

1. $\mathbb{Z}_2$ -graded spaces and algebras:

Def: A $\mathbb{Z}_2$ -graded space (or superspace) is a vector space with a $\mathbb{Z}_2$ -grading,
 $E = E^+ \oplus E^-$.

If E and F are superspaces, then $E \otimes F$ is a superspace w/ $\mathbb{Z}_2$ -grading

$$(E \otimes F) = (E \otimes F)^+ \oplus (E \otimes F)^-$$

$$(E \otimes F)^+ = (E^+ \otimes F^+) \oplus (E^- \otimes F^-)$$

$$(E \otimes F)^- = (E^+ \otimes F^-) \oplus (E^- \otimes F^+)$$

Def: A $\mathbb{Z}_2$ -graded algebra (or superalgebra) is an algebra A together w/ a $\mathbb{Z}_2$ -grading

$$A = A^+ \oplus A^-$$

in such way that

$$A^+ A^\pm \subset A^\pm \text{ and } A^- A^\pm \subset A^\mp.$$

• On a superalgebra A the $\mathbb{Z}_2$ -graded commutator (or supercommutator) is defined by

$$[a, b] = \begin{cases} ab - ba & \text{if } a \text{ or } b \text{ is in } A^+, \\ ab + ba & \text{if both } a \text{ and } b \text{ are in } A^-. \end{cases}$$

• Let E be a superspace. Then $\text{End}(E)$ is a superalgebra with $\mathbb{Z}_2$ -grading

$$\text{End}(E) = \text{End}^+(E) \oplus \text{End}^-(E)$$

$$\text{End}^+(E) = \{ T; T(E^\pm) \subset E^\pm \} = \left\{ \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix}; A^\pm \in \text{End}(E^\pm) \right\}$$

$$\text{End}^-(E) = \{ T; T(E^\pm) \subset E^\mp \} = \left\{ \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}; B^\pm \in \mathcal{L}(E^\pm, E^\mp) \right\}$$

Under the splitting $E = E^+ \oplus E^-$ the $\mathbb{Z}_2$ -grading operator is

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$\text{End}(E)^\pm = \{ T; \gamma T = \pm T \gamma \}$$

Def: The supertrace of $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{End}(E)$ is

$$\text{Str}_E(T) = \text{Tr}_E(A) - \text{Tr}_E(D)$$

$$= \text{Tr}_E(\gamma T)$$

Lemma 1.1: For all $S, T \in \text{End}(E)$,

$$\text{Str}_E [S, T] = 0.$$

Proof: It is enough to prove this when S and T are in $\text{End}(E)^\pm$.

If S is in $\text{End}(E)^+$ and $T \in \text{End}(E)^+$, then both ST , TS , and hence $[S, T]$, are in $\text{End}(E)^+$. (2)

Since the supertrace vanishes on $\text{End}(E)^+$, we see that in this case, $\text{Str}_E [S, T] = 0$.

Suppose that both S and T are in $\text{End}(E)^+$. Then $[S, T] = [S, T]$ and γ commutes with S and T . Thus,

$$\text{Str}_E [S, T] = \text{Tr}_E \{ \gamma [S, T] \} = \text{Tr}_E [\gamma S, T] = 0.$$

It remains to look at the case where both S and T are in $\text{End}(E)^-$. Write $S = \begin{pmatrix} 0 & A_- \\ A_+ & 0 \end{pmatrix}$ and

$T = \begin{pmatrix} 0 & B_- \\ B_+ & 0 \end{pmatrix}$. Then

$$[S, T] = ST + TS = \begin{pmatrix} 0 & A_- \\ A_+ & 0 \end{pmatrix} \begin{pmatrix} 0 & B_- \\ B_+ & 0 \end{pmatrix} + \begin{pmatrix} 0 & B_- \\ B_+ & 0 \end{pmatrix} \begin{pmatrix} 0 & A_- \\ A_+ & 0 \end{pmatrix} = \begin{pmatrix} A_- B_+ + B_- A_+ & 0 \\ 0 & A_+ B_- + B_+ A_- \end{pmatrix}$$

Thus,

$$\begin{aligned} \text{Str}_E [S, T] &= \text{Tr}_E^+ (A_- B_+ + B_- A_+) - \text{Tr}_E^- (A_+ B_- + B_+ A_-) \\ &= \{ \text{Tr}_E^+ (A_- B_+) - \text{Tr}_E^- (B_+ A_-) \} + \{ \text{Tr}_E^+ (B_- A_+) - \text{Tr}_E^- (A_+ B_-) \} \\ &= 0. \end{aligned}$$

The proof is complete. $\square$

Let $A = A_+ \oplus A_-$ be a superalgebra.

Def.: A $\mathbb{Z}_2$ -graded module over A is a left-module E together with a $\mathbb{Z}_2$ -grading,

$$E = E_+ \oplus E_-,$$

such that

$$A^+ E^\pm \subset E^\pm \quad \text{and} \quad A^- E^\pm \subset E^\mp.$$

Ex.: Any superspace E is a $\mathbb{Z}_2$ -graded module over the superalgebra $A := \text{End}(E)$.

If E and F are $\mathbb{Z}_2$ -graded modules over A , then $\text{Hom}_A(E, F)$ is a superspace with $\mathbb{Z}_2$ -grading,

$$\text{Hom}_A(E, F) = \text{Hom}_A(E, F)^+ \oplus \text{Hom}_A(E, F)^-$$

where

$$\begin{aligned} \text{Hom}_A(E, F)^+ &= \{ T \in \text{Hom}(E, F); T(E^\pm) \subset F^\pm \} \\ \text{Hom}_A(E, F)^- &= \{ T \in \text{Hom}(E, F); T(E^\pm) \subset F^\mp \} \end{aligned}$$

2. The Exterior Algebra $\Lambda^* \mathbb{R}^n$:

Let $\Lambda^* \mathbb{R}^n = \bigoplus_{j=0}^n \Lambda^j \mathbb{R}^n$ be the exterior algebra of $\mathbb{R}^n$. (In the sequel we shall freely identify $\Lambda^0 \mathbb{R}^n$ with $\mathbb{R}$).

The exterior algebra $\Lambda^* \mathbb{R}^n$ is a superalgebra with $\mathbb{Z}_2$ -grading

$$\Lambda^* \mathbb{R}^n = \Lambda^{\text{ev}} \mathbb{R}^n \oplus \Lambda^{\text{odd}} \mathbb{R}^n, \text{ where}$$

$$\Lambda^{\text{ev}} \mathbb{R}^n = \bigoplus_{j \text{ even}} \Lambda^j \mathbb{R}^n \text{ and } \Lambda^{\text{odd}} \mathbb{R}^n = \bigoplus_{j \text{ odd}} \Lambda^j \mathbb{R}^n.$$

Let $v \in \mathbb{R}^n$. We can associate to v two endomorphisms of $\Lambda^* \mathbb{R}^n$ as follows.

The first operator is the exterior multiplication by v :

$$E(v)\omega := v \wedge \omega \quad \forall \omega \in \Lambda^* \mathbb{R}^n.$$

The 2nd operator is the interior multiplication by v : *ω missing*

$$i(v)(\omega^1 \wedge \dots \wedge \omega^p) = \sum_{1 \leq j \leq p} (-1)^{j-1} \langle v, \omega^j \rangle \omega^1 \wedge \dots \wedge \widehat{\omega^j} \wedge \dots \wedge \omega^p.$$

Notice that $E(v)$ raises the degree of forms, while $i(v)$ decreases it.

Let $\{e^1, \dots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$ (e.g., the canonical basis). Let $\mathcal{I}$ be the family of ordered subsets $I = (i_1, \dots, i_p)$, $1 \leq i_1 < \dots < i_p \leq n$, including the empty set. For $I \in \mathcal{I}$ set

$$e^I = \begin{cases} e^{i_1} \wedge \dots \wedge e^{i_p} & \text{if } I = (i_1, \dots, i_p), \\ 1 & \text{if } I = \emptyset. \end{cases}$$

Then $\{e^I\}_{I \in \mathcal{I}}$ is an orthonormal basis of $\Lambda^* \mathbb{R}^n$.

In the sequel, for $I = (i_1, \dots, i_p) \in \mathcal{I} \setminus \{\emptyset\}$ and $j \in \{1, \dots, n\} \setminus I$ we define

$$E(j, I) := \sup \{q; i_q < j\}.$$

Therefore if $p = E(j, I)$ then $I \cup \{j\} = (i_1, \dots, i_p, j, i_{p+1}, \dots, i_n)$. If $I = \emptyset$ we set $E(j, \emptyset) = 1$.

Lemma 2.1. Let $j \in \{1, \dots, n\}$. Then, for all $I \in \mathcal{I}$,

$$E(e^j) e^I = \begin{cases} 0 & \text{if } j \in I, \\ E(j, I) e^{I \cup \{j\}} & \text{if } j \notin I, \end{cases}$$

and

$$i(e^j) e^I = \begin{cases} E(j, I) e^{I \setminus \{j\}} & \text{if } j \in I, \\ 0 & \text{if } j \notin I. \end{cases}$$

Using these formulas it is not difficult to check that

$$E(e^j) E(e^k) + E(e^k) E(e^j) = i(e^j) i(e^k) + i(e^k) i(e^j) = 0$$

$$E(e^j) i(e^k) + i(e^k) E(e^j) = \delta^{jk} = \langle e^j, e^k \rangle.$$

Therefore, by bilinearity we obtain:

Lemma 2.2: For all $v, w \in \mathbb{R}^n$,

$$\begin{aligned} \varepsilon(v) \varepsilon(w) + \varepsilon(w) \varepsilon(v) &= i(v) i(w) + i(w) i(v) = 0, \\ \varepsilon(v) i(w) + i(w) \varepsilon(v) &= \langle v, w \rangle. \end{aligned}$$

Observe also that the formulas of Lemma 2.1 show that, with respect to the orthonormal basis $\{e^i\}_{i=1}^n$, the matrices of the endomorphisms $\varepsilon(e^i)$ and $i(e^i)$ are transposes of each other. Thus,

$$\varepsilon(e^i)^* = i(e^i).$$

By linearity this gives:

Lemma 2.3: For all $v \in \mathbb{R}^n$,

$$\varepsilon(v)^* = i(v) \text{ and } i(v)^* = \varepsilon(v).$$

3. The Clifford algebra $\text{CPCl}(\mathbb{R}^n)$.

Def.: The Clifford algebra of $\mathbb{R}^n$, denoted $\text{CPCl}(\mathbb{R}^n)$, is the subalgebra of $\text{End}_{\mathbb{R}}(\wedge^* \mathbb{R}^n)$ generated by the operators,

$$c(v) := \varepsilon(v) - i(v), \quad v \in \mathbb{R}^n.$$

Remark: The operator $c(v)$ is called the Clifford multiplication by v .

Using Lemma 2.2 and Lemma 2.3 we get

Lemma 3.1: (i) For all $v, w \in \mathbb{R}^n$,

$$c(v)c(w) + c(w)c(v) = -2\langle v, w \rangle.$$

(ii) For all $v \in \mathbb{R}^n$,

$$c(v)^* = -c(v).$$

It follows from (i) that, for any orthonormal basis $\{e^1, \dots, e^n\}$ we have

$$c(e^i)c(e^j) + c(e^j)c(e^i) = -2\delta^{ij}.$$

This implies that, if $j \neq i$,
 $c(e^i)c(e^j) = -c(e^j)c(e^i)$,

and

$$c(e^i)^2 = -1.$$

Moreover, (ii) implies that $\text{CPCl}(\mathbb{R}^n)$ is closed under taking adjoints, i.e., $\text{CPCl}(\mathbb{R}^n)$ is an involutive sub-algebra of $\text{End}_{\mathbb{R}}(\wedge^* \mathbb{R}^n)$ (and so this is a C^* -algebra over $\mathbb{R}$).

Let $\{e^1, \dots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$. As $\{e^i\}_{i=1}^n$ is an orthonormal basis of $\mathbb{R}^n$, the linear map $\mathbb{R}^n \ni v \mapsto c(v) \in \text{CPCl}(\mathbb{R}^n)$ is uniquely extended into the linear map

$$c: \wedge^* \mathbb{R}^n \longrightarrow \text{CPCl}(\mathbb{R}^n)$$

such that

(5)

$$c(e^I) = \begin{cases} c(e^{i_1}) \cdots c(e^{i_k}) & \text{if } I = (i_1, \dots, i_k), \\ 1 & \text{if } I = \emptyset. \end{cases}$$

On the other, as $CP(\mathbb{R}^n)$ is an algebra of endomorphisms of $\Lambda^* \mathbb{R}^n$ we have a natural linear map $\sigma: CP(\mathbb{R}^n) \rightarrow \Lambda^* \mathbb{R}^n$ defined by

$$\sigma(a) = a \cdot 1 \quad \forall a \in CP(\mathbb{R}^n).$$

Prop. 3.2: The linear maps c and σ are inverses of each other.

Proof: It is clear that c is onto because $1, c(e^{i_1}), \dots, c(e^{i_n})$ generate $CP(\mathbb{R}^n)$.

