Noncommutative Geometry
Chapter 12:
The Local Index Formula
in Noncommutative Geometry

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Reminder: The Atiyah-Singer Index Theorem

Example

Assume M is spin, oriented, Riemannian and has even dimension.

1. For $C = \hat{A}(R^M)^\vee$ and the Dirac operator,
$$\left\langle \varphi_{\hat{A}(R^M)}, \mathcal{E} \right\rangle = \left\langle \hat{A}(R^M)^\vee, E \right\rangle \quad \text{and} \quad \mathrm{ind}_{\not{D}}[\mathcal{E}] = \mathrm{ind}_{\not{D}}[E].$$

2. By the K-theoretic version of the Atiyah-Singer Index Theorem explained in Chapter 10,
$$\mathrm{ind}_{\not{D}}[E] = (2i\pi)^{-\frac{n}{2}} \left\langle \hat{A}(R^M)^\vee, E \right\rangle.$$

3. Therefore, the Atiyah-Singer Index Theorem can be further restated as

Theorem

$$\mathrm{ind}_{\not{D}}[\mathcal{E}] = (2i\pi)^{-\frac{n}{2}} \left\langle \varphi_{\hat{A}(R^M)}, \mathcal{E} \right\rangle \quad \forall \mathcal{E} \in K_0(C^\infty(M)).$$

The Connes-Chern Character

Setup

- $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple, $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$
- $\gamma := 1_{\mathcal{H}^+} - 1_{\mathcal{H}^-}$ is the grading operator, $\gamma^2 = 1$, $\gamma^* = \gamma$.
- $F := D|D|^{-1}$ is the sign of D.

Assumption

$(\mathcal{A}, \mathcal{H}, D)$ is *p-summable* with $p \geq 1$, i.e.,

$$\mu_k(D^{-1}) = \mathrm{O}(k^{-\frac{1}{p}}).$$

That is, D^{-1} is an infinitesimal operator of order $1/p$.

Lemma

Let $q > p$. *Then*

$$\mathrm{Tr}\left[\left|[F, a^1]\cdots[F, a^q]\right|\right] < \infty \qquad \forall a^j \in \mathcal{A}.$$

Definition (Connes)

For $n > \frac{1}{2}(p+1)$ let τ_{2n} be the $2n$-cochain defined by

$$\tau_{2n}(a^0, \ldots, a^{2n}) = \frac{1}{2}\frac{n!}{(2n)!}\,\mathrm{Tr}\left[\gamma F[F, a^0]\cdots[F, a^{2n}]\right], \quad a^j \in \mathcal{A}.$$

The Connes-Chern Character

Lemma (Connes)

1. τ_{2n} *is a normalized cyclic cocycle.*
2. *The class of* τ_{2n} *in* $HC^{\text{even}}(\mathcal{A})$ *does not depend on* n*.*

Definition

The class of τ_{2n} in $HC^{\text{even}}(\mathcal{A})$ is called the *Connes-Chern character* and is denoted $\mathrm{Ch}(\mathcal{A}, D)$.

Theorem (Connes)

For all $\mathcal{E} \in K_0(\mathcal{A})$,

$$\mathrm{ind}_D[\mathcal{E}] = \langle \mathrm{Ch}(\mathcal{A}, D), \mathcal{E} \rangle .$$

Remark

Connes' cocycle τ_{2n} is difficult to compute in practice, because it definition involves

1. The operator F which is like a ψDO.
2. The operator trace which is not a local functional, i.e., it does not vanish on infinitesimals of a given order.

Therefore, it was sought for a more convenient representative of the Connes-Chern character.

Assumption

The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is θ-*summable*, i.e.,

$$\mathrm{Tr}\left[e^{-tD^2}\right] < \infty \qquad \forall t > 0.$$

Remark

$$p\text{-summability} \implies \theta\text{-summability.}$$

Definition (Jaffe-Lesniewski-Osterwlader)

For $t > 0$ define $\varphi^t_{\text{JLO}} = (\varphi_0, \varphi_2, \dots)$ by

$$\varphi^t_{2k}(a^0, \dots, a^{2k}) =$$
$$t^k \int_{\Delta_{2k}} \operatorname{Tr}\left[a^0 e^{-ts_0 D^2}[D, a^1]e^{-ts_1 D^2} \cdots [D, a^{2k}]e^{-ts_{2k} D^2}\right] ds, \quad a^j \in \mathcal{A},$$

where Δ_{2k} is the $2k$-simplex

$$\Delta_{2k} := \{(s_0, \dots, s_{2k}) \in \mathbb{R}^{2k+1};\ s_0 + \cdots + s_{2k} = 1,\ s_j \geq 0\}.$$

Remark

As observed by Quillen, φ^t_{JLO} can be interpreted as the Chern character of a superconnection on the algebra of cochains.

