f \Delta^{n/2}) = \lambda(f) = c(n) \int_{\mathbb{T}^n} f(x) dx.$$

This shows that  $f \Delta^{n/2}$  is strongly measurable, and we have

$$f \Delta^{n/2} = c(n) \int_{\mathbb{T}^n} f(x) dx.$$

The result is proved.  $\square$



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Integration Formula ( $L^p$ -version)If  $f \in C(\mathbb{T}^n)$ , then

$$\int_{\mathbb{T}^n} f \Delta^{n/2} = c(n) \int_{\mathbb{T}^n} f(x) dx.$$

As  $f$  is a trace,  $f \Delta^{n/2} = \int \Delta^{n/4} f \Delta^{n/4}$ , and so we can rewrite the formula as

$$\int \Delta^{n/4} f \Delta^{n/4} = c(n) \int_{\mathbb{T}^n} f(x) dx.$$

Question: Can the formula be all  $f \in L^1(\mathbb{T}^n)$ , or at least for  $L^p(\mathbb{T}^n)$  for some  $p \geq 1$ ?

Note: If  $f \in L^1(\mathbb{T}^n)$  and  $f$  is not bounded, then the multiplication by  $f$  is not a bounded operator on  $L^2(\mathbb{T}^n)$ !

If  $f \in C(\mathbb{T}^n)$ , then by Hölder's inequality

$$\|\Delta^{n/4} f \Delta^{n/4}\|_{1,\infty} \leq C \|\Delta^{n/4}\|_{2,\infty} \|f\|_{2,1} \|\Delta^{n/4}\|_{2,\infty}.$$

Prop. (Birman-Solomjag  $'72$ ) (let  $p > 1$ ).

1. If  $f \in L^p(\mathbb{T}^n)$ , then  $\Delta^{n/4} f \Delta^{n/4} \in L_{\text{loc}}$ .

2.  $\exists C_p > 0$  s.t.

$$\|\Delta^{n/4} f \Delta^{n/4}\|_{1,\infty} \leq C_p \|f\|_p \quad \forall f \in L^p(\mathbb{T}^n).$$

Re. The estimates do not hold for  $p=1$ !

Def. (Eggenmund).  $L(\text{Log } \mathbb{T}^n)$  is the space spanned by (equivalence classes of) measurable functions  $f: \mathbb{T}^n \rightarrow \mathbb{C}$  s.t.

$$\int_{\mathbb{T}^n} |f(x)| \log(1 + |f(x)|) dx < \infty.$$

Prop.  $L(\text{Log } \mathbb{T}^n)$  is a Banach space w.r.t. the norm

$$\|f\|_{L(\text{Log})} := \inf \left\{ \lambda > 0; \int_{\mathbb{T}^n} |\lambda f(x)| \log(1 + |\lambda f(x)|) dx \leq 1 \right\}$$



Rule.

1. We have continuous (strict) inclusions,  
 $L^p(\Pi^n) \not\subseteq L(\text{Bog } L(\Pi^n)) \not\subseteq L'(\Pi^n) \quad \forall p > 1.$

2.  $C(\Pi^n)$  is dense in  $L(\text{Bog } L(\Pi^n))$

Thm. (Solomyak, Sukochev-Zaicev).

1. If  $f \in L(\text{Bog } L(\Pi^n))$ , then  $\Delta^{n/4} f \Delta^{n/4} \in \mathcal{L}_{\text{mes}}$ .

2. There is  $C > 0$  s.t.

$$\|\Delta^{n/4} f \Delta^{n/4}\| \leq C \|f\|_{L(\text{Bog } L)} \quad \forall f \in L(\text{Bog } L(\Pi^n)).$$

As a consequence we get:

Thm.: If  $f \in L(\text{Bog } L(\Pi^n))$ , then  $\Delta^{n/4} f \Delta^{n/4}$  is strongly measurable, and we have

$$\int \Delta^{n/4} f \Delta^{n/4} = \alpha(n) \int_{\Pi^n} f(x) dx.$$

Proof. Let  $\varphi: \mathcal{L}^{\text{mes}} \rightarrow \mathbb{C}$  be a normalized positive trace. We know that:

$$\alpha) \quad \varphi(\Delta^{n/4} f \Delta^{n/4}) = \alpha(n) \int_{\Pi^n} f(x) dx \quad \forall f \in C(\Pi^n).$$

We prove that:

- The map  $f \mapsto \int_{\Pi^n} f(x) dx$  is a continuous linear form on  $L'(\Pi^n)$ , and hence it induces a continuous linear form on  $L(\text{Bog } L(\Pi^n))$ .