Furthermore, let $I \in \mathcal{I}$. If $I = \emptyset$, then $c(e^I) = 1$ and hence $c(e^I) \cdot 1 = 1$. Suppose let

$I = (i_1, \dots, i_k) \in \mathcal{I} \setminus \emptyset$. As $c(e^{i_p}) = E(e^{i_p}) - i(e^{i_p})$ carries the degree, we see that

$$\begin{aligned} c(e^I) &= c(e^{i_1}) \cdots c(e^{i_k}) = (E(e^{i_1}) - i(e^{i_1})) \cdots (E(e^{i_k}) - i(e^{i_k})) \\ &= E(e^{i_1}) \cdots E(e^{i_k}) + T, \end{aligned}$$

where T is an operator which carries the degree. In particular, $T \cdot 1 = 0$, and so we have

$$c(e^I) \cdot 1 = E(e^{i_1}) \cdots E(e^{i_k}) \cdot 1 = e^{i_1} \wedge \cdots \wedge e^{i_k} = e^I.$$

This shows that $\sigma \circ c(e^I) = e^I \quad \forall I \in \mathcal{I}$, and hence

$$\sigma \circ c = \text{id}_{\mathcal{I}}.$$

This proves that c is injective and σ is a left-inverse for c . As we already know that c is onto, it follows that c is a linear isomorphism with inverse σ . The proof is complete. $\square$

Remark: An immediate consequence of the fact that σ is the inverse of c , is the fact that the map c does not depend on the choice of the orthonormal basis $\{e^i\}$. In fact, for any orthogonal basis $\{f^i\}$, we have

$$c(f^{i_1} \wedge \cdots \wedge f^{i_k}) = c(f^{i_1}) \cdots c(f^{i_k}), \quad \forall (i_1, \dots, i_k) \in \mathcal{I} \setminus \emptyset.$$

It should be stressed that c and σ are not isomorphisms of algebras, because we have

Lemma 3.3: For all $v, w \in \mathbb{R}^n$,

$$c(v \wedge w) = c(v)c(w) + \langle v, w \rangle.$$

Proof: Let $\{e^1, \dots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$. If $j \neq k$, then

$$c(e^j \wedge e^k) = c(e^j)c(e^k) = c(e^j)c(e^k) + \langle e^j, e^k \rangle.$$

Furthermore, as $c(e^0)^2 = -1$, we see that

$$0 = c(e^0 \wedge e^0) = c(e^0)c(e^0) + \langle e^0, e^0 \rangle.$$

Thus the Lemma holds when u and v are elements of the basis $\{e^i\}$. By linearity it holds for all u and v in $\mathbb{R}^n$. The space is complete. $\square$

In general, we have

Lemma 3.4: Let $\xi \in \wedge^k \mathbb{R}^n$ and $\eta \in \wedge^p \mathbb{R}^n$ with $k+p \leq n$. Then

$$c(\xi \wedge \eta) = c(\xi)c(\eta) \text{ mod } c(\wedge^{k+p-2} \mathbb{R}^n).$$

Proof: Let $\{e^1, \dots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$. It is enough to prove the Lemma when

$\xi = e^I$ and $\eta = e^J$ for some $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_p)$ in $\mathcal{I}$.

Suppose that $I \cap J \neq \emptyset$. Then $e^I \wedge e^J = 0$, so we need to show that $c(e^I)c(e^J)$ is contained in $c(\wedge^{k+p-2} \mathbb{R}^n)$.

Let $j \in I \cap J$. As $c(e^0)c(e^0) = -c(e^0)c(e^0)$ we see that

$$c(e^I) = c(e^{i_1}) \dots c(e^{i_k}) = \pm c(e^{I \setminus \{j\}}) c(e^j),$$

$$c(e^J) = c(e^{j_1}) \dots c(e^{j_p}) = \pm c(e^{J \setminus \{j\}}) c(e^j).$$

Combining with the equality $c(e^0)^2 = -1$, then gives

$$\begin{aligned} c(e^I)c(e^J) &= \pm c(e^{I \setminus \{j\}}) c(e^j) c(e^{J \setminus \{j\}}) c(e^j) \\ &= \pm c(e^{I \setminus \{j\}}) c(e^{J \setminus \{j\}}) \in c(\wedge^{k+p-2} \mathbb{R}^n). \end{aligned}$$

Assume now that $I \cap J = \emptyset$. Write $\{1, \dots, n\} \setminus (I \cup J) = \{m_1, \dots, m_{n-(k+p)}\}$. As $\{e^{i_1}, \dots, e^{i_k}, e^{j_1}, \dots, e^{j_p}, e^{m_1}, \dots, e^{m_{n-(k+p)}}\}$ is an orthonormal basis of $\mathbb{R}^n$,

$$c(e^I \wedge e^J) = c(e^{i_1} \wedge \dots \wedge e^{i_k} \wedge e^{j_1} \wedge \dots \wedge e^{j_p}) = c(e^{i_1}) \dots c(e^{i_k}) c(e^{j_1}) \dots c(e^{j_p}) = c(e^I) c(e^J).$$

This completes the proof of the Lemma. $\square$

Next, the $\mathbb{Z}_2$ -grading of $\wedge^* \mathbb{R}^n$ gives rise to $\mathbb{Z}_2$ -grading on $\mathcal{C}P(\mathbb{R}^n)$. Namely,

$$\mathcal{C}P(\mathbb{R}^n) = \mathcal{C}P^+(\mathbb{R}^n) \oplus \mathcal{C}P^-(\mathbb{R}^n),$$

$$\mathcal{C}P^+(\mathbb{R}^n) = (\wedge^{\text{even}} \mathbb{R}^n) \text{ and } \mathcal{C}P^-(\mathbb{R}^n) = c(\wedge^{\text{odd}} \mathbb{R}^n).$$

Prop. 3.5: $\mathcal{C}P(\mathbb{R}^n)$ is a superalgebra.

Proof: For $u \in \mathbb{R}^n$ the endomorphisms $E(u)$ and $i(u)$ shift the degree by 1, so there are elements of $\text{End}(\wedge^* \mathbb{R}^n)^-$, and hence the Clifford multiplication $c(u)$ is in $\text{End}(\wedge^* \mathbb{R}^n)^-$ too. As a product of an even (resp., odd) number of elements of $\text{End}(\wedge^* \mathbb{R}^n)^-$ is contained in $\text{End}(\wedge^* \mathbb{R}^n)^+$ (resp., $\text{End}(\wedge^* \mathbb{R}^n)^-$), we see that $\mathcal{C}P^+(\mathbb{R}^n)$ (resp., $\mathcal{C}P^-(\mathbb{R}^n)$) is contained in $\text{End}(\wedge^* \mathbb{R}^n)^+$ (resp., $\text{End}(\wedge^* \mathbb{R}^n)^-$), and hence

$$\mathcal{C}P^\pm(\mathbb{R}^n) = \text{End}(\wedge^* \mathbb{R}^n)^\pm \cap \mathcal{C}P(\mathbb{R}^n).$$

Thus,

(7)

$$\begin{aligned} \mathcal{CP}^+(\mathbb{R}^n), \mathcal{CP}^\pm(\mathbb{R}^n) &\subset (\text{End}(\Lambda^* \mathbb{R}^n)^\pm, \text{End}(\Lambda^* \mathbb{R}^n)^\pm) \cap \mathcal{CP}(\mathbb{R}^n) \subset \mathcal{CP}^\pm(\mathbb{R}^n), \\ \mathcal{CP}^-(\mathbb{R}^n), \mathcal{CP}^\pm(\mathbb{R}^n) &\subset (\text{End}(\Lambda^* \mathbb{R}^n)^\pm, \text{End}(\Lambda^* \mathbb{R}^n)^\pm) \cap \mathcal{CP}(\mathbb{R}^n) \subset \mathcal{CP}^\mp(\mathbb{R}^n). \end{aligned}$$

This proves that $\mathcal{CP}(\mathbb{R}^n)$ is a superalgebra. $\square$

As $\mathcal{CP}(\mathbb{R}^n)$ is a superalgebra, we can look at supercommutators. The following result, although seemingly simple, will have many important consequences in the sequel.

Lemma 3.6 Let $\sigma \in \mathbb{R}^n$ and $\eta \in \Lambda^* \mathbb{R}^n$. Then

$$[c(\sigma), c(\eta)]' = -2c(\iota(\sigma)\eta).$$

Proof: Let $\{e^1, \dots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$. Then it is enough to prove the Lemma when $\eta = e^I$ and $\sigma = e^j$ for some $I = (i_1, \dots, i_r) \in \mathcal{I}$ and $j \in \{1, \dots, n\}$. Furthermore, upon reordering the basis $\{e^i\}$ we may assume that $j=1$ and either $I = (1, \dots, r)$ if $j \in I$ or $I = (2, \dots, r+1)$ otherwise.

Suppose that $j \notin I$. Then $\iota(e^j)c^I = 0$. Moreover, as $c(e^i)c(e^j) = -c(e^j)c(e^i)$ for $i \neq j$, we have

$$c(e^j)c(e^I) = c(e^1)c(e^2)\dots c(e^r) = (-1)^r c(e^2)\dots c(e^r)c(e^1) = (-1)^r c(e^j)c(e^I).$$

Thus,

$$[c(e^j), c(e^I)]' = c(e^j)c(e^I) - (-1)^r c(e^I)c(e^j) = 0 = -2c(\iota(e^j)e^I).$$

Assume now that $j \in I$. Then

$$\iota(e^j)c^I = \iota(e^1)(e^1 \wedge \dots \wedge e^r) = e^1 \wedge \dots \wedge e^r.$$

Furthermore, as $(c(e^1))^2 = -1$, we have

$$c(e^j)c(e^I) = c(e^1)c(e^1)c(e^2)\dots c(e^r) = -c(e^2)\dots c(e^r),$$

$$c(e^I)c(e^j) = c(e^1)\dots c(e^r)c(e^1) = (-1)^{r-1}c(e^1)^2c(e^2)\dots c(e^r) = (-1)^r c(e^2)\dots c(e^r).$$

Thus,

$$\begin{aligned} [c(e^j), c(e^I)]' &= c(e^j)c(e^I) - (-1)^r c(e^I)c(e^j) \\ &= -c(e^2)\dots c(e^r) - (-1)^r (-1)^r c(e^2)\dots c(e^r) \\ &= -2c(e^2)\dots c(e^r) \\ &= -2c(\iota(e^j)e^I). \end{aligned}$$

This completes the proof. $\square$

Observe that if $\sigma, \omega^1, \dots, \omega^r$ are orthonormal unit vectors, then

$$\sigma \wedge \omega^1 \wedge \dots \wedge \omega^r = \iota(\sigma)(\sigma \wedge \omega^1 \wedge \dots \wedge \omega^r).$$

It follows from this that any element of $\bigwedge^k \mathbb{R}^n$ for $k \leq n-1$ is a linear combination of elements of the form $i(v)z$ with $v \in \mathbb{R}^n$ and $z \in \bigwedge^{k-1} \mathbb{R}^n$. Combining this observation with the previous lemma we obtain:

Lemma 3.7: The supercommutator space of $CP(\mathbb{R}^n)$ is contained in $c\left(\bigoplus_{j=0}^{n-1} \bigwedge^j \mathbb{R}^n\right)$.

As $CP^+(\mathbb{R}^n)CP^+(\mathbb{R}^n) \subset CP^+(\mathbb{R}^n)$ we see that $CP^+(\mathbb{R}^n)$ is a subalgebra of $CP(\mathbb{R}^n)$.

In fact, we have

Lemma 3.8: If $n \geq 3$, there is an isomorphism of algebras,
 $CP^+(\mathbb{R}^n) \cong CP(\mathbb{R}^{n-1})$.

Proof: Let $\{e^1, \dots, e^n\}$ be the canonical basis of $\mathbb{R}^n$. Upon identifying $\mathbb{R}^{n-1}$ with the span of $e^1, \dots, e^{n-1}$ in $\mathbb{R}^n$ we can identify $CP(\mathbb{R}^{n-1})$ with the subalgebra of $CP(\mathbb{R}^n)$ generated by $c(e^1), \dots, c(e^{n-1})$. Then

$$CP(\mathbb{R}^n) = CP(\mathbb{R}^{n-1}) \oplus (CP(\mathbb{R}^{n-1})c(e^n))$$

$$CP^+(\mathbb{R}^n) = CP^+(\mathbb{R}^{n-1}) \oplus (CP^-(\mathbb{R}^{n-1})c(e^n)).$$

We also observe that, as $c(e^n)$ anticommutes with the $c(e^j)$, $1 \leq j \leq n-1$, we have

$$c(e^n)a = \pm c(e^n)c(a) \quad \forall a \in CP^{\pm}(\mathbb{R}^{n-1})$$

As $CP(\mathbb{R}^{n-1}) = CP^+(\mathbb{R}^{n-1}) \oplus CP^-(\mathbb{R}^{n-1})$ any $a \in CP(\mathbb{R}^{n-1})$ can be uniquely written as $a = a^+ + a^-$, $a^{\pm} \in CP^{\pm}(\mathbb{R}^{n-1})$.

We then define a linear map $\alpha: CP(\mathbb{R}^{n-1}) \rightarrow CP^+(\mathbb{R}^n)$ by
 $\alpha(a) := a^+ + a^-c(e^n)$.

This map is onto since $CP^+(\mathbb{R}^n) = CP^+(\mathbb{R}^{n-1}) \oplus CP^-(\mathbb{R}^{n-1})c(e^n)$. Moreover, if $\alpha(a) = 0$, then $a^+ = a^-c(e^n) = 0$, and hence $0 = a^-c(e^n)c(e^n) = -a^-$, so that $a = 0$. Therefore, α is a linear isomorphism.