Proposition (Jaffe-Lesniewski-Osterwlader, Connes, Getzler-Szenes)

1. $(b+B)\varphi^t_{\mathrm{JLO}} = 0$.
2. *For all* $t>0$ *and* $\mathcal{E} \in K_0(\mathcal{A})$,

$$\mathrm{ind}_D[\mathcal{E}] = \left\langle \varphi^t_{\mathrm{JLO}}, \mathcal{E} \right\rangle .$$

Remark

1. $\varphi^t_{2k} \neq 0$ for large k, so φ^t_{JLO} is NOT a cochain in $C^{\mathrm{even}}(\mathcal{A})$.
2. This is a cocycle in *entire cyclic cohomology*, i.e., in the cohomology of infinite cochains $\varphi = (\varphi_0, \varphi_2, \ldots)$ such that, for any finite subset $S \subset \mathcal{A}$, the power series,

$$\sum_{k\geq 0} \frac{z^k}{k!} \varphi_{2k}(a^0, \ldots, a^{2k}), \qquad a^j \in S,$$

are entire functions.

Retraction of the JLO Cocycle

Assumption

$(\mathcal{A}, \mathcal{H}, D)$ is p-summable.

Remark

This assumption ensures us the existence of the Connes-Chern character.

Theorem (Connes)

Connes's cocycle τ_{2n}^{D} *and the JLO cocycle* $\varphi_{\mathrm{JLO}}^{t}$ *are cohomologous in entire cyclic cohomology.*

Retraction of the JLO Cocycle

Assumption

As $t \to 0^+$,

$$\varphi_{2k}^{t} = \sum_{\substack{\alpha, l \geq 0 \\ \alpha + l > 0}} t^{-\alpha} (\log^l t) \varphi_{2k}^{(\alpha,l)} + \varphi_{2k}^{(0,0)} + \mathrm{o}(t),$$

where the $\varphi_k^{(\alpha,l)}$ are $2k$-cochains.

Definition

The finite part of the JLO cocycle is

$$\mathrm{FP}_{t \to 0^+} \varphi_{\mathrm{JLO}}^{t} := \left(\varphi_0^{(0,0)}, \varphi_2^{(0,0)}, \ldots \right).$$

Retraction of the JLO Cocycle

Theorem (Connes-Moscovici)

1. $\mathrm{FP}_{t\to 0^+}\,\varphi^t_{\mathrm{JLO}}$ *is an even periodic cyclic cocycle representing the Connes-Chern character.*
2. *For all* $\mathcal{E} \in K_0(\mathcal{A})$,

$$\mathrm{ind}_D[\mathcal{E}] = \left\langle \mathrm{FP}_{t\to 0^+}\,\varphi^t_{\mathrm{JLO}}, \mathcal{E} \right\rangle .$$

Example

For a Dirac spectral triple $\left(C^\infty(M), L^2(M, \not{S}), \not{D}\right)$,

$$\mathrm{FP}_{t\to 0^+}\,\varphi^t_{\mathrm{JLO}} = (\varphi_0, \varphi_{2k}, \ldots),$$

$$\varphi_{2k}(f^0, \ldots, f^{2k}) = \frac{(2i\pi)^{-\frac{n}{2}}}{(2k)!} \int_M f^0 df^1 \wedge \cdots \wedge df^k \wedge \hat{A}(R^M).$$

Dimension Spectrum

For $T \in \mathcal{L}(\mathcal{H})$ set

$$\delta^0(T) = T, \quad \delta^1(T) := [|D|, T], \qquad \delta^2(T) := [|D|, [|D|, T]],$$
$$\delta^j(T) = \underbrace{[|D|, [|D|, \ldots, [|D|, T] \cdots]}_{j \text{ times}}.$$

Definition

$\mathcal{B}$ is the algebra generated by γ and the $\delta^j(a)$ and $\delta^j([D, a])$, $a \in \mathcal{A}$.