- As  $\varphi$  is a positive linear trace, it is continuous on  $\mathcal{L}^{\text{mes}}$ .

- By the results of Solomyak and Sukochev-Zaicev, we have a continuous linear map  $L(\text{Bog } L(\Pi^n)) \ni f \mapsto \Delta^{n/4} f \Delta^{n/4} \in \mathcal{L}_{\text{mes}}$ .

- Therefore, we get a continuous linear form  $L(\text{Bog } L(\Pi^n)) \ni f \mapsto \varphi(\Delta^{n/4} f \Delta^{n/4})$ .



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This shows that both sides of (\*) are continuous linear forms on  $L \log L(H)$ .  
 As they agree on  $C(\pi^u)$  and  $C(\pi^u)$  is a dense subspace of  $L \log L(H)$ , it  
 follows that the two sides agree for all  $f \in L \log L(H)$ . ~~It is now~~

To sum up, if  $f \in L \log L(H)$ , then, for every positive normalized  
 trace  $\varphi: L^{\infty} \rightarrow \mathbb{C}$  we have

$$\varphi(\Delta^{u/4} f \Delta^{-u/4}) = c(u) \int_{\pi^u} f(x) dx.$$

This shows that  $\Delta^{u/4} f \Delta^{u/4}$  is strongly measurable, and

$$\int \Delta^{u/4} f \Delta^{u/4} = c(u) \int_{\pi^u} f(x) dx.$$



Differential Operators

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha, \quad D_{x_j} = \frac{1}{i} \frac{\partial}{\partial x_j}, \quad \Delta = D_{x_1}^2 + \dots + D_{x_n}^2$$

$$D_x^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n} \quad D_{x_j}(e^{\xi \cdot x}) = \xi_j e^{\xi \cdot x}$$

$$D_x^\alpha(e^{\xi \cdot x}) = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n} e^{\xi \cdot x} = \xi^\alpha e^{\xi \cdot x}$$

$$Pe^{\xi \cdot x} = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha(e^{\xi \cdot x}) =$$

$$= \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha e^{\xi \cdot x}.$$

$$u = \sum \hat{u}(\xi) e^{\xi \cdot x} \rightarrow Pu = \sum \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \hat{u}(\xi) e^{\xi \cdot x}.$$

If we set  $f(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$ , R.

$$Pu = \sum_{\xi \in \mathbb{R}^n} f(x, \xi) \hat{u}(\xi) e^{\xi \cdot x}.$$

For a priori for  $P = \Delta$ , we let  $f(x, \xi) = |\xi|^2$ . We call  $f(x, \xi)$  the symbol of  $P$ .  
(Symbols on  $\Pi^n$ )

Def.  $S^m(\Pi^n \times \mathbb{R}^n)$ ,  $m \in \mathbb{R}$ , consists of functions  $\sigma(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  that are an asymptotic expansion,

$$\sigma(x, \xi) \sim \sum_{j \geq 0} \sigma_{m-j}(x, \xi),$$

where:

- $\sigma_{m-j}(x, \xi) \in C^\infty(\Pi^n \times (\mathbb{R}^n \setminus 0))$ , and
- $\sigma_{m-j}(x, \lambda \xi) = \lambda^{m-j} \sigma_{m-j}(x, \xi) \quad \forall \lambda > 0.$

$\nu$  means let  $\forall \alpha, \beta \quad \forall N \exists C_{N, \alpha, \beta} > 0$  s.t.

$$|\partial_x^\alpha \partial_\xi^\beta (\sigma(x, \xi) - \sum_{j < N} \sigma_{m-j}(x, \xi))| \leq C_{N, \alpha, \beta} |\xi|^{m-N-|\beta|}$$

$$\forall x \in \Pi^n \quad \forall \xi \in \mathbb{R}^n, |\xi| > 1.$$



Rule. For  $N=0$  this implies the estimates,

$$| \partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi) | \leq C_{\alpha\beta} (1 + |\xi|)^{m - |\beta|} \quad \forall (x, \xi) \in \mathbb{T}^n \times \mathbb{R}^n.$$