It remains to show that α is a morphism of algebras. Let $a, b \in CP(\mathbb{R}^n)$. Then

$$\begin{aligned} \alpha(a)\alpha(b) &= (a^+ + a^-c(e^n))(b^+ + b^-c(e^n)) \\ &= a^+b^+ + a^-c(e^n)b^-c(e^n) + (a^+b^-c(e^n) + a^-c(e^n)b^+) \end{aligned}$$

As $c(e^n)^2 = -1$ and $c(e^n)$ commutes (resp. anticommutes) with the elements of $CP^+(\mathbb{R}^n)$ (resp., $CP^-(\mathbb{R}^n)$) we see that

$$a^-c(e^n)b^-c(e^n) = -a^-c(e^n)^2b^- = a^-b^-,$$

$$a^-c(e^n)b^+ = a^-b^+c(e^n)$$

(9)

Therefore,

$$\begin{aligned}\alpha(a)d(e) &= (a^+p^+ + a^-p^-) + (a^+p^- + a^-p^+)c(e^u) \\ &= (ap)^+ + (ap)^-c(e^u) \\ &= \alpha(ap).\end{aligned}$$

This proves that α is a morphism of algebras and completes the proof. $\square$

Next, it is convenient to complicate $CP(\mathbb{R}^n)$. We introduce

$$CP_{\mathbb{C}}(\mathbb{R}^n) = CP(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C} \quad \text{and} \quad CP_{\mathbb{C}}^{\pm}(\mathbb{R}^n) = CP^{\pm}(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}.$$

Upon identifying $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$ with $\mathbb{C}^n$ and $(\wedge^k \mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$ with $\wedge^k \mathbb{C}^n$ we extend the $\mathbb{R}$ -linear isomorphisms c and σ into $\mathbb{C}$ -linear isomorphisms

$$\wedge^k \mathbb{C}^n \xrightarrow{c} CP_{\mathbb{C}}(\mathbb{R}^n)$$

We observe that, for all $v, w \in \mathbb{C}^n$,

$$(1) \quad c(v)c(w) + c(w)c(v) = -2\langle \bar{v}, w \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product of $\mathbb{C}^n$:

$$\langle v, w \rangle := \sum \bar{v}_j w_j \quad \forall v = (v_j) \in \mathbb{C}^n \quad \forall w = (w_j) \in \mathbb{C}^n.$$

If $\{e^1, \dots, e^n\}$ is any oriented orthonormal basis, the volume element of $\mathbb{R}^n$ is

$$\omega_{\mathbb{R}^n} = e^1 \wedge \dots \wedge e^n \in \wedge^n \mathbb{R}^n.$$

Def. The chirality operator is the element of $CP_{\mathbb{C}}(\mathbb{R}^n)$ defined by

$$\Gamma = (i)^{\frac{n+1}{2}} c(\omega_{\mathbb{R}^n})$$

where $\{e^1, \dots, e^n\}$ is any oriented orthonormal basis of $\mathbb{R}^n$.

Lemma 39: (i) $\Gamma^2 = 1$ and $\Gamma^* = \Gamma$.

$$(ii) \quad c(v)\Gamma = (-1)^{n+1} \Gamma c(v) \quad \forall v \in \mathbb{R}^n. \quad (iii) \quad \Gamma^* = \Gamma.$$

Proof. Let $\{e^1, \dots, e^n\}$ be an oriented orthonormal basis of $\mathbb{R}^n$. Then

$$\begin{aligned}c(\omega_{\mathbb{R}^n})^2 &= c(\omega_{\mathbb{R}^n}) \cdot c(\omega_{\mathbb{R}^n}) = c(e^1) \dots c(e^n) c(e^1) \dots c(e^n) \\ &= (-1)^{n-1} c(e^1) c(e^1) c(e^2) \dots c(e^n) c(e^2) \dots c(e^n) \\ &= (-1)^{\frac{(n-1)n}{2}} c(e^1) c(e^1) c(e^2) c(e^2) \dots c(e^n) c(e^n) \\ &= (-1)^{\frac{(n-1)n}{2}} (-1) \dots (-1) = (-1)^{\frac{(n-1)n}{2}} (-1)^n = (-1)^{\frac{(n+1)n}{2}}.\end{aligned}$$

If n is even, then $\Gamma = i^{\frac{n}{2}} c(\omega_n)$, and hence

$$\Gamma^2 = (-1)^{\frac{n}{2}} c(\omega_n)^2 = (-1)^{\frac{n}{2}} \cdot (-1)^{\frac{(n+1)n}{2}} = (-1)^{\frac{(n+2)n}{2}} = 1,$$

since both n and $n+2$ are even numbers.

If n is odd, then $\Gamma = i^{\frac{n+1}{2}} c(\omega_n)$, and hence

$$\Gamma^2 = (-1)^{\frac{n+1}{2}} c(\omega_n)^2 = (-1)^{\frac{n+1}{2}} \cdot (-1)^{\frac{(n+1)n}{2}} = (-1)^{\frac{(n+1)^2}{2}} = 1,$$

since $(n+1)$ is even and so $(n+1)^2$ is a multiple of 4.

In any case we see that $\Gamma^2 = 1$, so the first part of the Lemma is proved.

In order to prove the 2nd part, it is enough to do it when $\alpha = e^j$ for some $j \in \{1, \dots, n\}$.

In fact, upon reordering the basis $\{e^1, \dots, e^n\}$ we may assume that $j=1$. In that case this is a byproduct of the proof of the 1st part. Indeed, as we have shown above,

$$c(\omega_n) c(e^1) = c(e^1) \dots c(e^n) c(e^1) = (-1)^{n-1} c(e^1) c(e^1) \dots c(e^n) = (-1)^{n-1} c(e^1) c(\omega_{n-1}),$$

and hence $\Gamma c(e^1) = (-1)^{n-1} c(e^1) \Gamma$.

Let us now compute Γ^* . As by Lemma 3.1 $c(e^j)^* = -c(e^j)$, we get

$$\begin{aligned} \Gamma^* &= (i^{\frac{n+1}{2}})^* c(e^n)^* \dots c(e^1)^* = i^{\frac{n+1}{2}} (-1)^{\frac{(n+1)}{2} + n} c(e^n) \dots c(e^1) = i^{\frac{n+1}{2}} (-1)^{\frac{(n+1)}{2} + n + \frac{n(n-1)}{2}} c(e^1) \dots c(e^n) \\ &= (-1)^{\frac{(n+1)}{2} + \frac{n(n+1)}{2}} \Gamma. \end{aligned}$$

If n is even, then $\frac{(n+1)}{2} = \frac{n}{2}$, and hence $(-1)^{\frac{(n+1)}{2} + \frac{n(n+1)}{2}} = (-1)^{\frac{n}{2} + \frac{n(n+1)}{2}} = (-1)^{\frac{n(n+2)}{2}} = 1$. If n is odd, then $\frac{(n+1)}{2} = \frac{n+1}{2}$, and hence $(-1)^{\frac{(n+1)}{2} + \frac{n(n+1)}{2}} = (-1)^{\frac{n+1}{2} + \frac{n(n+1)}{2}} = (-1)^{\frac{(n+1)^2}{2}} = 1$. In both cases we see that $\Gamma^* = \Gamma$.

The proof is complete. $\square$

Lemma 3.10: Assume n odd. Then Γ is a central element of $Cl_0(\mathbb{R}^n)$ and we have

$$Cl_0^-(\mathbb{R}^n) = \Gamma Cl_0^+(\mathbb{R}^n) \quad \text{and} \quad Cl_0(\mathbb{R}^n) = Cl_0^+(\mathbb{R}^n) \oplus \Gamma Cl_0^+(\mathbb{R}^n),$$

where $Cl_0^{\pm}(\mathbb{R}^n)$ are the even/odd subalgebras of $Cl_0(\mathbb{R}^n)$.

4. The Spinor Representation:

Thm. 4.1 (Spinor Representation, n even): Assume n even. Then:

1. Up to equivalence, the algebra $Cl_0(\mathbb{R}^n)$ has a unique irreducible representation, namely, the spinor representation,

$$\rho: Cl_0(\mathbb{R}^n) \longrightarrow \text{End}(\mathcal{S}),$$

where $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is the space of spinors of $\mathbb{R}^n$ (see definition in page).

2. The representation ρ gives rise to an isomorphism of algebras,

$$Cl_0(\mathbb{R}^n) \cong \text{End}(\mathcal{S}).$$

Rmk.: The unicity up to equivalence means that if $\pi: C_0(\mathbb{R}^n) \rightarrow \text{End}(E)$ is another irreducible representation then $\mathcal{F}$ and E are isomorphic $C_0(\mathbb{R}^n)$ -modules, i.e., there exists a linear isomorphism $\varphi: \mathcal{F} \rightarrow E$ s.t.

$$\varphi(p(a)\sigma) = \pi(a)\varphi(\sigma) \quad \forall \sigma \in \mathcal{F} \quad \forall a \in C_0(\mathbb{R}^n).$$

Rmk.: As we shall $\mathcal{F}$ has dimension $2^{n/2}$, so we get the following isomorphism of algebras,

$$C_0(\mathbb{R}^n) \cong \text{End}(\mathcal{F}) \cong M_{2^{n/2}}(\mathbb{R}).$$

Since $M_{2^{n/2}}(\mathbb{R})$ is a simple algebra, it is clear that $C_0(\mathbb{R}^n)$ has a unique irreducible representation.

The next 10 pages are devoted to proving Theorem 4.1. PLEASE TURN OVER.

Proof of Theorem 4.1:

Construction of the spinor representation:

Set $m = \frac{n}{2}$ and let $\{e^1, \dots, e^m\}$ be an orthonormal basis of $\mathbb{R}^m$.

$$z^j = \frac{1}{\sqrt{2}}(e^j + ie^{j+m}) \quad \text{and} \quad \bar{z}^j = \overline{z^j} = \frac{1}{\sqrt{2}}(e^j - ie^{j+m}).$$

Then $\{z^j, \bar{z}^j\}_{1 \leq j \leq m}$ is an orthonormal basis of $\mathbb{C}^m$ and we get the orthogonal splitting,

$$\mathbb{C}^m = \Lambda^{1,0} \mathbb{C}^m \oplus \Lambda^{0,1} \mathbb{C}^m,$$

where

$$\Lambda^{1,0} \mathbb{C}^m = \text{Span}\{z^1, \dots, z^m\} \quad \text{and} \quad \Lambda^{0,1} \mathbb{C}^m = \text{Span}\{\bar{z}^1, \dots, \bar{z}^m\} = \overline{\Lambda^{1,0} \mathbb{C}^m}.$$

This splitting then gives rise to the orthogonal decomposition

$$\Lambda^k \mathbb{C}^m = \bigoplus_{p+q=k} \Lambda^{p,q} \mathbb{C}^m,$$

where

$$\begin{aligned} \Lambda^{p,q} \mathbb{C}^m &= (\Lambda^{1,0} \mathbb{C}^m)^p \wedge (\Lambda^{0,1} \mathbb{C}^m)^q \\ &= \text{Span}\left\{ z^{i_1} \wedge \dots \wedge z^{i_p} \wedge \bar{z}^{j_1} \wedge \dots \wedge \bar{z}^{j_q} \mid 1 \leq i_1 < \dots < i_p \leq m, \right. \\ &\quad \left. 1 \leq j_1 < \dots < j_q \leq m \right\}. \end{aligned}$$

Furthermore, we see that the algebra $\mathcal{C}l(\mathbb{R}^n)$ is generated by $\{e^j\}$ and $\{e^{j+m}\}$. Moreover, by using the formula (1) on page 9 we see that

$$d(z^j) d(\bar{z}^k) + d(\bar{z}^k) d(z^j) = c(\bar{z}^k) c(z^j) + c(z^j) c(\bar{z}^k) = 0,$$

$$d(z^j) d(\bar{z}^j) + d(\bar{z}^j) d(z^j) = -2 \delta^{jj}, \quad 1 \leq j \leq m \quad (\text{See Remark page 5})$$

The spinor bundle is defined to be

$$S_m := \Lambda^{0,*} \mathbb{C}^m = \bigoplus_{q=0}^m \Lambda^{0,q} \mathbb{C}^m$$

We define a linear map ρ from $\mathcal{C}l(\mathbb{R}^n) = \mathcal{C}l(\Lambda^{1,0} \mathbb{C}^m) \oplus \mathcal{C}l(\Lambda^{0,1} \mathbb{C}^m)$ to $\text{End}(S_m)$. If $\bar{z} \in \Lambda^{0,1} \mathbb{C}^m$, then

$$\rho(d(\bar{z})) \sigma = \sqrt{2} E(\bar{z}) \sigma,$$

where $E(\bar{z})$ is the exterior product by $\bar{z}$, i.e.,

$$E(\bar{z}) \sigma := \bar{z} \wedge \sigma \quad \forall \sigma \in \Lambda^{0,*} \mathbb{C}^m.$$

If $z \in \Lambda^{1,0} \mathbb{C}^m$, then

$$\rho(c(z)) = -\sqrt{2} i(z),$$

where $i(z)$ is the interior product by z , that is,

$$i(z)(\omega^1 \wedge \dots \wedge \omega^q) = \sum_{j=1}^q (-1)^{j-1} \langle \bar{z}, \omega^j \rangle \omega^1 \wedge \dots \wedge \hat{\omega}^j \wedge \dots \wedge \omega^q \quad \forall \omega^j \in \Lambda^{0,1} \mathbb{C}^m.$$