Fact

For any $b \in \mathcal{B}$, the function $\zeta_b(z) := \operatorname{Tr}\left[b|D|^{-z}\right]$ is analytic for $\Re z \gg 1$.

Dimension Spectrum

Definition

The *dimension spectrum* is the maximal subset $\Sigma \subset \mathbb{C}$ such that all the functions $\zeta_b(z)$, $b \in \mathcal{B}$, have an analytic continuation to $\mathbb{C} \setminus \Sigma$.

Example

A Dirac spectral triple $\left(C^\infty(M), L^2(M, \not{S}), \not{D}\right)$ has dimension spectrum,

$$\Sigma = \{k \in \mathbb{Z};\ k \leq \dim M\}.$$

Key Assumptions

Assumption

The dimension spectrum Σ is *discrete* and is *simple*, i.e., the zeta functions $\zeta_b(z)$, $b \in \mathcal{B}$, have at worst simple pole singularities on Σ.

Assumption

The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is *regular*, i.e., all the operators in $\mathcal{B}$ are bounded.

Residual Trace

Definition

$\Psi_D^q(\mathcal{A})$, $q \in \mathbb{C}$, is the space of operators such that

$$P \simeq b_0|D|^m + b_1|D|^{m-1} + \dots, \qquad b_j \in \mathcal{B},$$

where $\simeq$ means that, for all $N \in \mathbb{N}$ and $s \in \mathbb{R}$,

$$|D|^{s-m}\Big(P - \sum_{j<N} b_j|D|^{q-j}\Big)|D|^{N-s} \text{ is bounded.}$$

Proposition (Connes-Moscovici)

Under the previous assumptions:

1. $\Psi_D^\bullet(\mathcal{A}) := \bigcup_{q\in\mathbb{C}} \Psi_D^q(\mathcal{A})$ *is an algebra.*
2. *The following formula defines a trace on* $\Psi_D^\bullet(\mathcal{A})$,

$$⨍ P := \mathrm{Res}_{z=0} \mathrm{Tr}\left[P|D|^{-z}\right], \qquad P \in \Psi^\bullet(\mathcal{A}).$$

The CM Cocycle

Theorem (Connes-Moscovici)

1. *The following formulas define a normalized even cocycle* $\varphi_{\text{CM}} = (\varphi_0, \varphi_2, \ldots)$ *in the periodic cyclic cohomology of* $\mathcal{A}$,

$$\varphi_0(a^0) = \text{Res}_{z=0} \left\{ \Gamma(z) \operatorname{Tr}\left[\gamma a^0 |D|^{-z}\right] \right\},$$

$$\varphi_{2k}(a^0, \ldots, a^{2k}) = \sum c_{k,\alpha} \int\!\!\!\!\!- a^0 [D, a^1]^{[\alpha_1]} \cdots [D, a^{2k}]^{[\alpha_{2k}]} |D|^{-2(|\alpha|+k)},$$

where the sum is finite, the $c_{k,\alpha}$ *are universal constants, and we have set*

$$T^{[j]} := \overbrace{[D^2, [D^2, \ldots [D^2, T] \cdots]]}^{j \text{ times}}.$$

2. *The CM cocycle represents the Connes-Chern character, and so we have*

$$\text{ind}_D[\mathcal{E}] = \langle \varphi_{\text{CM}}, \mathcal{E} \rangle \qquad \forall \mathcal{E} \in K_0(\mathcal{A}).$$

The CM Cocycle

Example

For a Dirac spectral triple $\left(C^\infty(M), L^2(M, \not{S}), \not{D}\right)$,

$$\varphi_{\mathrm{CM}} = (\varphi_0, \varphi_2, \ldots),$$

$$\varphi_{2k}(f^0, \ldots, f^{2k}) = \frac{(2i\pi)^{-\frac{n}{2}}}{(2k)!} \int_M f^0 df^1 \wedge \cdots \wedge df^{2k} \wedge \hat{A}(R^M).$$

Classical	NCG
Manifold M	Spectral triple $(\mathcal{A}, \mathcal{H}, D)$
Vector bundles over M	F.g. projective modules over $\mathcal{A}$
$\mathrm{ind} \not{D}_E$	$\mathrm{ind}_D[\mathcal{E}]$
Differential forms	Cyclic cocycles
Atiyah-Singer Index Formula	Connes-Chern character & CM cocycle
Characteristic classes	Cyclic cohomology for Hopf algebras