If  $\sigma(x, \xi) \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$ , we define  $P_\sigma: C^\infty(\mathbb{T}^n) \rightarrow C^\infty(\mathbb{T}^n)$  by

$$(P_\sigma u)(x) = \int \sigma(x, \xi) \hat{u}(\xi) e^{ix \cdot \xi} d\xi, \quad u = \int \hat{u}(\xi) e^{ix \cdot \xi} d\xi \in C^\infty(\mathbb{T}^n).$$

In particular,  $P_\sigma e^{ix \cdot \xi} = \sigma(x, \xi) e^{ix \cdot \xi}$

Lemma.  $P_\sigma$  is well-defined and maps continuously  $C^\infty(\mathbb{T}^n)$  to itself.

Def.  $(\Phi_m, \sigma_m)$ ,  $\Phi_m(\mathbb{T}^n)$ ,  $m \in \mathbb{R}$ , consists of linear operators  $P: C^\infty(\mathbb{T}^n) \rightarrow C^\infty(\mathbb{T}^n)$

for which  $\exists \sigma(x, \xi) \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$  s.t.

$$P = P_\sigma.$$

Rule. The function  $\sigma(x, \xi)$  is called the symbol of  $P$ . However, it is NOT unique!

• If we expand  $\sigma(x, \xi) = \sum_{j \geq 0} \sigma_{m-j}(x, \xi)$ , with  $\sigma_{m-j}(x, \xi)$  homogeneous of  $\xi$ , then the homogeneous symbols are uniquely determined by  $P$ !

• We call  $\sigma_m(x, \xi)$  the principal symbol of  $P$ .

• We have

$$\sigma_m(x, \xi) = \lim_{\lambda \rightarrow \infty} \lambda^{-m} \sigma(x, \lambda \xi) \quad \forall (x, \xi) \in \mathbb{T}^n \times (\mathbb{R}^n \setminus \{0\})$$

Prop. Let  $P_1 \in \Phi^{m_1}(\mathbb{T}^n)$  and  $P_2 \in \Phi^{m_2}(\mathbb{T}^n)$  have principal symbols  $\sigma_{m_1}(x, \xi)$  and  $\sigma_{m_2}(x, \xi)$ , respectively. Then:

$$1. P_1 P_2 \in \Phi^{m_1+m_2}(\mathbb{T}^n).$$

$$2. \text{The principal symbol of } P_1 P_2 \text{ is } \sigma_{m_1}(x, \xi) \sigma_{m_2}(x, \xi).$$

Example. If  $q \in C^\infty$ , then  $\Delta^q = P_q$  w/  $\sigma(x, \xi) = \begin{cases} |\xi|^2 q & \text{for } |\xi| \geq 1, \\ 0 & \text{near } \xi = 0. \end{cases}$  so  $\Delta^q \in \Phi^{2q}(\mathbb{T}^n)$ .



• Prop. Let  $P \in \Psi^m(\mathbb{T}^n)$ ,  $m \leq 0$ . Then:

1.  $P$  uniquely extends to a continuous linear operator,  
 $P: L^2(\mathbb{T}^n) \rightarrow L^2(\mathbb{T}^n)$ .

2. If  $m < 0$ , this operator is compact.

Proof. In other words,

$$\Psi^0(\mathbb{T}^n) \subset \mathcal{L}(L^2(\mathbb{T}^n)) \text{ and } \Psi^m(\mathbb{T}^n) \subset \mathcal{K} \quad \forall m < 0.$$

Prop. If  $P \in \Psi^{-m}(\mathbb{T}^n)$ ,  $m > 0$ , then  
 $P \in \mathcal{L}^{\frac{n}{m}, \infty}$

Proof. We know that  $\Delta^{n/2} \in \mathcal{L}^{\frac{n}{2}, \infty}$ . We have

$$P = P \Delta^{n/2} \Delta^{-n/2}$$

Here  $P \in \Psi^{-m}(\mathbb{T}^n)$  and  $\Delta^{n/2} \in \Psi^m(\mathbb{T}^n)$ , and so  $P \Delta^{n/2} \in \Psi^0(\mathbb{T}^n) \subset \mathcal{L}(L^2(\mathbb{T}^n))$ .

As  $\mathcal{L}^{\frac{n}{2}, \infty}$  is an ideal of  $\mathcal{L}(L^2(\mathbb{T}^n))$ , we deduce that  $P \in \mathcal{L}^{\frac{n}{m}, \infty}$ .