(See Remark page 5)

By using Lemma 2.2 it is not difficult to check that, for $j, k = 1, \dots, m$,
 $i(z^j) i(z^k) + i(z^k) i(z^j) = \varepsilon(z^j) \varepsilon(z^k) + \varepsilon(z^k) \varepsilon(z^j) = 0$
 $i(z^j) \varepsilon(z^k) + \varepsilon(z^k) i(z^j) = \delta^{jk}$,

and hence

$$\begin{aligned} \rho(\alpha(z^j)) \rho(\alpha(z^k)) + \rho(\alpha(z^k)) \rho(\alpha(z^j)) &= 0 \\ \rho(\alpha(z^j)) \rho(\alpha(z^k)) + \rho(\alpha(z^j)) \rho(\alpha(z^k)) &= 0 \\ \rho(\alpha(z^j)) \rho(\alpha(z^k)) + \rho(\alpha(z^k)) \rho(\alpha(z^j)) &= -2\delta^{jk} \end{aligned}$$

By bilinearity it then follows that, for all $v, w \in \mathbb{C}^n$,

$$\rho(\alpha(v)) \rho(\alpha(w)) + \rho(\alpha(w)) \rho(\alpha(v)) = -2\langle \bar{v}, w \rangle.$$

Lemma 4.2: Let A be a (unital) algebra over $\mathbb{C}$ and let $\pi: \mathcal{C}(\mathbb{C}^n) \rightarrow A$ be a $\mathbb{C}$ -linear map s.t.

$$(4.1) \quad \pi(\alpha(v)) \pi(\alpha(w)) + \pi(\alpha(w)) \pi(\alpha(v)) = -2\langle \bar{v}, w \rangle \quad \forall v, w \in \mathbb{C}^n$$

Then π uniquely extends to a ^{unital} $\ast$ -morphism of algebras,

$$\pi: \mathcal{C}_0(\mathbb{R}^n) \rightarrow A.$$

In addition, for any $I = (i_1, \dots, i_n) \in \mathcal{I}_n$,

$$\pi(\alpha(e^{i_1} \wedge \dots \wedge e^{i_n})) = \pi(\alpha(e^{i_1})) \dots \pi(\alpha(e^{i_n})).$$

Proof: The unicity follows from the fact that the $\alpha(v)$'s, $v \in \mathbb{C}^n$, generate $\mathcal{C}_0(\mathbb{R}^n)$. Therefore, we only have to show that the map $\pi: \mathcal{C}(\mathbb{C}^n) \rightarrow A$ can be extended into a $\ast$ -morphism of algebras $\pi: \mathcal{C}_0(\mathbb{R}^n) \rightarrow A$.

We extend π into the $\mathbb{C}$ -linear map $\pi: \mathcal{C}_0(\mathbb{R}^n) \rightarrow A$ s.t., for all $I = (i_1, \dots, i_n) \in \mathcal{I}_n$,

$$\pi(1) = 1_A$$

$$\pi(\alpha(e^{i_1} \wedge \dots \wedge e^{i_n})) = \pi(\alpha(e^{i_1})) \dots \pi(\alpha(e^{i_n})) \quad \forall I = (i_1, \dots, i_n) \in \mathcal{I}_n \setminus \emptyset.$$

As $\alpha(e^{i_1} \wedge \dots \wedge e^{i_n}) = \alpha(e^{i_1}) \dots \alpha(e^{i_n})$, we see that, by definition,

$$\pi(\alpha(e^{i_1}) \dots \alpha(e^{i_n})) = \pi(\alpha(e^{i_1})) \dots \pi(\alpha(e^{i_n})) \quad \text{if } 1 \leq i_1 < \dots < i_n \leq n.$$

Observe also that (4.1) implies that

$$\pi(\alpha(e^j)) \pi(\alpha(e^k)) + \pi(\alpha(e^k)) \pi(\alpha(e^j)) = -2\delta^{jk},$$

and hence

$$(4.2) \quad \pi(\alpha(e^j)) \pi(\alpha(e^k)) = -\pi(\alpha(e^k)) \pi(\alpha(e^j)) \quad \text{if } j \neq k$$

$$(4.3) \quad \pi(\alpha(e^j))^2 = -1$$

cPaim: Let $v \in \mathbb{C}^n$ and $a \in C_P(\mathbb{R}^n)$. Then

$$\pi(c(v)a) = \pi(c(v))\pi(a)$$

Proof of the cPaim: It is enough to prove the cPaim when $v = e^{\hat{j}}$ and $a = c(e^{\hat{I}})$ for some $j \in \{1, \dots, n\}$ and $I = (i_1, \dots, i_k) \in \mathcal{I}_n$.

Assume that $j \notin I$. If $j < i_1$, then

$$\begin{aligned}\pi(c(e^{\hat{j}})c(e^{\hat{I}})) &= \pi(c(e^{\hat{j}})c(e^{\hat{i}_1}) \dots c(e^{\hat{i}_k})) \\ &= \pi(c(e^{\hat{j}}))\pi(c(e^{\hat{i}_1})) \dots \pi(c(e^{\hat{i}_k})) \\ &= \pi(c(e^{\hat{j}}))\pi(c(e^{\hat{I}}))\end{aligned}$$

Suppose that $j > i_1$ and let $p = \sup\{q; i_q < j\}$. Then

$$\begin{aligned}\pi(c(e^{\hat{j}})c(e^{\hat{I}})) &= c(e^{\hat{j}})c(e^{\hat{i}_1}) \dots c(e^{\hat{i}_k}) \\ &= (-1)^p c(e^{\hat{i}_1}) \dots c(e^{\hat{i}_p}) c(e^{\hat{j}}) \dots c(e^{\hat{i}_k})\end{aligned}$$

Thus,

$$\pi(c(e^{\hat{j}})c(e^{\hat{I}})) = (-1)^p \pi(c(e^{\hat{i}_1}) \dots c(e^{\hat{i}_p})) \pi(c(e^{\hat{j}}) \dots c(e^{\hat{i}_k})) = \pi(c(e^{\hat{j}}))\pi(c(e^{\hat{I}})).$$

Therefore, using (4.2) we get

$$\pi(c(e^{\hat{j}})c(e^{\hat{I}})) = \pi(c(e^{\hat{j}}))\pi(c(e^{\hat{i}_1}) \dots c(e^{\hat{i}_k})) = \pi(c(e^{\hat{j}}))\pi(c(e^{\hat{I}})).$$

Assume now that $j \in I$ and let p be the index s.t. $i_p = j$. Then

$$\begin{aligned}c(e^{\hat{j}})c(e^{\hat{I}}) &= c(e^{\hat{i}_1})c(e^{\hat{i}_2}) \dots c(e^{\hat{i}_k}) \\ &= (-1)^{p-1} c(e^{\hat{i}_1}) \dots c(e^{\hat{i}_{p-1}}) c(e^{\hat{i}_p}) \dots c(e^{\hat{i}_k}) \\ &= (-1)^p c(e^{\hat{i}_1}) \dots \widehat{c(e^{\hat{i}_p})} \dots c(e^{\hat{i}_k}).\end{aligned}$$

Conversely, by using (4.2)-(4.3) we get

$$\begin{aligned}\pi(c(e^{\hat{j}}))\pi(c(e^{\hat{I}})) &= \pi(c(e^{\hat{i}_1}) \dots c(e^{\hat{i}_k})) \\ &= (-1)^p \pi(c(e^{\hat{i}_1}) \dots \widehat{c(e^{\hat{i}_p})} \dots c(e^{\hat{i}_k})).\end{aligned}$$

As $\pi(c(e^{\hat{i}_1}) \dots \widehat{c(e^{\hat{i}_p})} \dots c(e^{\hat{i}_k})) = \pi(c(e^{\hat{i}_1}) \dots \pi(c(e^{\hat{i}_p})) \dots c(e^{\hat{i}_k}))$, we deduce that

$$\pi(c(e^{\hat{j}})c(e^{\hat{I}})) = \pi(c(e^{\hat{j}}))\pi(c(e^{\hat{I}})).$$

This completes the proof of the cPaim. $\square$

Let us now go back to the proof of Lemma 4.2. Let $a \in C_P(\mathbb{R}^n)$. Then combining the above cPaim with an easy induction shows that

$$\pi(c(e^{\hat{i}_1}) \dots c(e^{\hat{i}_k}), a) = \pi(c(e^{\hat{i}_1}) \dots \pi(c(e^{\hat{i}_k})) \pi(a)) \quad \text{if } i_1 < \dots < i_k.$$

As $c(e^{i_1}) \cdots c(e^{i_k}) = c(e^{i_1} \wedge \cdots \wedge e^{i_k})$ and $\pi(c(e^{i_1})) \cdots \pi(c(e^{i_k})) = \pi(c(e^{i_1} \wedge \cdots \wedge e^{i_k}))$, we

see that, for all $I \in \mathcal{I}_n$,

$$\pi(c(e^I) a) = \pi(c(e^I)) \pi(a).$$

By linearity it follows that

$$\pi(p a) = \pi(p) \pi(a) \quad \forall a, p \in C_0(\mathbb{R}^n)$$

This proves that π is a morphism of algebras. $\square$

Let us go back to the construction of the spinor representation. Using the above we see that the linear map $\rho: C(\mathbb{R}^n) \rightarrow \text{End}(\mathbb{S})$ uniquely extends to a morphism of algebras

$$\rho: C_0(\mathbb{R}^n) \longrightarrow \text{End}(\mathbb{S}),$$

in such way that

$$\rho(c(e^{i_1} \wedge \cdots \wedge e^{i_k})) = \rho(c(e^{i_1})) \cdots \rho(c(e^{i_k})) \quad \text{if } i_1 < \cdots < i_k.$$

In particular, ρ is a representation of $C_0(\mathbb{R}^n)$ in $\mathbb{S}$.

B. Irreducibility of the spinor representation (even):

Let $\mathcal{I}$ be the family consisting of the empty subset of $\{1, \dots, m\}$ and of all the ordered subsets $I = (i_1, \dots, i_m)$ of $\{1, \dots, m\}$ w/ $1 \leq i_1 < \dots < i_m \leq 1$. For $I \in \mathcal{I}$ we set

$$z^I = z^{i_1} \wedge \dots \wedge z^{i_m} \quad \text{and} \quad z^{\bar{I}} = z^{\bar{i}_1} \wedge \dots \wedge z^{\bar{i}_q} \quad \text{if } I = (i_1, \dots, i_q) \neq \emptyset, \\ z^\emptyset = z^{\bar{\emptyset}} = 1.$$

Moreover, the length of any $I \in \mathcal{I}$ is defined to be

$$|I| = \begin{cases} 0 & \text{if } I = \emptyset, \\ q & \text{if } I = (i_1, \dots, i_q) \neq \emptyset. \end{cases}$$

Then:

- $\{z^I \wedge z^{\bar{J}}\}_{I, J \in \mathcal{I}}$ is an orthonormal basis of $\wedge \mathbb{C}^n$,
- $\{z^{\bar{I}}\}_{I \in \mathcal{I}}$ is an orthonormal basis of $\mathcal{S} = \wedge^{0,*} \mathbb{C}^n$,
- $\{z^{\bar{I}}\}_{I \in \mathcal{I}}$ is an orthonormal basis of $\wedge^{0,q} \mathbb{C}^n$.

For brevity, if $I = (i_1, \dots, i_q) \neq \emptyset$, then

$$c(z^I) = c(z^{i_1}) \dots c(z^{i_q}) \quad \text{and} \quad c(z^{\bar{I}}) = c(z^{\bar{i}_1}) \dots c(z^{\bar{i}_q}).$$

Claim. Let $\bar{z}, \bar{w} \in \mathcal{S}$. Then there always exists $a \in C^0(\mathbb{R}^n)$ s.t.

$$\rho(a) \bar{z} = \bar{w}.$$

Proof of the claim: Write $\bar{z} = \sum \alpha_I z^I$, $\alpha_I \in \mathbb{C}$, and define

$$q = \sup \{ |I|; \alpha_I \neq 0 \}.$$

Assume that $q = 0$. Then $\bar{z} = \alpha \cdot 1$, with $\alpha = \alpha_\emptyset \in \mathbb{C} \setminus \{0\}$. Set $\alpha_2 = \bar{\alpha}'$. Then

$$\rho(a) \bar{z} = \rho(\bar{\alpha}') \alpha = \bar{\alpha}' \cdot \alpha = 1.$$

Suppose that $q \geq 1$ and let $I^\circ = (i_1, \dots, i_q) \in \mathcal{I}$ be s.t. $|I^\circ| = q$ and $\alpha_{I^\circ} \neq 0$. Observe that

$$i(z^{i_q}) \dots i(z^{i_1}) z^{\bar{I}} = \begin{cases} 0 & \text{if } |I| \leq q \text{ or } |I| = q \text{ and } I \neq I^\circ, \\ 1 & \text{if } I = I^\circ. \end{cases}$$

Set

$$\alpha_2 = \alpha_{I^\circ}^{-1} \left(\frac{-1}{\sqrt{2}} \right)^q c(z^{i_q}) \dots c(z^{i_1}) = \alpha_{I^\circ}^{-1} \left(\frac{-1}{\sqrt{2}} \right)^q c(z^{i_q} \wedge \dots \wedge z^{i_1}).$$

