7. Spin Structures and the Dirac Operator:

Let M^n be a C^∞ manifold

Def.: The frame bundle of M is the bundle over M,

$$F(M) := \coprod_{x\in M} F_x(M), \quad F_x(M) := \{\text{Invertible linear map } f: \mathbb{R}^n \to T_xM\}.$$
$$\simeq \{\text{bases of } T_xM\}$$

$F(M)$ is a $GL_n(\mathbb{R})$-principal bundle.

Orientation on M = datum of a non-zero section of $\Lambda^n T^*M$.
= reduction of structure group of $F(M)$ to $GL_n^+(\mathbb{R}) = \{A\in GL_n(\mathbb{R});\ \det A>0\}$.

Here, $F(M) \simeq F^+(M) \times_{GL_n^+(\mathbb{R})} GL_n(\mathbb{R})$

$F_x^+(M) = \{\text{orientation-preserving invertible linear maps } f: \mathbb{R}^n \to T_xM\}$.
$\simeq \{\text{oriented bases of } T_xM\}$ $\quad F^+(M) = GL_n^+(\mathbb{R})$-principal bundle

Riemannian structure on M = datum of a Riemannian metric on TM
= reduction of structure group of $F(M)$ to $O(n)$

$F(M) \simeq O(M) \times_{O(n)} GL_n(\mathbb{R})$

$O(M) = \{\text{orthonormal bases}\}$ $\quad O(M) = O(n)$-principal bundle

Oriented Riemannian structure on M = reduction of structure group of $F(M)$ to $SO(n) = O(n)\cap GL_n^+(\mathbb{R})$

$F(M) \simeq SO(M) \times_{SO(n)} GL_n(\mathbb{R})$

$SO_x(M) = \{\text{orientation isometric linear map } f: \mathbb{R}^n \to T_xM\}$

Def.: A spin structure on M is a reduction of a structure group of $F(M)$ to $Spin(n)$, i.e., $\exists$ $Spin(n)$-principal bundle $Spin(M)$ s.t. $F(M) \simeq Spin(M) \times_{Spin(n)} GL_n(\mathbb{R})$.

If M is spin, then the spinor bundle can be realized as an associated bundle

$$\$ = \$(M) := Spin(M) \times_{Spin(n)} \$(\mathbb{R}^n)$$

spinor space of $\mathbb{R}^n$

Rmk: Spinor representation $\rho: Spin(n) \to \$(\mathbb{R}^n)$ unitary so get natural Hermitian inner product on $\$$.

Suppose M oriented + Riem.:

Def.: The Clifford bundle $Cl(M)$ is the subbundle of $End(\Lambda^* T^*M)$ whose fiber at x is the algebra spanned by

$$c(v) := \varepsilon(v) - i(v), \qquad v\in T_x^*M$$

$$\varepsilon(v)\eta := v\wedge\eta \quad \forall \eta\in\Lambda^* T_x^*M$$

$$i(v)(\eta^1\wedge\cdots\wedge\eta^\ell) = \sum_{j=1}^{\ell} (-1)^{j-1}\langle v,\eta^j\rangle\, \eta^1\wedge\cdots\wedge\widehat{\eta^j}\wedge\cdots\wedge\eta^\ell \quad \forall \eta^j\in T_x^*M.$$

$Cl(M)$ is a $Cl(\mathbb{R}^n)$-principal bundle and a $SO(M)$-principal bundle.

$$Cl(M) = SO(M) \times_{SO(n)} Cl(\mathbb{R}^n).$$

We have vector bundle isomorphisms,

$$\Lambda T^*M \underset{\sigma}{\overset{c}{\rightleftarrows}} Cl(M). \qquad \Lambda T^*M \simeq SO(M) \times_{SO(n)} \Lambda^\bullet \mathbb{R}^n$$

Suppose in addition that M is spin. Then

$$SO(M) \simeq Spin(M) \times_{Spin(n)} SO(n)$$

$$\rightsquigarrow \quad \underset{Cl(M)\otimes\mathbb{C}}{\underset{\parallel}{Cl_{\mathbb{C}}(M)}} \simeq Spin(M) \times_{Spin(n)} Cl_{\mathbb{C}}(\mathbb{R}^n) \simeq Spin(M) \times_{Spin(n)} End(S(\mathbb{R}^n)) \simeq End(S).$$

