Noncommutative Geometry
Chapter 10:
K-Theory and Atiyah-Singer Index Theorem

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Atiyah-Singer Index Theorem

Setup

- (M^n, g) is a compact spin oriented Riemannian manifold (n even).
- E is a Hermitian vector bundle over M with connection ∇^E.

Definition (Twisted Dirac Operator)

The operator $\not{D}_E = \not{D}_{E,\nabla^E} : C^\infty(M, \$ \otimes E) \to C^\infty(M, \$ \otimes E)$ is

$$\not{D}_E = \not{D} \otimes 1_E + c \circ \nabla^E,$$

where $c \circ \nabla^E$ is given by the composition,

$$\begin{array}{ccccc} C^\infty(M, \$ \otimes E) & \stackrel{1\otimes\nabla^E}{\rightarrow} & C^\infty(M, \$ \otimes T^*M \otimes E) & \stackrel{c\otimes 1}{\rightarrow} & C^\infty(M, \$ \otimes E) \\ & & \sigma \otimes \xi \otimes s & \longrightarrow & (c(\xi)\sigma) \otimes s. \end{array}$$

Atiyah-Singer Index Theorem

Definition

The *Fredholm index* of $\not{D}_E$ is

$$\operatorname{ind} \not{D}_E := \dim\ker\left[(\not{D}_E)_{|\$^+ \otimes E}\right] - \dim\ker\left[(\not{D}_E)_{|\$^- \otimes E}\right].$$

Theorem (Atiyah-Singer)

$$\operatorname{ind} \not{D}_E = (2i\pi)^{-\frac{n}{2}} \int_M \hat{A}(R^M) \wedge \operatorname{Ch}(F^E),$$

where:

- $\hat{A}(R^M) := \det^{\frac{1}{2}}\left[\frac{R^M/2}{\sinh\left(R^M/2\right)}\right]$ *is called the* $\hat{A}$*-class of the curvature* R^M *of* M.
- $\operatorname{Ch}(F^E) := \operatorname{Tr}\left[e^{-F^E}\right]$ *is called the Chern form of the curvature* F^E *of* ∇^E.

Atiyah-Singer Index Theorem

Remark

The index formula can be proved by heat kernel arguments.

- By the McKean-Singer formula,

$$\begin{aligned} \operatorname{ind} \not{D}_E &= \operatorname{Tr}\left[\gamma e^{-t\not{D}_E^2}\right] \quad \forall t>0 \\ &= \int_M \operatorname{Tr}\left[\gamma e^{-t\not{D}_E^2}(x,x)\right] \operatorname{vol}_g(x) \quad \forall t>0. \end{aligned}$$

  where $\gamma := 1_{\$^+\otimes E} - 1_{\$^-\otimes E}$ is the grading operator.

- The proof is then completed by using:

Theorem (Local Index Theorem; Atiyah-Bott-Patodi, Gilkey)

$$\operatorname{Tr}\left[\gamma e^{-t\not{D}_E^2}(x,x)\right] \operatorname{vol}_g(x) \xrightarrow[t\to 0^+]{} \left[\hat{A}(R^M)\wedge \operatorname{Ch}(F^E)\right]^{(n)}.$$

K-Theory

Setup

- M is a compact manifold.

Definition

Two vector bundles E_1 and E_2 over M are *stably equivalent* if there exists a vector bundle F such that

$$E_1 \oplus F \simeq E_2 \oplus F.$$

Remark

There is an addition on stable equivalence classes of vector bundles given by

$$[E_1] + [E_2] := [E_1 \oplus E_2].$$

This turns the set of stable equivalence classes into a monoid.

Definition

$K^0(M)$ is the Abelian group of formal differences

$$[E_1] - [E_2]$$

of stable equivalence classes of vector bundles over M.

Remark

Let G be an Abelian group and $\varphi : \mathrm{Vect}(M) \to G$ a map such that

$$\varphi(E_1 \oplus E_2) = \varphi(E_1) + \varphi(E_2) \qquad \forall E_j \in \mathrm{Vect}(M).$$

Then φ gives rise to a unique additive map,

$$\varphi : K^0(M) \longrightarrow G,$$
$$\varphi([E]) := \varphi(E) \qquad \forall E \in \mathrm{Vect}(M).$$

Index Map of a Dirac Operator

Setup

- M^n is a compact spin oriented Riemannian manifold (n even).
- $\not{D} : C^\infty(M, \$) \to C^\infty(M, \$)$ is the Dirac operator of M.

Lemma

If E_1 and E_2 are vector bundles over M, then

$$\operatorname{ind} \not{D}_{E_1 \oplus E_2} = \operatorname{ind} \not{D}_{E_1} + \operatorname{ind} \not{D}_{E_2}.$$

Proposition

The Dirac operator gives rise to a unique additive index map,

$$\operatorname{ind}_{\not{D}} : K^0(M) \longrightarrow \mathbb{Z},$$
$$\operatorname{ind}_{\not{D}}[E] := \operatorname{ind} \not{D}_E.$$

de Rham Currents

Setup

M is a compact manifold.