Then

$$\begin{aligned} \rho(\alpha_2) \bar{z} &= \left\{ \alpha_{I^\circ}^{-1} \left(\frac{-1}{\sqrt{2}} \right)^q \left(\frac{1}{\sqrt{2}} \right) i(z^{i_q}) \dots \left(\frac{-1}{\sqrt{2}} \right) c(z^{i_1}) \right\} \sum_{|I| \leq q} \alpha_I z^{\bar{I}} \\ &= \alpha_{I^\circ}^{-1} \sum_{|I| \leq q} \alpha_I c(z^{i_q}) \dots c(z^{i_1}) z^{\bar{I}} = \alpha_{I^\circ}^{-1} \alpha_{I^\circ} = 1. \end{aligned}$$

Then, whether $q=0$ or $q \geq 1$ there always exists $a_2 \in (C_0(\mathbb{R}^n))^*$ s.t.

$$\rho(a_2)\bar{z} = 1.$$

Next, let us write $\bar{\omega} = \sum \beta_I z^I$ and define

$$a_1 = \sum \bar{z}^{|\mathbb{I}|/2} \beta_I c(z^{\bar{I}}).$$

Choose that $I = (i_1, \dots, i_q) \in \mathcal{I} \setminus \emptyset$, then

$$\begin{aligned} \rho(c(z^{\bar{I}})) &= \rho(c(z^{\bar{i}_1}) \dots c(z^{\bar{i}_q})) \\ &= \rho(c(z^{\bar{i}_1})) \dots \rho(c(z^{\bar{i}_q})) \\ &= \sqrt{2} \mathcal{E}(z^{\bar{i}_1}) \dots \sqrt{2} \mathcal{E}(z^{\bar{i}_q}) \\ &= 2^{q/2} \mathcal{E}(z^{\bar{i}_1} \wedge \dots \wedge z^{\bar{i}_q}) \\ &= 2^{|\mathbb{I}|/2} \mathcal{E}(z^{\bar{I}}). \end{aligned}$$

Thus,

$$\begin{aligned} \rho(a_1) &= \sum \bar{z}^{|\mathbb{I}|/2} \beta_I \cdot \rho(c(z^{\bar{I}})) \\ &= \sum \bar{z}^{|\mathbb{I}|/2} \beta_I \cdot 2^{|\mathbb{I}|/2} \mathcal{E}(z^{\bar{I}}) \\ &= \mathcal{E}\left(\sum \beta_I z^{\bar{I}}\right) \\ &= \mathcal{E}(\bar{\omega}). \end{aligned}$$

Now, define

$$a = a_1 a_2.$$

As $\rho(a_2)\bar{z} = 1$ and $\rho(a_1) = \mathcal{E}(\bar{\omega})$, we see that

$$\rho(a)\bar{z} = \rho(a_1)\rho(a_2)\bar{z} = \mathcal{E}(\bar{\omega}) \cdot 1 = \bar{\omega}.$$

The claim is thus proved. $\square$

Now, let E be a subspace of $\mathcal{S}$ which is invariant under ρ and suppose that $E \neq \{0\}$. Let $\bar{\omega} \in \mathcal{S} \setminus \{0\}$ and let $\bar{z} \in E \setminus \{0\}$. Then by Claim 1 there exists $a \in (C_0(\mathbb{R}^n))^*$ s.t. $\rho(a)\bar{z} = \bar{\omega}$. As E is ρ -invariant we see that $\bar{\omega}$ must be contained. It then follows that $E = \mathcal{S}$. Thus ρ has no non-trivial invariant subspaces, that is, ρ is an irreducible representation.

Invertibility of the spinor representation (new):

As $\{z^{\bar{I}} \wedge z^{\bar{J}}\}_{I, J \in \mathcal{J}}$ is a basis of $\Lambda^2 \mathbb{C}^n$, we see that $\{c(z^{\bar{I}} \wedge z^{\bar{J}})\}_{I, J \in \mathcal{J}}$ is a basis of $C_0(\mathbb{R}^n)$. Observe also that as, for all $\xi \in \Lambda^i \mathbb{C}^n$ and $\eta \in \Lambda^j \mathbb{C}^n$ it is true that

$$c(\xi \wedge \eta) = c(\xi)c(\eta) \mod c(\Lambda^{i+j-2} \mathbb{C}^n),$$

we see that, for all $I, J \in \mathcal{J}$,

$$c(z^{\bar{I}} \wedge z^{\bar{J}}) = c(z^{\bar{I}})c(z^{\bar{J}}) \mod c(\Lambda^{|I|+|J|-2} \mathbb{C}^n).$$

Therefore, upon fixing a suitable ordering of the elements of $\mathcal{J}$, the matrices of the coordinates of the $c(z^{\bar{I}})c(z^{\bar{J}})$ with respect to the basis $\{c(z^{\bar{I}} \wedge z^{\bar{J}})\}_{I, J \in \mathcal{J}}$ can be written as an upper-triangular matrix whose diagonal entries are all equal to 1. In particular, this matrix is invertible and hence $\{c(z^{\bar{I}})c(z^{\bar{J}})\}_{I, J \in \mathcal{J}}$ is a basis of $C_0(\mathbb{R}^n)$. Therefore, any $a \in C_0(\mathbb{R}^n)$ can be uniquely written as

$$a = \sum \alpha_{I, J} c(z^{\bar{I}})c(z^{\bar{J}}), \quad \alpha_{I, J} \in \mathbb{C}.$$

Claim 2: Let $a \in C_0(\mathbb{R}^n) \setminus \{0\}$. Then there always exists $\sigma \in \mathcal{J}$ s.t.

$$\rho(c(a))\sigma \neq 0.$$

Proof: let us write

$$a = \sum \alpha_{I, J} c(z^{\bar{I}})c(z^{\bar{J}}), \quad \alpha_{I, J} \in \mathbb{C},$$

and define

$$j_0 = \min_{I, J} |J|; \quad \exists I \in \mathcal{J} \text{ s.t. } \alpha_{I, J} \neq 0 \{$$

Observe also that if $J = (j_1, \dots, j_k) \in \mathcal{J} \setminus \{\emptyset\}$, then

$$c(z^{\bar{J}}) = c(z^{\bar{j}_1}) \dots c(z^{\bar{j}_k})$$

$$\begin{aligned} \rho(c(z^{\bar{J}})) &= \rho(c(z^{\bar{j}_1})) \dots \rho(c(z^{\bar{j}_k})) \\ &= (-\sqrt{2})^k i(z^{\bar{j}_1}) \dots i(z^{\bar{j}_k}) \end{aligned}$$

Suppose now that $j_0 = 0$. Then, for all $J = (j_1, \dots, j_k) \in \mathcal{J} \setminus \{\emptyset\}$,

$$\rho(c(z^{\bar{J}})) \cdot 1 = (-\sqrt{2})^k i(z^{\bar{j}_1}) \dots i(z^{\bar{j}_k}) \cdot 1 = 0$$

Thus,

$$\begin{aligned} \rho(a) \cdot 1 &= \sum \alpha_{I, J} \rho(c(z^{\bar{I}})) \rho(c(z^{\bar{J}})) \cdot 1 \\ &= \sum_{I \in \mathcal{J}} \alpha_{I, \emptyset} \rho(c(z^{\bar{I}})) \rho(c(1)) \cdot 1 \\ &= \sum_{I \in \mathcal{J}} \alpha_{I, \emptyset} \rho(c(z^{\bar{I}})) \cdot 1 \end{aligned}$$

As observed before,

$$\rho(c(z^{\bar{I}})) = 2^{-|I|/2} \varepsilon(z^{\bar{I}}).$$

Therefore,

$$\rho(a)1 = \sum_{I \in \mathcal{I}} \alpha_{I,0} 2^{-|I|/2} \varepsilon(z^{\bar{I}})1 = \sum_{I \in \mathcal{I}} \alpha_{I,0} z^{\bar{I}} \neq 0,$$

since at least one of the $\alpha_{I,0}$ is $\neq 0$ due to the fact that $p_0 \neq 0$. This proves the claim.

Suppose now that p_0 is ≥ 1 and let $J_0 = (j_1, \dots, j_{p_0}) \in \mathcal{I} \setminus \emptyset$ be such that $\exists J \in \mathcal{I}$ so that $\alpha_{I,J_0} \neq 0$. Let $J = (j_1, \dots, j_p) \in \mathcal{I} \setminus \emptyset$ with $p \geq p_0$. Then

$$\begin{aligned} \rho(c(z^{\bar{J}})) z^{\bar{j}_{p_0}} \wedge \dots \wedge z^{\bar{j}_1} &= (-\sqrt{2})^{p_0} i(z^{\bar{j}_1}) \dots i(z^{\bar{j}_{p_0}}) z^{\bar{j}_{p_0}} \wedge \dots \wedge z^{\bar{j}_1} \\ &= \begin{cases} 0 & \text{if } p > p_0 \text{ or } p = p_0 \text{ and } J \neq J_0, \\ (-\sqrt{2})^{p_0} & \text{if } J = J_0. \end{cases} \end{aligned}$$

Therefore, if we set

$$\sigma = (-\frac{1}{\sqrt{2}})^{p_0} z^{\bar{j}_{p_0}} \wedge \dots \wedge z^{\bar{j}_1},$$

then we have

$$\rho(a)\sigma = \sum_{\substack{I, J \in \mathcal{I} \\ |J| \geq p_0}} \alpha_{I,J_0} \rho(c(z^{\bar{I}})) \rho(c(z^{\bar{J}})) \cdot (-\frac{1}{\sqrt{2}})^{p_0} z^{\bar{j}_{p_0}} \wedge \dots \wedge z^{\bar{j}_1}$$

$$= \sum_{I \in \mathcal{I}} \alpha_{I,J_0} \rho(c(z^{\bar{I}})) \cdot 1$$

$$= \sum_{I \in \mathcal{I}} \alpha_{I,J_0} z^{\bar{I}} \quad (\text{as in the case } p_0 = 0)$$

$\neq 0$.

Thus the claim holds even when $p_0 \neq 0$. The proof of the claim is complete.

This claim shows that, for all $a \in C_b(\mathbb{R}^n) \setminus \{0\}$, the endomorphism $\rho(a)$ is $\neq 0$. To, $\lim_{a \rightarrow 0} \rho(a) = \{0\}$.

Observe now that $\mathcal{I} = \bigoplus_{q=0}^n \wedge^{0,q} \mathbb{C}^n$ has dimension $2^n = 2^{n/2}$, and so $\dim \text{End}(\mathcal{I}) = (2^{n/2})^2 = 2^n$. Moreover, the isomorphism between $C_b(\mathbb{R}^n)$ and $\wedge^* \mathbb{C}^n$ shows that $\dim C_b(\mathbb{R}^n) = 2^n$. Since $C_b(\mathbb{R}^n)$ and $\text{End}(\mathcal{I})$ have same dimension, the injectivity of ρ implies that ρ is an isomorphism.

D. Uniqueness of the spinor representation (n even):

(2)

Before getting to the uniqueness of the spinor representation, let us briefly recall why the matrix algebras are simple, i.e., they have only one irreducible representation up to equivalence.

Let $k \in \mathbb{N}$ and let us denote by $\lambda: M_k(\mathbb{C}) \rightarrow \text{End}(\mathbb{C}^k)$ the representation of $M_k(\mathbb{C})$ in $\mathbb{C}^k$ defined by

$$\lambda(A)\xi := A \cdot \xi \quad \forall A \in M_k(\mathbb{C}) \quad \forall \xi \in \mathbb{C}^k.$$

This representation is irreducible because, for all $\xi, \eta \in \mathbb{C}^k \setminus \{0\}$, there is $A \in M_k(\mathbb{C})$ s.t. $A\xi = \eta$.

Prop. 4.3: Up to equivalence, λ is the unique irreducible representation of $M_k(\mathbb{C})$.

Proof: Let $\pi: M_k(\mathbb{C}) \rightarrow \text{End}(V)$ be an irreducible representation of $M_k(\mathbb{C})$ in some vector space V .

For $\xi, \eta \in \mathbb{C}^k$ we shall denote by $E_{\xi, \eta}$ the elementary matrices s.t.

$$(E_{\xi, \eta})\zeta = \langle \eta, \zeta \rangle \xi \quad \forall \zeta \in \mathbb{C}^k.$$

Then $M_k(\mathbb{C})$ is spanned by the $E_{\xi, \eta} \propto \xi$ and η range over $\mathbb{C}^k \setminus \{0\}$ (or over a basis of $\mathbb{C}^k$). Furthermore, observe that

$$A E_{\xi, \eta} = E_{A\xi, \eta} \quad \forall A \in M_k(\mathbb{C}) \quad \forall \xi, \eta \in \mathbb{C}^k.$$

As π is non-zero, there exist $\xi_0, \eta_0 \in \mathbb{C}^k \setminus \{0\}$ and $v \in V \setminus \{0\}$ s.t.

$$\pi(E_{\xi_0, \eta_0})v \neq 0.$$

Let $\varphi: \mathbb{C}^k \rightarrow V$ be the linear map defined by

$$\varphi(\xi) := \pi(E_{\xi, \eta_0})v \quad \forall \xi \in \mathbb{C}^k.$$

Let $S \in \mathbb{C}^k$ and $A \in M_k(\mathbb{C})$, then

$$\begin{aligned} \varphi(\lambda(A)\xi) &= \varphi(A\xi) = \pi(E_{A\xi, \eta_0})v \\ &= \pi(A E_{\xi, \eta_0})v = \pi(A) \pi(E_{\xi, \eta_0})v = \pi(A) \varphi(\xi). \end{aligned}$$

Thus φ is an $M_k(\mathbb{C})$ -equivariant linear map. This implies that $\ker \varphi$ and $\text{im } \varphi$ are invariant subspaces of λ and π respectively. As λ and π are irreducible representations and φ is non-zero (since $\varphi(\xi_0) = \pi(E_{\xi_0, \eta_0})v \neq 0$), we then deduce that $\ker \varphi = \{0\}$ and $\text{im } \varphi = V$, that is, φ is an $M_k(\mathbb{C})$ -equivariant linear isomorphism. Therefore, the representations λ and π are equivalent, proving the proposition. $\square$

The above proposition means that $M_k(\mathbb{C})$ is a simple algebra. By corollary any algebra isomorphic (as algebra) to some $M_k(\mathbb{C})$ is simple.