Get $*$-action

$$c: \Lambda T^*M \otimes S \longrightarrow S \qquad c(\xi)^* = -c(\xi) \quad \forall \xi \in C^\infty(M, T^*M)$$
$$(\xi, \sigma) \longrightarrow c(\xi)\sigma.$$

In addition, Levi-Civita connection on $SO(M)$ lifts to a connection on $Spin(M)$ by using

$$\tau^{-1}: so(n) \longrightarrow Cl^2(\mathbb{R}^n) = \text{Lie alg. of } Spin(n)$$
$$\tau^{-1}(A) = \frac{1}{2} \sum_{i<j} \langle Ae^i, e^j \rangle c(e^i) c(e^j).$$

Consequence: The Levi-Civita connection (covariant derivative) ∇^{TM} lifts to a unitary connection (covariant derivative) on S:

$\{e_i\}$ local orthonormal frame of TM $\qquad \omega_{ijk} = \langle \nabla^{TM}_{e_i} e_j, e_k \rangle \qquad \nabla^{TM}: C^\infty(M, TM) \to C^\infty(M, T^*M \otimes TM)$

$\{e^i\}$ dual orth. frame of T^*M $\qquad \nabla^S: C^\infty(M, S) \longrightarrow C^\infty(M, T^*M \otimes S)$

$$\nabla^S_{\partial_i} = \partial_i + \frac{1}{4} \sum \omega_{ijk} \underbrace{c(e^j) c(e^k)}_{\in End(S)}.$$

Def. (Dirac Operator): The Dirac operator is the first-order differential operator

$$D: C^\infty(M, S) \longrightarrow C^\infty(M, S),$$

given by the composition

$$D: C^\infty(M, S) \xrightarrow{\nabla^S} C^\infty(M, T^*M \otimes S) \xrightarrow{c} C^\infty(M, S).$$

Rmk: Locally,

$$D = \sum c(dx^i) \nabla^S_{\partial_i} = \sum c(e^j) \nabla^S_{e_j}$$

Prop 7.1: (1) D is (formally) selfadjoint.

(2) $\sigma_1(D)(x, \xi) = i c(\xi) \leftarrow$ Clifford multiplication by $\xi \in T^*_x M$

(3) $D\!\!\!/$ is elliptic.

(4) $[D\!\!\!/, f] = c(df)$ $\forall f \in C^\infty(M)$.

Proof: (2): $D\!\!\!/ = \sum c(dx^i) \nabla^{\$}_{\partial_i}$ $\quad \nabla^{\$}_{\partial_i} = \partial_i + \text{l.o.t.}$

$$\sigma_1(D\!\!\!/) = \sum c(dx^j)\, i\xi_j = ic\big(\underbrace{\sum \xi_j dx^j}_{\xi}\big) = ic(\xi)$$

(3) $\underset{\sigma_1(D\!\!\!/^2)}{\sigma_1(D\!\!\!/)^2} = (ic(\xi))(ic(\xi)) = -\underset{\langle \xi, \xi\rangle}{c(\xi)^2} = |\xi|^2 \Rightarrow \sigma_1(D\!\!\!/)$ invertible

$\Rightarrow D\!\!\!/$ is elliptic

(4) follows from (1) $[D\!\!\!/, f] \underset{\text{general fact for 1st order diff. op.}}{=} \sigma_1(x, df) = ic(df)$

Thm 22 (Lichnerowicz Formula): We have

$$D\!\!\!/^2 = (\nabla^{\$})^* \nabla^{\$} + \frac{1}{4}\kappa,$$

where κ is the scalar curvature of M.

Assume M even. Then $\$ = \$^+ \oplus \$^-$ compatible with action of $Cl(M)$

$\rightsquigarrow \$$ Clifford module.

$\nabla^{\$}$ preserves splitting $\$ = \$^+ \oplus \$^-$

$c(e^i)$ switches $\$^\pm$ and $\$^\mp$.

Therefore,

$$D\!\!\!/ = \begin{pmatrix} 0 & D\!\!\!/_- \\ D\!\!\!/_+ & 0 \end{pmatrix} \quad \text{w/ } D\!\!\!/_\pm : C^\infty(M, \$^\pm) \longrightarrow C^\infty(M, \$^\mp).$$

8. Dirac Operators on Clifford-module bundle:

M^n = Riem. mfld of even dim. $Cl_{\mathbb{C}}(M)$ = complexified Clifford algebra bundle of M.