Prop. Let  $P \in \Psi^m(\mathbb{T}^n)$ ,  $m < -n$ . Then:

1.  $P$  is trace-class.

2. If  $\sigma \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$  is s.t.  $P = P_\sigma$ , then

$$\text{Tr}[P] = (2\pi)^{-n} \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{T}^n} \sigma(x, \xi) dx.$$

Proof. We know that

$$\Psi^m(\mathbb{T}^n) = \Psi^{-|m|}(\mathbb{T}^n) \subset \mathcal{L}^{\frac{n}{|m|}, \infty}$$

Thus, if  $P \in \Psi^m(\mathbb{T}^n)$ , then

$$\mu_j(P) = O(j^{-\frac{|m|}{n}})$$

As  $m < -n$ ,  $|m| > n$ , i.e.,  $\frac{|m|}{n} > 1$ , and

$$\sum \mu_j(P) < \infty$$

This means that  $P$  is trace-class.



We have

$$\text{Tr}[P] = \sum_{e \in \mathbb{R}^n} \langle P e e | e e \rangle$$

If  $P = P_{\sigma, \mu} / \sigma(x, \xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$  then

$$P e e = (\sigma) e e = \sigma(x, \xi) e e.$$

Then,

$$\langle P e e | e e \rangle = \langle \sigma(x, \xi) e e | e e \rangle$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(x, \xi) e e(x) \overline{e e(x)} dx. \quad e e(x) \overline{e e(x)} = e^{i h \cdot x} e^{-i h \cdot x} = 1.$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(x, \xi) dx$$

Therefore, we get

$$\text{Tr}[P] = \sum_{e \in \mathbb{R}^n} (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(x, \xi) dx$$

Rule. It is possible to choose  $\sigma$  s.t.

$$\sum_{e \in \mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi) dx = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \sigma(x, \xi) dx d\xi.$$

In this case, we get the formula,

$$\text{Tr}[P] = (2\pi)^{-n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \sigma(x, \xi) dx d\xi.$$

Rule. Recall that if  $P \in \mathcal{I}^{-m}(\mathbb{R}^n)$ ,  $m > 0$ , then  $P \in \mathcal{L}^{\frac{m}{m+1}}$ .

In particular, if  $m = n$ , then  $P \in \mathcal{L}^{1, \infty}$ .

Thm. (Cordes Trace Thm. for  $\mathbb{R}^n$ ). Every  $P \in \mathcal{I}^{-n}(\mathbb{R}^n)$  is strongly measurable, and

$$\text{Tr}[P] = \frac{(2\pi)^{-n}}{n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \sigma_{-n}(x, \xi) dx d\xi,$$

where  $\sigma_{-n}(x, \xi)$  is the principal symbol of  $P$ .



Rule. For  $P = f\Delta^{u/2}$  with  $f \in C(\mathbb{R}^n)$ , let  $f\Delta^{u/2} \in \mathcal{F}^{u,u}(\mathbb{R}^n)$  and  $f\Delta^{u/2} = P_\sigma$ , w/

$$\sigma(x, \xi) = \begin{cases} f(x) |\xi|^{-u}, & |\xi| \geq 1 \\ 0, & \text{near } \xi = 0. \end{cases}$$

In fact, its principal symbol is  $\sigma_{-u}(x, \xi) = f|\xi|^{-u}$ . So, here we have

$$\begin{aligned} \frac{(2\pi)^{-n}}{\mu} \iint_{\mathbb{R}^n \times \mathbb{B}^{n+1}} \sigma_{-u}(x, \xi) &= \frac{(2\pi)^{-n}}{\mu} \iint_{\mathbb{R}^n \times \mathbb{B}^n} f(x) |\xi|^{-u} d\text{vol}^{n+1} \xi = \frac{(2\pi)^{-n}}{\mu} \int_{\mathbb{R}^n} f(x) dx \int_{\mathbb{B}^n} d^{n+1} \xi \\ &= \frac{(2\pi)^{-n}}{\mu} |\mathbb{B}^{n+1}| \int_{\mathbb{R}^n} f(x) dx. \end{aligned}$$

$$\begin{aligned} &= c(u) \int_{\mathbb{R}^n} f(x) dx, \text{ where} \\ c(u) &= (2\pi)^{-n} |\mathbb{B}^{n+1}| = (2\pi)^{-n} \cdot \frac{1}{\mu} |\mathbb{B}^{n+1}| \end{aligned}$$