Now, as we have isomorphisms of algebras,

$$C_0(\mathbb{R}^n) \subseteq \text{End}(\mathbb{S}) \simeq M_2(\mathbb{R}^n) \quad \text{w/ } 2 = 2^{n/2},$$

we see that the Clifford algebra $C_0(\mathbb{R}^n)$ is simple. Therefore, the spinor representation is the unique irreducible representation of $C_0(\mathbb{R}^n)$ up to equivalence. This

completes the proof of the Theorem. $\square$.

When n is odd, the following holds.

Theorem 4.4 (Spinor Representation, odd case): Assume n odd. Then

(1) Up to equivalence, there is a unique irreducible representation of $C_0(\mathbb{R}^n)$ mapping Γ to the identity, namely, the spinor representation,

$$\rho: C_0(\mathbb{R}^n) \longrightarrow \mathbb{S}_n,$$

where $\mathbb{S}_n$ is the spinor bundle of $\mathbb{R}^n$ (see definition before).

(2) The spinor representation induces an isomorphism of algebras,

$$C_0^+(\mathbb{R}^n) \simeq \text{End}(\mathbb{S}_n).$$

(3) Under the spinor representation, the chirality operator Γ acts like the identity on $\mathbb{S}_n$.

Remark: As we shall see, $\mathbb{S}_n$ has dimension $2^{\frac{n-1}{2}}$, so the isomorphism $C_0^+(\mathbb{R}^n) \simeq \text{End}(\mathbb{S}_n)$ and the equality $C_0(\mathbb{R}^n) = C_0^+(\mathbb{R}^n) \oplus \Gamma C_0^+(\mathbb{R}^n)$ imply that

$$C_0(\mathbb{R}^n) \simeq M_{2^{\frac{n-1}{2}}}(\mathbb{R}^n) \oplus \Gamma M_{2^{\frac{n-1}{2}}}(\mathbb{R}^n).$$

Proof of the Theorem 4.4 (let us denote by $\gamma: C_0(\mathbb{R}^{n+1}) \longrightarrow C_0^+(\mathbb{R}^n)$ the algebra isomorphism from $C_0(\mathbb{R}^{n+1})$ into $C_0^+(\mathbb{R}^n)$ as constructed in pages 2 and 3. Let $\mathbb{S}_n$ be the space of spinors of $\mathbb{R}^n$ and let us denote by $\rho_n: C_0(\mathbb{R}^{n+1}) \longrightarrow \text{End}(\mathbb{S}_n)$ the spinor representation of $C_0(\mathbb{R}^{n+1})$. Set $\mathbb{S}_{n-1} = \mathbb{S}_n$ and let $\rho_n: C_0(\mathbb{R}^n) \longrightarrow \text{End}(\mathbb{S}_n)$ be the linear map defined by

$$\rho_n(a' + \Gamma a'') = \rho_{n-1}(\gamma(a')) + \rho_{n-1}(\gamma(a'')) \quad \forall a', a'' \in C_0^+(\mathbb{R}^n).$$

Observe that $\rho_n(\Gamma) = \text{id}_{\mathbb{S}_n}$.

Let $a, b \in C_0(\mathbb{R}^n)$ and write $a = a' + \Gamma a''$ and $b = b' + \Gamma b''$ with $a, a', b, b'' \in C_0^+(\mathbb{R}^n)$. Then, as $\Gamma^2 = 1$ and Γ commutes with all the elements of $C_0^+(\mathbb{R}^n)$,

$$ab = (a' + \Gamma a'')(b' + \Gamma b'') = (a'b' + a''b'') + \Gamma(a'b'' + a''b').$$

Thus,

$$\begin{aligned}
 \rho_n(a'b') &= \rho_n((a'b' + a''b'') + \Gamma(a'b'' + a''b')) = \rho_{n+1} \circ \gamma^{-1}(a'b' + a''b'') + \rho_{n+1} \circ \gamma^{-1}(a'b'' + a''b') \quad (22) \\
 &= \rho_{n+1} \circ \gamma(a') \rho_{n+1} \circ \gamma(b') + \rho_{n+1} \circ \gamma(a'') \rho_{n+1} \circ \gamma(b'') + \rho_{n+1} \circ \gamma(a') \rho_{n+1} \circ \gamma(b'') + \rho_{n+1} \circ \gamma(a'') \rho_{n+1} \circ \gamma(b') \\
 &= \rho_n(a') \rho_n(b') + \rho_n(a'') \rho_n(b'') + \rho_n(a') \rho_n(b'') + \rho_n(a'') \rho_n(b') \\
 &= (\rho_n(a') + \rho_n(a'')) (\rho_n(b') + \rho_n(b'')) \\
 &= \rho_n(a' + \Gamma a'') \rho_n(b' + \Gamma b'') = \rho_n(a) \rho_n(b).
 \end{aligned}$$

Therefore ρ_n is a representation of the algebra $\mathcal{C}_0(\mathbb{R}^n)$ in $\mathcal{H}_n$.

Observe that in the subalgebra $\mathcal{C}_0^+(\mathbb{R}^n)$, the spinor representation ρ_n agrees with $\rho_{n+1} \circ \gamma^{-1}$, which is the spinor representation of $\mathcal{C}_0(\mathbb{R}^{n-1})$ under the algebra isomorphism $\gamma: \mathcal{C}_0(\mathbb{R}^{n-1}) \xrightarrow{\sim} \mathcal{C}_0^+(\mathbb{R}^n)$. As the representation ρ_{n+1} is irreducible, it then follows that $\mathcal{H}_n$ has no non-trivial subspaces invariant under $\rho_n(\mathcal{C}_0^+(\mathbb{R}^n))$, and hence ρ_n has no non-trivial invariant subspaces. That is, ρ_n is an irreducible representation.

Let us now prove the uniqueness of the spinor representation ρ_n . Let $\pi: \mathcal{C}_0(\mathbb{R}^n) \rightarrow \text{End}(V)$ be an irreducible representation of $\mathcal{C}_0(\mathbb{R}^n)$ in some vector space V in such a way that $\pi(\Gamma) = \text{id}_V$.

Let W be a subspace of V which is invariant under the action of $\pi(\mathcal{C}_0^+(\mathbb{R}^n))$, i.e.,

$$\pi(a)w \in W \quad \forall a \in \mathcal{C}_0^+(\mathbb{R}^n) \quad \forall w \in W.$$

Let $a \in \mathcal{C}_0(\mathbb{R}^n)$ and let us write $a = a' + \Gamma a''$ w/ $a', a'' \in \mathcal{C}_0^+(\mathbb{R}^n)$. As $\pi(\Gamma) = \text{id}_V$, we see that

$$\pi(a) = \pi(a' + \Gamma a'') = \pi(a') + \pi(\Gamma) \pi(a'') = \pi(a') + \pi(a'') = \pi(a' + a'').$$

As $a' + a'' \in \mathcal{C}_0^+(\mathbb{R}^n)$, we deduce that, for all $w \in W$,

$$\pi(a)w = \pi(a' + a'')w \in W.$$

Therefore, W is an invariant subspace of π , and hence W is either $\{0\}$ or V . Thus, we have shown that $\pi|_{\mathcal{C}_0^+(\mathbb{R}^n)}$ is an irreducible representation of $\mathcal{C}_0^+(\mathbb{R}^n)$.

Observe that ρ_{n+1} is an algebra isomorphism from $\mathcal{C}_0(\mathbb{R}^{n-1})$ onto $\text{End}(\mathcal{H}_{n+1}) = \text{End}(\mathcal{H}_n)$, and $\rho_n|_{\mathcal{C}_0^+(\mathbb{R}^n)} = \rho_{n+1} \circ \gamma^{-1}$, we see that ρ_n induces an algebra isomorphism,

$$\mathcal{C}_0^+(\mathbb{R}^n) \xrightarrow{\sim} \text{End}(\mathcal{H}_n).$$

As $\dim \mathcal{H}_n = \dim \mathcal{H}_{n+1} = 2^{\frac{n}{2}}$, we thus have an algebra isomorphism,

$$\mathcal{C}_0^+(\mathbb{R}^n) \xrightarrow{\sim} M_{2^{\frac{n}{2}}}(\mathbb{C}).$$

It then follows that, up to equivalence, $\mathcal{C}_0^+(\mathbb{R}^n)$ has a unique irreducible representation. Thus

(23)

$\rho|_{\mathcal{C}_0(\mathbb{R}^n)}$ and $\pi|_{\mathcal{C}_0(\mathbb{R}^n)}$ are equivalent representations of $\mathcal{C}_0^*(\mathbb{R}^n)$, i.e., $\exists$ linear isomorphism $\phi: \mathcal{H} \rightarrow V$ s.t. $\phi(\rho(a)\sigma) = \pi(a)\phi(\sigma) \quad \forall \sigma \in \mathcal{H} \quad \forall a \in \mathcal{C}_0^*(\mathbb{R}^n)$.

Let $a \in \mathcal{C}_0(\mathbb{R}^n)$ and write $a = a' + \Gamma a'$. We have

$$\rho(a) = \rho(a' + \Gamma a') = \rho(a' + a'') \quad \text{and} \quad \pi(a) = \pi(a' + \Gamma a') = \pi(a' + a'').$$

As $a' + a'' \in \mathcal{C}_0^*(\mathbb{R}^n)$, we deduce that, for all $\sigma \in \mathcal{H}$,

$$\phi(\rho(a)\sigma) = \phi(\rho(a' + a'')\sigma) = \pi(a' + a'')\phi(\sigma) = \pi(a)\phi(\sigma).$$

This proves that ϕ is an $\mathcal{C}_0(\mathbb{R}^n)$ -equivariant linear isomorphism from $\mathcal{H}$ to V , so π and ρ are equivalent representations of $\mathcal{C}_0(\mathbb{R}^n)$. The proof of the Theorem is complete. $\square$

As $\mathcal{C}_0(\mathbb{R}^n)$ is a subalgebra of $\text{End}_{\mathbb{C}}(\mathbb{R}^n)$, we can regard $\mathcal{C}_0(\mathbb{R}^n)$ as a subalgebra of $\text{End}_{\mathbb{C}}(\mathbb{R}^n)$. It then follows from Lemma 3.1 that $\mathcal{C}_0(\mathbb{R}^n)$ is an involutive subalgebra of $\text{End}_{\mathbb{C}}(\mathbb{R}^n)$.

Prop. 4.5: The spinor representation is a $*$ -representation, i.e.,

$$\rho(a^*) = \rho(a)^* \quad \forall a \in \mathcal{C}_0(\mathbb{R}^n).$$

Proof: Let us first assume that n is even.

Using Lemma 2.3 we see that, for $j=1, \dots, m$,

$$\begin{aligned} E(g\bar{\partial})^* &= (E(e\partial) - i E(e^m\partial))^* = E(e\partial)^* + i E(e^m\partial)^* \\ &= i(e\partial) + i \cdot i(e^m\partial) = i(g\partial). \end{aligned}$$

Thus,

$$\begin{aligned} \rho(\alpha(g\bar{\partial}))^* &= \sqrt{2} E(g\bar{\partial})^* = \sqrt{2} i(g\partial) = -\rho(\alpha(g\partial)) \\ \rho(\alpha(g\partial))^* &= -\rho(\alpha(g\bar{\partial})). \end{aligned}$$

Moreover, using Lemma 3.1 we also get

$$\begin{aligned} \alpha(g\bar{\partial})^* &= (\alpha(e\partial) - i\alpha(e^m\partial))^* = \alpha(e\partial)^* + i\alpha(e^m\partial)^* \\ &= -\alpha(e\partial) - i\alpha(e^m\partial) = -\alpha(g\partial), \end{aligned}$$

and hence $\alpha(g\partial)^* = -\alpha(g\bar{\partial})$. Thus,

$$\rho(\alpha(g\bar{\partial}))^* = \rho(-\alpha(g\partial)) = \rho(\alpha(g\bar{\partial}))^*$$

$$\rho(\alpha(g\partial)) = \rho(-\alpha(g\bar{\partial})) = \rho(\alpha(g\partial)).$$

It then follows that

$$\rho(\alpha(a)^*) = \rho(\alpha(a))^* \quad \forall a \in \mathbb{C}^n.$$

Therefore, for any $I = (e_1, \dots, e_n) \in \mathcal{I}_n$,

$$\begin{aligned} \rho((c(I))^*) &= \rho((c(e_1) \dots c(e_n))^*) = \rho((c(e_1))^* \dots (c(e_n))^*) \\ &= \rho(c(e_1)^*) \dots \rho(c(e_n)^*) \\ &= \rho(c(e_1)) \dots \rho(c(e_n)) \\ &= (\rho(c(e_1)) \dots \rho(c(e_n)))^* \\ &= \rho(c(I))^* \end{aligned}$$

It then follows that $\rho(a^*) = \rho(a)^* \forall a \in \mathcal{C}_0(\mathbb{R}^n)$. This proves the proposition when n is even.