Definition

$\mathcal{D}'_k(M)$ is the space of de Rham currents of dimension k, i.e., continuous linear forms on $C^\infty(M, \Lambda^k_{\mathbb{C}} T^*M)$.

Example

Let N be an oriented submanifold of dimension k. Then N defines a k-dimensional current C_N on M by

$$\langle C_N, \eta \rangle := \int_N \iota^* \eta \qquad \forall \eta \in C^\infty(M, \Lambda^k_{\mathbb{C}} T^*M),$$

where $\iota : N \to M$ is the inclusion of N into M.

Poincaré Duality

Definition

Assume M oriented and set $n = \dim M$. The *Poincaré dual* of an $n-k$- form ω on M is the k-dimensional current $\omega^\wedge$ defined by

$$\langle \omega^\wedge, \eta \rangle := \int_M \omega \wedge \eta \qquad \forall \eta \in C^\infty(M, \Lambda^k_{\mathbb{C}} T^*M).$$

Example

The Poincaré dual of $\hat{A}(R^M)$ is

$$\left\langle \hat{A}(R^M)^\wedge, \eta \right\rangle = \int_M \hat{A}(R^M) \wedge \eta.$$

This is an even (resp., odd) current if $\dim M$ is even (resp., odd).

de Rham Homology

Definition (de Rham Boundary)

The *de Rham boundary* $d^t : \mathcal{D}'_k(M) \to \mathcal{D}'_{k-1}(M)$ is defined by

$$\langle d^t C, \eta \rangle := \langle C, d\eta \rangle \qquad \forall \eta \in C^\infty(M, \Lambda^{k-1}_{\mathbb{C}} T^* M).$$

Definition

The *de Rham homology of M* is the homology of the chain complex $(\mathcal{D}'_\bullet(M), d^t)$. It is denoted $H_\bullet(M)$.

Remark

If M is oriented, then Poincaré duality yields an isomorphism,

$$H^{n-k}(M) \simeq H_k(M).$$

Pairing with K-Theory

Definition

Let $C = C_0 + C_2 + \cdots$ be an even current, and let E be a vector bundle over M. The pairing of C and E is

$$\langle C, E\rangle := \left\langle C, \mathrm{Ch}(F^E)\right\rangle,$$

where F^E is the curvature of any connection on E.

Lemma

The value of $\langle C, E\rangle$ depends only the homology class of C and the K-theory class of E.

Proposition

The above pairing descends to a bilinear pairing,

$$\langle \cdot, \cdot\rangle : H_{\mathrm{even}}(M) \times K^0(M) \longrightarrow \mathbb{C}.$$

Setup

- M^n is a compact spin oriented Riemannian manifold (n even).
- $\not{D} : C^\infty(M, \not{S}) \to C^\infty(M, \not{S})$ is the Dirac operator of M.
- E is a vector bundle over M.

- For the Poincaré dual $C = \hat{A}(R^M)^\wedge$ we get
$$\left\langle \hat{A}(R^M)^\wedge, E \right\rangle = \left\langle \hat{A}(R^M)^\wedge, \mathrm{Ch}(F^E) \right\rangle = \int_M \hat{A}(R^M) \wedge \mathrm{Ch}(F^E).$$
- By Atiyah-Singer Index Theorem,
$$\mathrm{ind}_{\not{D}}[E] = \mathrm{ind}\,\not{D}_E = (2i\pi)^{-\frac{n}{2}} \int_M \hat{A}(R^M) \wedge \mathrm{Ch}(F^E).$$
- Therefore, Atiyah-Singer Index Theorem can be restated as

Theorem (Atiyah-Singer)

$$\mathrm{ind}_{\not{D}}[E] = (2i\pi)^{-\frac{n}{2}} \left\langle \hat{A}(R^M)^\wedge, E \right\rangle \qquad \forall E \in K^0(M).$$

Remark

$\mathrm{Ch}(\not{D}) := (2i\pi)^{-\frac{n}{2}} \left[\hat{A}(R^M)^\wedge\right] \in H_{\mathrm{even}}(M)$ is called the *Chern character* of $\not{D}$.

Noncommutative Vector Bundles

Definition

A *finitely generated projective module* over an algebra $\mathcal{A}$ is a (right-)module of the form,

$$\mathcal{E} = e\mathcal{A}^N, \qquad e \in M_N(\mathcal{A}),\ e^2 = e.$$

Theorem (Serre-Swan)

For $\mathcal{A} = C^\infty(M)$ (with M compact manifold), there is a one-to-one correspondence:

$$\begin{array}{ccc} \{\textit{Vector Bundles over } M\} & \longleftrightarrow & \{\textit{f.g. proj. modules over } C^\infty(M)\} \\ E & \longrightarrow & C^\infty(M, E). \end{array}$$