Therefore, we recover the formula,

$$ff\Delta^{u/2} = c(u) \int_{\mathbb{R}^n} f(x) dx.$$



Proof of Comm Trace Thm. for FDOs on  $\Pi^u$ :

For  $j=1 \rightarrow u$  and  $\sigma(x, \xi) \in S^m(\Pi^u \times \mathbb{R}^u)$  define

$$\Delta_{\xi_j} \sigma(x, \xi) = \sigma(x, \xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_u), \quad x \in \Pi^u, \xi = (\xi_1, \dots, \xi_u) \in \mathbb{R}^u.$$

Def.  $S^{-\infty}(\Pi^u \times \mathbb{R}^u) = \bigcap_{m \in \mathbb{R}} S^m(\Pi^u \times \mathbb{R}^u).$

Rule.  $\sigma(x, \xi) \in S^{-\infty}(\Pi^u \times \mathbb{R}^u)$  consists of all functions  $\sigma(x, \xi) \in C^\infty(\Pi^u \times \mathbb{R}^u)$  s.t.,  $\forall N \geq 0 \forall \alpha, \beta \in \mathbb{N}^u \exists C_{\alpha\beta} > 0$  s.t.

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| \leq C_{\alpha\beta} (1 + |\xi|)^{-N} \quad \forall (x, \xi) \in \Pi^u \times \mathbb{R}^u.$$

Lemma 1: Let  $\sigma \in S^m(\Pi^u \times \mathbb{R}^u)$ .

1. There is  $\sigma_j(x, \xi) \in S^m(\Pi^u \times \mathbb{R}^u)$  s.t.

$$\partial_{\xi_j} \sigma = \Delta_{\xi_j} \sigma_j \quad \text{mod } S^{-\infty}(\Pi^u \times \mathbb{R}^u).$$

2.  $\Delta_{\xi_j} \sigma(x, \xi) \in S^{m-1}(\Pi^u \times \mathbb{R}^u).$

3. We have

$$P_{\Delta_{\xi_j} \sigma} = [P_{e^{-i\xi_j}}, e^{i\xi_j}].$$

Consequence: If  $\sigma(x, \xi) \in S^{-u+1}(\Pi^u \times \mathbb{R}^u)$ , then  $P_{\Delta_{\xi_j} \sigma} \in \text{Com}(\mathcal{L}^{1100})$ .

Proof. By Part 1 we have  $\partial_{\xi_j} \sigma = \Delta_{\xi_j} \sigma_j + \rho$  w/  $\sigma_j \in S^{-u+1}(\Pi^u \times \mathbb{R}^u)$  and  $\rho(x, \xi) \in S^{-\infty}$ . Then,

$$P_{\partial_{\xi_j} \sigma} = P_{\Delta_{\xi_j} \sigma_j} + P_\rho.$$

By Part 3  $P_{\Delta_{\xi_j} \sigma_j} \in \text{Com}(\mathcal{L}^{1100})$ . Moreover  $P_\rho \in \mathcal{F}^{-u+1}(\Pi^u) \subseteq \mathcal{L} \subseteq \text{Com}(\mathcal{L}^{1100})$ .



(Fedosov et al., Guillemin)

Lemma. Let  $\sigma \in S^{-u}(\Pi^u \times \mathbb{R}^u)$  have a principal sym bol  $\sigma_{-u}(x, \xi)$  s.t.

$$\int_{\mathbb{R}^{u+1}} \sigma_{-u}(x, \xi) d^{u+1}\xi = 0$$

Then there exist  $\sigma_1(x, \xi), \dots, \sigma_u(x, \xi)$  in  $S^{-u+1}(\Pi^u \times \mathbb{R}^u)$  s.t.

$$\sigma = \partial_{\xi_1} \sigma_1 + \dots + \partial_{\xi_u} \sigma_u \quad \text{mod } S^{-\infty}(\Pi^u \times \mathbb{R}^u)$$

Combining this with the previous consequence we get:

Proof. Let  $\sigma \in S^{-u}(\Pi^u \times \mathbb{R}^u)$  be as above. Then

$$P_\sigma \in \text{Com}(\mathcal{L}^{1, \infty}).$$

Proof. By the previous Lemma, there are  $\sigma_j(x, \xi) \in S^{-u+1}(\Pi^u \times \mathbb{R}^u)$  and  $\rho(x, \xi) \in S^{-u}(\Pi^u \times \mathbb{R}^u)$  such that

$$P_\sigma = P_{\partial_{\xi_1} \sigma_1} + \dots + P_{\partial_{\xi_u} \sigma_u} + P_\rho, \quad \text{with } \sigma_j \in S^{-u+1}(\Pi^u \times \mathbb{R}^u)$$

By the previous consequence each  $P_{\partial_{\xi_j} \sigma_j}$  is in  $\text{Com}(\mathcal{L}^{1, \infty})$ . Moreover, we know that  $P_\rho \in \mathcal{L}' \subseteq \text{Com}(\mathcal{L}^{1, \infty})$ . Thus,  $P_\sigma \in \text{Com}(\mathcal{L}^{1, \infty})$ .

Proof of Connes Trace Theorem: Let  $P \in \mathcal{F}^{-u}(\Pi^u)$ . Put  $P = P_\sigma$  with  $\sigma(x, \xi) \in S^{-u}(\Pi^u \times \mathbb{R}^u)$ ,  $\sigma(x, \xi) \sim \sum_{j=0}^\infty \sigma_{-u+j}(x, \xi)$ . Set

$$a(x) = \frac{1}{|\mathbb{B}^{u+1}|} \int_{\Pi^u} \sigma_{-u}(x, \xi) d^{u+1}\xi \in C(\Pi^u)$$

We have

$$P = a \Delta^{u/2} + Q \quad \text{with } Q = P - a \Delta^{u/2}.$$

We have  $a \Delta^{u/2} = P_\rho$  with

$$\rho(x, \xi) = \begin{cases} a(x) |\xi|^{-u} & \text{if } |\xi| \geq 1, \text{ and} \\ 0 & \text{near } \xi = 0. \end{cases}$$

In particular, the principal symbol of  $a \Delta^{u/2}$  is  $\rho_{-u}(x, \xi) = a(x) |\xi|^{-u}$ .



Then,  $Q \in \mathcal{I}^{-u}(\mathbb{R}^n)$  and its principal symbol is

$$\begin{aligned}\omega_{-u}(x, \xi) &= \sigma_{-u}(x, \xi) - \rho_{-u}(x, \xi) \\ &= \sigma_{-u}(x, \xi) - a(x) |\xi|^{-u}.\end{aligned}$$

We then have

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \omega_{-u}(x, \xi) d\xi &= \int_{\mathbb{R}^{n-1}} \sigma_{-u}(x, \xi) d\xi - \int_{\mathbb{R}^{n-1}} a(x) |\xi|^{-u} d\xi \\ &= \int_{\mathbb{R}^{n-1}} \sigma_{-u}(x, \xi) d\xi - a(x) |\mathbb{R}^{n-1}|.\end{aligned}$$

$$= 0.$$

The previous proposition then ensures that  $Q \in \text{Com}(\mathcal{L}^{1, \infty})$ .

Let  $\varphi: \mathcal{L}^{1, \infty} \rightarrow \mathbb{R}$  be a normalized positive trace. We have

$$\varphi(P) = \varphi(a \Delta^{u/2}) + \varphi(Q) = \varphi(a \Delta^{u/2}).$$

By Goursat's Integration Formula, we have

$$\varphi(a \Delta^{u/2}) = \int a \Delta^{u/2} = a(u) \int_{\mathbb{R}^n} a(x) dx.$$

$$\text{Here } a(x) = \frac{1}{|\mathbb{R}^{n-1}|} \int_{\mathbb{R}^{n-1}} \sigma_{-u}(x, \xi) dx \text{ and } a(u) = \frac{(2\pi)^{-u}}{|\mathbb{R}^n|} = \frac{1}{(2\pi)^u |\mathbb{R}^{n-1}|}.$$

$$\varphi(a \Delta^{u/2}) = \frac{(2\pi)^{-u}}{u} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} \sigma_{-u}(x, \xi) dx d\xi.$$

It follows that

$$\varphi(P) = \varphi(a \Delta^{u/2}) = \frac{(2\pi)^{-u}}{u} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} \sigma_{-u}(x, \xi) dx d\xi.$$

This shows that  $P$  is strongly measurable and

$$\int P = \frac{(2\pi)^{-u}}{u} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} \sigma_{-u}(x, \xi) dx d\xi$$

The result is proved.  $\square$