Let us now assume that n is odd. Let $\varphi: \mathcal{C}_0(\mathbb{R}^{n+1}) \rightarrow \mathcal{C}_0^+(\mathbb{R}^n)$ be the linear isomorphism

$$\varphi(a) = a^+ + a^- c(e^n),$$

where $e^n = (0, \dots, 0, 1)$ and $a = a^+ + a^-$ is the decomposition in terms of components in $\mathcal{C}_0^+(\mathbb{R}^n)$. Then

by definition,

$$\rho(a) = \rho_{n+1}(\varphi(a)) \quad \forall a \in \mathcal{C}_0^+(\mathbb{R}^n),$$

where ρ_{n+1} is the spinor representation of $\mathcal{C}_0(\mathbb{R}^{n+1})$.

Observe that Lemma 3.1 implies that $\mathcal{C}_0^+(\mathbb{R}^{n+1})$ and $\mathcal{C}_0^-(\mathbb{R}^{n+1})$ are involutive subspaces of $\mathcal{C}_0(\mathbb{R}^{n+1})$. Thus,

$$(a^\pm)^* = (a^\pm)^\pm \quad \forall a \in \mathcal{C}_0(\mathbb{R}^n).$$

As pointed out in the proof of Lemma 3.8, all the elements of $\mathcal{C}_0(\mathbb{R}^{n+1})$ anticommutes with $c(e^n)$. As $c(e^n)^* = -c(e^n)$ we see that, for all $a \in \mathcal{C}_0(\mathbb{R}^{n+1})$,

$$\begin{aligned} \varphi(a)^* &= (a^+ + a^- c(e^n))^* = (a^+)^* + c(e^n)^* (a^-)^* \\ &= (a^+)^* - c(e^n) (a^-)^* = (a^+)^* + (a^-)^* c(e^n) = \varphi(a^*). \end{aligned}$$

This proves that φ is a $*$ -isomorphism, so φ^{-1} is a $*$ -isomorphism. Therefore, for all $a \in \mathcal{C}_0^+(\mathbb{R}^n)$,

$$\rho(a^*) = \rho_{n+1}(\varphi^{-1}(a^*)) = \rho_{n+1}(\varphi^{-1}(a)^*) = \rho_{n+1}(\varphi^{-1}(a))^* = \rho(a)^*.$$

Let $a \in \mathcal{C}_0^-(\mathbb{R}^n)$. Then $\Gamma a \in \mathcal{C}_0^+(\mathbb{R}^n)$. As $\rho(\Gamma) = 1$ we get

$$\rho((\Gamma a)^*) = \rho(\Gamma a)^* = (\rho(\Gamma) \rho(a))^* = \rho(a)^*.$$

As Lemma 3.10 tells us that $\Gamma^* = \Gamma$ we also get

$$\rho((\Gamma a)^*) = \rho(a^* \Gamma) = \rho(a^*) \rho(\Gamma) = \rho(a^*),$$

and hence $\rho(a^*) = \rho(a)^*$.

All this shows that ρ is a $*$ -representation even when n is odd. The proof is complete. $\square$

From now on and until the end of the section, we assume that m is even.

As $\Gamma^2 = 1$ and $\Gamma^* = \Gamma$, we see that

$$\rho(\Gamma)^2 = \rho(\Gamma^2) = 1 \quad \text{and} \quad \rho(\Gamma)^* = \rho(\Gamma^*) = \rho(\Gamma).$$

Therefore, we get that we have an algebraic decomposition

$$\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-,$$

given by the eigenspaces of $\rho(\Gamma)$, i.e.,

$$\mathcal{S}^+ = \ker(\rho(\Gamma) - 1) \quad \text{and} \quad \mathcal{S}^- = \ker(\rho(\Gamma) + 1).$$

We thus have a $\mathbb{Z}_2$ -grading on $\mathcal{S}$. In fact, we have:

Prop. 4.6: Equipped w/ the above $\mathbb{Z}_2$ -grading $\mathcal{S}$ is a $\mathbb{Z}_2$ -graded module over $\mathcal{C}\ell_0(\mathbb{R}^m)$.

Proof: As m is even, $\Gamma(c_i) = -c_i \Gamma \quad \forall c_i \in \mathbb{C}^m$. It then follows that Γ commutes with the elements of $\mathcal{C}\ell_0^+(\mathbb{R}^m)$ and anticommutes with those of $\mathcal{C}\ell_0^-(\mathbb{R}^m)$. Therefore, if $a \in \mathcal{C}\ell_0^+(\mathbb{R}^m)$ and $\sigma \in \mathcal{S}^+$, then

$$\Gamma \rho(\Gamma)(\rho(a)\sigma) = \rho(\Gamma a)\sigma = \rho(a)\Gamma\sigma = \rho(a)(\rho(\Gamma)a)\sigma = \pm \rho(a)\sigma.$$

Likewise, if $a \in \mathcal{C}\ell_0^-(\mathbb{R}^m)$ and $\sigma \in \mathcal{S}^+$, then

$$\rho(\Gamma)(\rho(a)\sigma) = \rho(\Gamma a)\sigma = -\rho(a)\Gamma\sigma = -\rho(a)(\rho(\Gamma)a)\sigma = \mp \rho(a)\sigma.$$

This shows that the action of $\mathcal{C}\ell_0^+(\mathbb{R}^m)$ preserves $\mathcal{S}^\pm$, while that of $\mathcal{C}\ell_0^-(\mathbb{R}^m)$ maps $\mathcal{S}^\pm$ to $\mathcal{S}^\mp$. This proves that $\mathcal{S}$ is a $\mathbb{Z}_2$ -graded module over $\mathcal{C}\ell_0(\mathbb{R}^m)$.

By construction $\mathcal{S} = \Lambda^{0,*} \mathbb{C}^m = \bigoplus_{q=0}^m \Lambda^{0,q} \mathbb{C}^m$ (where $m = \frac{m}{2}$). Therefore, we have another $\mathbb{Z}_2$ -grading,

$$\mathcal{S} = \Lambda^{0,\text{ev}} \mathbb{C}^m \oplus \Lambda^{0,\text{odd}} \mathbb{C}^m,$$

$$\Lambda^{0,\text{ev}} \mathbb{C}^m = \bigoplus_{\substack{q=0 \\ q \text{ even}}}^m \Lambda^{0,q} \mathbb{C}^m, \quad \Lambda^{0,\text{odd}} \mathbb{C}^m = \bigoplus_{\substack{q=0 \\ q \text{ odd}}}^m \Lambda^{0,q} \mathbb{C}^m.$$

As it turns, we actually have only one $\mathbb{Z}_2$ -grading. Namely, we have:

Lemma 4.7: $\mathcal{S}^+ = \Lambda^{0,\text{ev}} \mathbb{C}^m$ and $\mathcal{S}^- = \Lambda^{0,\text{odd}} \mathbb{C}^m$.

Proof: What we need to show is that $\rho(\Gamma) = 1$ on $\Lambda^{0,\text{ev}} \mathbb{C}^m$ and $\rho(\Gamma) = -1$ on $\Lambda^{0,\text{odd}} \mathbb{C}^m$, or equivalently,

$$\rho(\Gamma) z = (-1)^q z \quad \forall z \in \Lambda^{0,q} \mathbb{C}^m \quad \forall q \in \{0, \dots, m\}.$$

For $j=1, \dots, m$ let π_j be the orthogonal projection onto $\text{im}(\varepsilon(z^{\bar{0}})) = z^{\bar{0}} \wedge \Lambda^{0,*} \mathbb{C}^m$. As $\text{im} \varepsilon(z^{\bar{0}})$ is spanned by $\{z^{\bar{I}}\}_{I \in \mathcal{I}_0}$, we see that $\text{ker} \pi_j$ is spanned by $\{z^{\bar{I}}\}_{I \notin \mathcal{I}_0}$. In addition, define

$$F = \prod_{j=1}^m (1 - 2\pi_j)$$

Notice that

$$(1 - 2\pi_j) z^{\bar{I}} = \begin{cases} -z^{\bar{I}} & \text{if } j \in I, \\ z^{\bar{I}} & \text{if } j \notin I. \end{cases}$$

Therefore,

$$F z^{\bar{I}} = (-1)^{|I|} z^{\bar{I}},$$

that is, for all $q=0, \dots, m$

$$F|_{\Lambda^{0,q} \mathbb{C}^m} = (-1)^q.$$

Observe also that if $j \in I$, then

$$\varepsilon(z^{\bar{0}}) \varepsilon(z^{\bar{0}}) z^{\bar{I}} = 0,$$

but if $j \notin I$, then

$$\varepsilon(z^{\bar{0}}) \varepsilon(z^{\bar{0}}) z^{\bar{I}} = \varepsilon(z^{\bar{0}}) (z^{\bar{0}} \wedge z^{\bar{I}}) = z^{\bar{I}}.$$

Therefore, we see that $\varepsilon(z^{\bar{0}}) \varepsilon(z^{\bar{0}})$ is the orthogonal projection onto $\text{span}\{z^{\bar{I}}\}_{I \notin \mathcal{I}_0} = \text{ker} \pi_j$, that is, $\varepsilon(z^{\bar{0}}) \varepsilon(z^{\bar{0}}) = 1 - \pi_j$. Thus,

$$1 - 2\pi_j = 2\varepsilon(z^{\bar{0}}) \varepsilon(z^{\bar{0}}) - 1 = -\rho(c(z^{\bar{0}}))\rho(c(z^{\bar{0}})) - 1 = -\rho(c(z^{\bar{0}})c(z^{\bar{0}}) + 1).$$

Observe further that

$$\begin{aligned} c(z^{\bar{0}})c(z^{\bar{0}}) &= \frac{1}{2} \left\{ c(e^{2\bar{0}}) + i c(e^{2\bar{0}+1}) \right\} \left\{ c(e^{2\bar{0}}) - i c(e^{2\bar{0}+1}) \right\} \\ &= \frac{1}{2} \left\{ c(e^{2\bar{0}})^2 + c(e^{2\bar{0}+1})^2 - i c(e^{2\bar{0}})c(e^{2\bar{0}+1}) + i c(e^{2\bar{0}+1})c(e^{2\bar{0}}) \right\} \\ &= -(1 + i c(e^{2\bar{0}})c(e^{2\bar{0}+1})) \end{aligned}$$

Thus,

$$\rho^{-1}(1 - 2\pi_j) = -(c(z^{\bar{0}})c(z^{\bar{0}}) + 1) = i c(e^{2\bar{0}})c(e^{2\bar{0}+1}),$$

and hence

$$\rho^{-1}(F) = \prod_{j=1}^m i c(e^{2\bar{0}})c(e^{2\bar{0}+1}) = i^m c(e^1)c(e^2) \dots c(e^{2m-1})c(e^{2m}) = \Gamma.$$

It then follows that

$$\rho(\Gamma)|_{\Lambda^{0,q}} = F|_{\Lambda^{0,q}} = (-1)^q \quad \text{for all } q \in \{0, \dots, m\}$$

This completes the proof. $\square$

We shall now give a formula for the supertrace of an endomorphism $T \in \text{End}(\mathbb{S})$. Recall that $\text{Str}_{\mathbb{S}}(T) = \text{Tr}_{\mathbb{S}^+}(A) - \text{Tr}_{\mathbb{S}^-}(D)$ (22)

We also observe that

$$\rho(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so we see that

$$\text{Str}_{\mathbb{S}}(T) = \text{Tr}_{\mathbb{S}}(\rho(T)T).$$

Notice further that the fact that $\mathbb{S}$ is a $\mathbb{Z}_2$ -graded module exactly means that the spinor representation $\rho: \mathcal{C}_0(\mathbb{R}^n) \rightarrow \text{End}(\mathbb{S})$ gives rise to an isomorphism of $\mathbb{Z}_2$ -graded algebras. Thus, at the level of supercommutators,

$$[\rho(a), \rho(b)]' = \rho([a, b]')$$

In addition, as $\rho: \mathcal{C}_0(\mathbb{R}^n) \rightarrow \text{End}(\mathbb{S})$ is an isomorphism and $\sigma: \mathcal{C}_0(\mathbb{R}^n) \rightarrow \Lambda^*\mathbb{C}^n$, we get a linear isomorphism,

$$\sigma \circ \rho': \text{End}(\mathbb{S}) \rightarrow \Lambda^*\mathbb{C}^n$$

This map is sometimes referred to as the symbol map.

In the sequel, we denote by $\omega_{\mathbb{R}^n}$ the volume form of $\mathbb{R}^n$, i.e.,

$$\omega_{\mathbb{R}^n} = e^1 \wedge \dots \wedge e^n, \quad \{e^i\} \text{ oriented orthonormal basis of } \mathbb{R}^n.$$

Proposition 4.8: For all $T \in \text{End}(\mathbb{S})$,

$$\text{Str}_{\mathbb{S}}(T) = (-2i)^{n/2} \langle \omega_{\mathbb{R}^n}, \sigma \circ \rho'(T)^{(n)} \rangle,$$

where $\sigma \circ \rho'(T)^{(n)}$ denotes the component of $\sigma \circ \rho'(T)$ in $\Lambda^n \mathbb{C}^n$.