Def.: A Hermitian vector bundle $E \to M$ is $\mathbb{Z}_2$-graded if it admits an orthogonal splitting

$$E = E^+ \oplus E^-, \quad E^\pm \text{ subbundle of } E.$$

Def.: Let E be a $\mathbb{Z}_2$-graded Hermitian vector bundle. A Clifford bundle structure on E is given by a fiberwise action, i.e. algebra-bundle map,

$$\pi : Cl_{\mathbb{C}}(M) \longrightarrow \mathrm{End}(E), \quad \pi(a)\xi = a\cdot\xi \quad a \in Cl_{\mathbb{C}}(M),\ \xi \in E$$

such that:

(i) π is a graded action, i.e., $\pi(Cl_{\mathbb{C}}^\pm(M)) \subset \mathrm{End}(E)^\pm$

(ii) c is a $*$-morphism, i.e., $\pi(a^*) = \pi(a)^*$ $\forall a \in Cl(M)$ ($\Leftrightarrow \pi(c(\omega))^* = -\pi(c(\omega))$ $\forall \omega \in T^*M$)

Ex1: $E = \Lambda^\bullet_{\mathbb{C}} T^*M = \Lambda^{ev} T^*M \oplus$ with π = inclusion $Cl_{\mathbb{C}}(M) \hookrightarrow \mathrm{End}(\Lambda^\bullet_{\mathbb{C}} T^*M)$.

(M oriented)

Ex.2: $E = \Lambda^{\bullet}_{\mathbb{C}} T^* M$ w/ same action as in Ex. 1, but w/ grading

$\Lambda^{\bullet}_{\mathbb{C}} T^* M = \Lambda^+ T^* M \oplus \Lambda^- T^* M$, $\quad \Lambda^{\pm} T^* M = \ker(\Gamma \mp 1)$,

Γ = chirality op. $= i^{\frac{n}{2}} c(\omega_M)$ $\quad \omega_M$ = volume form $= \sqrt{\det g(x)}\, dx^1 \wedge \cdots \wedge dx^n$

$\Gamma \in C^\infty(M, \mathrm{End}(\Lambda^{\bullet}_{\mathbb{C}} T^* M))$ $\quad = e^1 \wedge \cdots \wedge e^n$ $\{e^i\}$ oriented orth. frame of T^*M

Rmk: $\Gamma = i^{\frac{n}{2}} *$, $*$ = Hodge star operator $\quad * : \Lambda^k T^* M \longrightarrow \Lambda^{n-k} T^* M$

$*\alpha \wedge \beta = \langle \alpha, \beta \rangle \omega_M(x)$. $\quad \alpha, \beta \in \Lambda^k T^*_x M$

Ex.3: M spin, oriented. W = $\mathbb{Z}_2$-graded oriented Hermitian vector bundle

$E = \$ \otimes W$ $\quad (\$ \otimes W)^{\pm} = (\$^{\pm} \otimes W^+) \oplus (\$^{\mp} \otimes W^-)$

" $a.(\sigma \otimes w) = (\rho(a)\sigma) \otimes w$ $\quad \rho : Cl_c(M) \longrightarrow \mathrm{End}(\$)$ spinor representation.

$E = \$ \otimes W$ = twisted Clifford-module bundle.

Rmk. Locally, $M \simeq U \subset \mathbb{R}^n$ $\quad \mathbb{R}^n$ is spin so locally spinor bundle \$ always exists

Prop. Locally any Clifford-module bundle is of the form $\$ \otimes W$.

Def.: Let E be a Clifford-module bundle. A Clifford connection on E is a connection $\nabla^E : C^\infty(M, E) \longrightarrow C^\infty(M, T^*M \otimes E)$ which preserves its Clifford-module bundle structure, that is, for all $X \in C^\infty(M, TM)$,

(i) $\nabla_X (C^\infty(M, E^{\pm})) \subset C^\infty(M, E^{\pm})$.

(ii) $\langle \nabla_X \xi, \eta \rangle + \langle \xi, \nabla_X \eta \rangle = X(\langle \xi, \eta \rangle)$ $\quad \forall \xi, \eta \in C^\infty(M, E)$

(iii) $[\nabla_X, \pi(a)] = \pi(\nabla_X^{Cl(M)} a)$ $\quad \forall a \in C^\infty(M, Cl_c(M))$ $\quad$ ($\nabla^{Cl(M)}$ = lift of LC connection to $Cl(M)$).