Suppose that $E = \mathrm{ran}(e)$ with $e = e^* = e^2 \in C^\infty(M, M_q(\mathbb{C}))$. Then

$$C^\infty(M, E) = \{\xi = (\xi_j) \in C^\infty(M, \mathbb{C}^q);\ e\xi = \xi\} = eC^\infty(M)^q.$$

Thus,

$$C^\infty(M, \$ \otimes E) = C^\infty(M, \$) \otimes_{C^\infty(M)} C^\infty(M, E) = eC^\infty(M, \$)^q.$$

Definition

The *Grassmanian connection* ∇_0^E of E is defined by

$$\nabla_0^E \xi := e(d\xi_j) \qquad \forall \xi = (\xi_j) \in C^\infty(M, E).$$

Lemma

Under the identification $C^\infty(M, \not{S} \otimes E) = eC^\infty(M, \not{S})^q$, *the twisted Dirac operator* $\not{D}_E = \not{D}_{E,\nabla_0^E}$ *agrees with*

$$e(\not{D} \otimes 1) : eC^\infty(M, \not{S})^q \longrightarrow eC^\infty(M, \not{S})^q,$$
$$[e(\not{D} \otimes 1)]\, s := e(\not{D} s_j) \qquad \forall s = (s_j) \in eC^\infty(M, \not{S})^q.$$

Index Map of a Spectral Triple

Setup

- $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple with $\mathcal{A}$ unital.
- $\mathcal{E} = e\mathcal{A}^q$, $e^2 = e \in M_q(\mathcal{A})$, is a f.g. projective module.

Remark

$e\mathcal{H}^q$ is a Hilbert space with grading $e\mathcal{H}^q = e\,(\mathcal{H}^+)^q \oplus e\,(\mathcal{H}^-)^q$.

Definition

$D_{\mathcal{E}}$ is the (unbounded) operator of $e\mathcal{H}^q$ with domain $e\,(\mathrm{dom}\,D)^q$ and defined by

$$D_{\mathcal{E}}\sigma := e(D_{\sigma_j}) \qquad \forall \sigma = (\sigma_j) \in e\,(\mathrm{dom}\,D)^q \,.$$

Index Map of a Spectral Triple

Lemma

The operator $D_{\mathcal{E}}$ is Fredholm.

Definition

The index of $D_{\mathcal{E}}$ is

$$\operatorname{ind} D_{\mathcal{E}} := \dim\ker(D_{\mathcal{E}})_{|e(\mathcal{H}^+)^q} - \dim\ker(D_{\mathcal{E}})_{|e(\mathcal{H}^-)^q}.$$

Example

For a Dirac spectral triple $\left(C^\infty(M), L^2(M,\$), \not{D}\right)$, as we saw before

$$\not{D}_{\mathcal{E}} = \not{D}_E \quad \text{with } E := \operatorname{ran}(e).$$

Thus,

$$\operatorname{ind}\not{D}_{\mathcal{E}} = \operatorname{ind}\not{D}_E.$$

K-Theory of $\mathcal{A}$

Definition

Two f.g. projective modules $\mathcal{E}_1$ and $\mathcal{E}_2$ over $\mathcal{A}$ are *stably equivalent* if there exists a f.g. projective module such that

$$\mathcal{E}_1 \oplus \mathcal{F} \simeq \mathcal{E}_2 \oplus \mathcal{F}.$$

Definition

$K_0(\mathcal{A})$ is the Abelian group of formal differences

$$[\mathcal{E}_1] - [\mathcal{E}_2]$$

of stable equivalence classes of f.g. projective modules over $\mathcal{A}$.

Remark

For $\mathcal{A} = C^\infty(M)$, the Serre-Swan theorem implies that

$$K_0\left(C^\infty(M)\right) \simeq K^0(M).$$

The Index Map of a Spectral Triple

Lemma

If $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are f.g. projective modules over* $\mathcal{A}$*, then*

$$\text{ind}\, D_{\mathcal{E}_1 \oplus \mathcal{E}_2} = \text{ind}\, D_{\mathcal{E}_1} + \text{ind}\, D_{\mathcal{E}_2}$$

Proposition

The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ *defines a unique additive index map,*

$$\text{ind}_D : K_0(\mathcal{A}) \longrightarrow \mathbb{Z},$$

such that, for any f.g. projective module $\mathcal{E}$ *over* $\mathcal{A}$*,*

$$\text{ind}_D[\mathcal{E}] = \text{ind}\, D_{\mathcal{E}}.$$

The Index Map of a Spectral Triple

Example

For a Dirac spectral triple $\left(C^\infty(M), L^2(M, \not{S}), \not{D}\right)$, under the Serre-Swan isomorphism

$$K_0\left(C^\infty(M)\right) \simeq K^0(M),$$

the index map $\mathrm{ind}_{\not{D}} : K_0\left(C^\infty(M)\right) \to \mathbb{Z}$ agrees with the Atiyah-Singer index map,

$$\mathrm{ind}_{\not{D}} : K^0(M) \longrightarrow \mathbb{Z}.$$