Ex.1: $E = \Lambda^{\bullet} T^* M$, $\nabla^{\Lambda^{\bullet} T^* M}$ = Levi-Civita connection lifted to $\Lambda^{\bullet} T^* M$ so $\nabla^{\Lambda^{\bullet} T^* M}$ is a Clifford connection.

Ex2: M spin + oriented $\quad$ W = $\mathbb{Z}_2$-graded Herm. ∇^W connection on W satisfying (i) + (ii).

$\nabla^{\$ \otimes W} := \nabla^{\$} \otimes 1 + 1 \otimes \nabla^W$ is a Clifford connection on $\$ \otimes W$.

Prop. Locally, any Clifford connection is of the form $\nabla^{\$ \otimes W}$.

Let E be a Clifford-module bundle w/ Clifford connection ∇^E.

$R^M \in C^\infty(M, \Lambda^2 T^* M \otimes \mathrm{End}(TM))$ = Riemannian curvature

We can lift R^M to E as $R^E \in C^\infty(M, \Lambda^2 T^* M \otimes \mathrm{End}(E))$ $\quad \in \Lambda^2 T^* M$, $\in \mathrm{End}(E)$

$$R^E = \frac{1}{4} \sum_{e, \ell} \langle R(\partial_i, \partial_j) e_e, e_\ell \rangle (dx^i \wedge dx^j) \otimes (\pi(c(e^e)) \pi(c(e^\ell)))$$

Ex: $E = \$$, $\quad \mathcal{R}^{\$}$ = curvature of spin connection $\nabla^{\$}$.

Prop. The curvature $F^E \in C^\infty(M, \Lambda^2 T^*M \otimes \mathrm{End}(E))$ of F^E is of the form

$$F^E = R^E + F^{E/\$},$$

where $F^{E/\$} \in C^\infty(M, \Lambda^2 T^*M \otimes \mathrm{Hom}_{Cl(M)}(E,E))$ is called the twisting curvature.

Rmk: $\mathrm{Hom}_{Cl(M)}(E,E)_x = \{ T \in \mathrm{End}(E)_x ;\ [T, \pi(a)] = 0 \ \forall a \in Cl_x(M)\}$.

Ex: $E = \$ \otimes W \rightsquigarrow F^{E/\$} = 1 \otimes F^W$, F^W = curvature of ∇^W.

$\nabla^E = \nabla^\$ \otimes 1 + 1 \otimes \nabla^W$

Def. The Dirac operator associated to (E, ∇^E) is the 1st order differential operator $D_E : C^\infty(M,E) \to C^\infty(M,E)$ given by the composition,

$$C^\infty(M,E) \xrightarrow{\nabla^E} C^\infty(M, T^*M \otimes E) \xrightarrow{\pi \circ c} C^\infty(M,E)$$

$$v \otimes \xi \longmapsto \pi(c(v))\xi$$

Rmk: Locally, $D_E = \sum \pi(c(dx^i)) \nabla^E_i$.

Ex.1: $E = \Lambda^*_{\mathbb{C}} T^*M \rightsquigarrow D_E = d + d^*$ (de Rham operator) d = exterior differential

$d : C^\infty(M, \Lambda^q T^*M) \to C^\infty(M, \Lambda^{q+1} T^*M)$.

Ex.2: $E = \$ \otimes W$ $\nabla^E = \nabla^\$ \otimes 1 + 1 \otimes \nabla^W$

Dirac op. $\not{D}_M : C^\infty(M, \$) \xrightarrow{\nabla^\$} C^\infty(M, T^*M \otimes \$) \xrightarrow{c} C^\infty(M, \$)$

$\omega \otimes \sigma \longmapsto \rho(c(\omega))\sigma$

$D_E = \not{D}_W : C^\infty(M, \$ \otimes W) \xrightarrow{\nabla^\$ \otimes 1 + 1 \otimes \nabla^W} C^\infty(M, T^*M \otimes \$ \otimes W) \xrightarrow{c \otimes 1} C^\infty(M, \$ \otimes W)$

$\omega \otimes \sigma \otimes w \longmapsto (\rho(c(\omega))\sigma) \otimes w$

$\rightsquigarrow \not{D}_W = \not{D}_M \otimes 1 + (c \otimes 1)(1 \otimes \nabla^W)$ = twisted Dirac op.

Rmk: Locally, D_E is of the $\not{D}_W$.

Prop. (1) D_E is selfadjoint and elliptic.

(2) $D_E = \begin{pmatrix} 0 & D_E^- \\ D_E^+ & 0 \end{pmatrix}$, $D_E^\pm : C^\infty(M, E^\pm) \to C^\infty(M, E^\mp)$, $D_E^- = (D_E^+)^*$

Prop. (Lichnerowicz Formula):

$$(D_E)^2 = (\nabla^E)^* \nabla^E + \frac{1}{4} \kappa + \mathcal{F}^{E/\$},$$

$\mathcal{F}^{E/\$} \in C^\infty(M, \mathrm{End}(E))$

$\mathcal{F}^{E/\$} = \frac{1}{2} \sum_{i<j} F^{E/\$}(e_i, e_j) \pi(c(e^i) c(e^j))$ $\{e_i\}$ loc. orth. frame of TM

$\{e^i\}$ dual coframe of T^*M

9. The Index of a Fredholm Operator:

$\mathcal{H}_1, \mathcal{H}_2 =$ Hilbert spaces

Def.: (1) An operator $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is Fredholm when

$$\dim \ker T < \infty \quad \text{and} \quad \dim \operatorname{coker} T < \infty$$

(2) The set of Fredholm op. $T: \mathcal{H}_1 \to \mathcal{H}_2$ $\overset{\shortparallel}{\mathcal{H}_2 / \operatorname{im} T}$ is denoted $\operatorname{Fred}(\mathcal{H}_1, \mathcal{H}_2)$.

(3) If $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is Fredholm, it index is

$$\operatorname{ind} T := \dim(\ker T) - \dim(\operatorname{coker} T).$$

Rmk: $T \in \operatorname{Fred}(\mathcal{H}_1, \mathcal{H}_2) \Rightarrow \operatorname{im} T$ is closed and $\ker T^* \simeq \operatorname{coker} T$. Then

$T \in \operatorname{Fred}(\mathcal{H}_1, \mathcal{H}_2) \Rightarrow$

$$\boxed{\operatorname{ind} T = \dim \ker T - \dim \ker T^*}$$

Prop.: (1) $\operatorname{Fred}(\mathcal{H}_1, \mathcal{H}_2)$ is an open subset of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$

(2) $\operatorname{ind}: \operatorname{Fred}(\mathcal{H}_1, \mathcal{H}_2) \to \mathbb{Z}$ is continuous and constant of path-connected components of $\operatorname{Fred}(\mathcal{H}_1, \mathcal{H}_2)$.

(3) If $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_3)$ and $S \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_2)$ are Fredholm, the $ST \in \operatorname{Fred}(\mathcal{H}_1, \mathcal{H}_2)$ and

$$\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$$

(4) If $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is invertible modulo compact operators, then T is Fredholm and

$$\operatorname{ind}(T+R) = \operatorname{ind}(T) \quad \forall R \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2).$$

$M =$ compact $\overset{(\text{Riemann})}{\text{manifold}}$

$E_1, E_2 = \overset{(\text{Hermitian})}{\text{vector bundles}}$ over M

Prop.: Let $P: C^\infty(M, E_1) \to C^\infty(M, E_2)$ be an elliptic operator of order m, $m > 0$ and let $P^*: C^\infty(M, E_2) \to C^\infty(M, E_1)$ be its formal adjoint.

(1) $\forall s \in \mathbb{R}$ the operator $P: L^2_{s+m}(M, E_1) \to L^2_s(M, E_2)$ is Fredholm and its index does not depend on s.

(2) $\ker P$ and $\ker P^*$ are contained in $C^\infty(M, E_1)$ and $C^\infty(M, E_2)$ respectively and

$$\operatorname{ind} P = \dim \ker P - \dim \ker P^*$$

(3) $\operatorname{ind}(P+R) = \operatorname{ind} P \quad \forall R \in \Psi^{m-1}(M, E_1, E_2)$.

Rmk: (3) shows that $\operatorname{ind} P$ is an invariant of the principal symbol $\sigma_m(P)$ of P. This is an a homotopy invariant of $\sigma_m(P)$.

Question (Gel'fand): Compute the index of a general elliptic ΨDO.

- Solution provided by the Atiyah-Singer index theorem (ASIT).
- Geometric meaning of ASIT is reached by the Local Index Formula (LIF). The LIF is actually equivalent to the ASIT.