Proof: Let $\eta \in \Lambda^*\mathbb{C}^n$ and let us write $\eta = \eta^{(n)} + \xi$ with $\eta^{(n)} \in \Lambda^n \mathbb{C}^n$ and $\xi \in \bigoplus_{j=0}^{n-1} \Lambda^j \mathbb{C}^n$. By Lemma 3.7 $\bigoplus_{j=0}^{n-1} \Lambda^j \mathbb{C}^n$ is contained in the supercommutator space of $\mathcal{C}_0(\mathbb{R}^n)$, so under ρ it is mapped to the supercommutator space of $\text{End}(\mathbb{S})$. As the supertrace vanishes on supercommutators (cf. Lemma 1.1), we deduce that

$$\text{Str}_{\mathbb{S}}[\rho(\alpha(\xi))] = 0$$

and hence

$$\text{Str}_{\mathbb{S}}[\rho(\alpha(\eta))] = \text{Str}_{\mathbb{S}}[\rho(\alpha(\eta^{(n)}))].$$

As $\rho^{(u)} = \langle \omega, \rho^{(u)} \rangle$ and $\Gamma = i^{u/2} c(\omega, \rho^{(u)})$, we see that
 $c(\rho^{(u)}) = \langle \omega, \rho^{(u)} \rangle c(\omega, \rho^{(u)}) = (-i)^{u/2} \langle \omega, \rho^{(u)} \rangle \Gamma$,
 and hence

$$\text{Str}_g \rho(c(\rho)) = (-i)^{u/2} \langle \omega, \rho^{(u)} \rangle \text{Str}_g(\rho(\Gamma)).$$

As $\rho(\Gamma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we see that

$$\text{Str}_g \rho(\Gamma) = \text{Tr}_{g^+} \text{id}_{g^+} - \text{Tr}_{g^-} (-\text{id}_{g^-}) = \text{Tr}_{g^+} \text{id}_{g^+} + \text{Tr}_{g^-} \text{id}_{g^-} = \dim g^+ + \dim g^- = \dim g = 2^{u/2}.$$

Therefore, we see that

$$\text{Str}_g[\rho(c(\rho))] = (-2i)^{u/2} \langle \omega, \rho^{(u)} \rangle \quad \forall \rho \in \Lambda^u \mathbb{C}^n.$$

Applying this to $\rho = \sigma \circ \bar{\rho}^*(\Gamma)$ with $\Gamma \in \text{End}(g)$ gives the proposition. $\square$

5. Modules over $\mathcal{C}(\mathbb{R}^n)$:

In the section, we assume n even and we look at $\mathbb{Z}_2$ -graded modules over $\mathcal{C}(\mathbb{R}^n)$.

Thanks to Prop. 4.6 we know that g is $\mathbb{Z}_2$ -graded module over $\mathcal{C}(\mathbb{R}^n)$. In the sequel, we shall simply denote by $(\alpha, \sigma) \rightarrow \alpha \cdot \sigma$ the action of $\mathcal{C}(\mathbb{R}^n)$ on g , i.e., $\alpha \cdot \sigma = \rho(\alpha) \sigma$.

Moreover, by Prop. 4.5 the spinor representation is a $*$ -representation. As mentioned earlier this implies that the grading operator $\rho(\Gamma)$ is selfadjoint and its eigenspaces $g^\pm = \ker(\rho(\Gamma) \mp 1)$ are orthogonal complements of each other.

Def.: Let E be a $\mathcal{C}(\mathbb{R}^n)$ -module, i.e., there is a representation $\pi: \mathcal{C}(\mathbb{R}^n) \rightarrow \text{End}_0(E)$. Then we say that the module E is selfadjoint when it is endowed with a Hermitian inner product with respect to which π is a $*$ -representation, i.e., $\pi(a^*) = \pi(a)^* \quad \forall a \in \mathcal{C}(\mathbb{R}^n)$.

$$= E_+ \oplus E_-$$

Def.: Let E be a $\mathbb{Z}_2$ -graded $\mathcal{C}(\mathbb{R}^n)$ -module associated to the $\mathbb{Z}_2$ -graded representation $\pi: \mathcal{C}(\mathbb{R}^n) \rightarrow \text{End}(E)$.

We say that E is a selfadjoint $\mathbb{Z}_2$ -graded module over $\mathcal{C}(\mathbb{R}^n)$ when E is equipped with a Hermitian product with respect to which

- The $\mathbb{Z}_2$ -grading $E = E_+ \oplus E_-$ is an orthogonal decomposition.
- The representation π is a $*$ -representation.

As explained above, the space of spinors $g = g^+ \oplus g^-$ is an example of a selfadjoint $\mathbb{Z}_2$ -graded $\mathcal{C}(\mathbb{R}^n)$ -module. Other examples are obtained as follows.

Let W be a Hermitian space. Then $g \otimes W$ is a $\mathcal{C}(\mathbb{R}^n)$ -module with respect to the

action of $C_0(\mathbb{R}^n)$ defined by

$$a \cdot (\sigma \otimes \omega) := (a\sigma) \otimes \omega \quad \forall (\sigma, \omega) \in \mathcal{S} \times W \quad \forall a \in C_0(\mathbb{R}^n).$$

That is, the corresponding representation is $\rho \otimes \iota_W: C_0(\mathbb{R}^n) \rightarrow \text{End}(\mathcal{S} \otimes W)$.

If we denote by $\langle \cdot, \cdot \rangle_{\mathcal{S}}$ and $\langle \cdot, \cdot \rangle_W$ the respective Hermitian inner products of $\mathcal{S}$ and W , then we can endow $\mathcal{S} \otimes W$ with the Hermitian inner product defined by

$$\langle \sigma_1 \otimes \omega_1, \sigma_2 \otimes \omega_2 \rangle_{\mathcal{S} \otimes W} = \langle \sigma_1, \sigma_2 \rangle_{\mathcal{S}} \langle \omega_1, \omega_2 \rangle_W \quad \forall \sigma_j \in \mathcal{S} \quad \forall \omega_j \in W.$$

Let $a \in C_0(\mathbb{R}^n)$. Then, for all $\sigma_j \in \mathcal{S}$ and $\omega_j \in W$, $j=1,2$, we have

$$\begin{aligned} \langle (\rho \otimes \iota_W)(a)^* (\sigma_1 \otimes \omega_1), \sigma_2 \otimes \omega_2 \rangle_{\mathcal{S} \otimes W} &= \langle \sigma_1 \otimes \omega_1, (\rho \otimes \iota_W)(\sigma_2 \otimes \omega_2) \rangle_{\mathcal{S} \otimes W} \\ &= \langle \sigma_1 \otimes \omega_1, (\rho(a)\sigma_2) \otimes \omega_2 \rangle_{\mathcal{S} \otimes W} \\ &= \langle \sigma_1, \rho(a)\sigma_2 \rangle_{\mathcal{S}} \langle \omega_1, \omega_2 \rangle_W \\ &= \langle \rho(a)^* \sigma_1, \sigma_2 \rangle_{\mathcal{S}} \langle \omega_1, \omega_2 \rangle_W \\ &= \langle \rho(a)^* \sigma_1, \sigma_2 \rangle_{\mathcal{S}} \langle \omega_1, \omega_2 \rangle_W \\ &= \langle (\rho \otimes \iota_W)(a)^* (\sigma_1 \otimes \omega_1), \sigma_2 \otimes \omega_2 \rangle_{\mathcal{S} \otimes W} \end{aligned}$$

This proves that

$$(\rho \otimes \iota_W)(a)^* = (\rho \otimes \iota_W)(a^*) \quad \forall a \in C_0(\mathbb{R}^n).$$

That is, $\rho \otimes \iota_W$ is a $*$ -representation and $\mathcal{S} \otimes W$ is a selfadjoint $C_0(\mathbb{R}^n)$ -module.

Suppose now that W has an orthogonal $\mathbb{Z}_2$ -grading $W = W^+ \oplus W^-$. Then $\mathcal{S} \otimes W$ is equipped with a $\mathbb{Z}_2$ -grading given in Section 1, i.e., $\mathcal{S} \otimes W = (\mathcal{S} \otimes W)^+ \oplus (\mathcal{S} \otimes W)^-$, with

$$(\mathcal{S} \otimes W)^+ = (\mathcal{S}^+ \otimes W^+) \oplus (\mathcal{S}^- \otimes W^-),$$

$$(\mathcal{S} \otimes W)^- = (\mathcal{S}^+ \otimes W^-) \oplus (\mathcal{S}^- \otimes W^+).$$

Then $(\mathcal{S} \otimes W)^+$ and $(\mathcal{S} \otimes W)^-$ are orthogonal subspaces. Moreover,

$$C_0^+(\mathbb{R}^n)(\mathcal{S} \otimes W) = (C_0^+(\mathbb{R}^n)\mathcal{S}) \otimes W,$$

and hence

$$\begin{aligned} C_0^+(\mathbb{R}^n)(\mathcal{S} \otimes W)^{\pm} &= C_0^+(\mathbb{R}^n)(\mathcal{S}^{\pm} \otimes W^+ \oplus \mathcal{S}^{\mp} \otimes W^-) \\ &\subset \{ (C_0^+(\mathbb{R}^n)\mathcal{S}^{\pm}) \otimes W^+ \} \oplus \{ (C_0^+(\mathbb{R}^n)\mathcal{S}^{\mp}) \otimes W^- \} \\ &\subset (\mathcal{S}^{\pm} \otimes W^+) \oplus (\mathcal{S}^{\pm} \otimes W^-) = (\mathcal{S} \otimes W)^{\pm}. \end{aligned}$$

Similarly,

$$C_0^-(\mathbb{R}^n)(\mathcal{S} \otimes W)^{\pm} \subset (\mathcal{S} \otimes W)^{\mp}.$$

Therefore, $\mathcal{S} \otimes W$ is a selfadjoint $\mathbb{Z}_2$ -graded $C_0(\mathbb{R}^n)$ -module.

We shall be module $\mathcal{S} \otimes W$ a twisted $C_0(\mathbb{R}^n)$ -module. Above shall now see any $C_0(\mathbb{R}^n)$ -module (of finite dimension) is isomorphic to a twisted module.

In the sequel by $\mathcal{C}_0(\mathbb{R}^n)$ -module we will exclusively mean a finite dimensional module.

Lemma 5.1: Let E be a $\mathcal{C}_0(\mathbb{R}^n)$ -module.

(i) We always can endow E with a Hermitian inner product with respect to which E becomes a selfadjoint $\mathcal{C}_0(\mathbb{R}^n)$ -module.

(ii) If E is $\mathbb{Z}_2$ -graded, then we even can choose the inner product so that E is a selfadjoint $\mathbb{Z}_2$ -graded $\mathcal{C}_0(\mathbb{R}^n)$ -module.

Proof: Let $\{e^1, \dots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$ and define:

$$G = \{ \pm 1 \} \cup \{ c(e^{i_1}), \dots, c(e^{i_k}) ; 1 \leq i_1 < \dots < i_k \leq n \}.$$

Then G spans $\mathcal{C}_0(\mathbb{R}^n)$. Moreover, the Clifford relations,

$$c(e^i)^2 = -1, \quad c(e^i)c(e^j) = -c(e^j)c(e^i), \quad i \neq j,$$

imply that G is actually a finite subgroup of $\mathcal{C}_0(\mathbb{R}^n)$.

Let $\langle \cdot, \cdot \rangle_E$ be a Hermitian inner product on E , and define a new Hermitian inner product

$$\langle \cdot, \cdot \rangle_E \text{ on } E \text{ by}$$

$$\langle x, y \rangle_E = \frac{1}{|G|} \sum_{g \in G} \langle gx, gy \rangle_E.$$

Then $\langle \cdot, \cdot \rangle_E$ is a G -invariant meaning that $\pi|_G : G \rightarrow \text{End}(E)$ is a unitary representation. In particular, for $j = 1, \dots, n$.

$$\pi(c(e^j))^* = \pi(c(e^j))^{-1}.$$

As $c(e^j)^2 = -1$ and $c(e^j)^* = -c(e^j)$ (cf. Lemma 3.1) we see that

$$\pi(c(e^j))^2 = \pi(c(e^j)^2) = \pi(-1) = -1,$$

and hence

$$\pi(c(e^j))^{-1} = -\pi(c(e^j)) = \pi(-c(e^j)) = \pi(c(e^j)^*).$$

Thus,

$$\pi(c(e^j))^* = \pi(c(e^j)^*) \quad \forall j \in \{1, \dots, n\}.$$

We then can argue as in the proof of Prop. 4.5 to prove that

$$\pi(a)^* = \pi(a^*) \quad \forall a \in \mathcal{C}_0(\mathbb{R}^n).$$

That is, π is a $*$ -representation and E is a selfadjoint $\mathcal{C}_0(\mathbb{R}^n)$ -module.

Suppose now that E is a $\mathbb{Z}_2$ -graded module with $\mathbb{Z}_2$ -grading $E = E^+ \oplus E^-$ and let us choose the inner product $\langle \cdot, \cdot \rangle_E$ so that E^+ and E^- are orthogonal subspaces. (Let $(x, y) \in E^+ \times E^-$ and let $x \in G$. If $x \in \mathcal{C}P^+(\mathbb{R}^n)$, then $(x, y) \in E^+ \times E^-$, and hence

$$\langle x, y \rangle_E = 0.$$

∃ $\xi \in C_0^-(\mathbb{R}^n)$, then $(\xi_1, \xi_2) \in E^- \times E^+$, and so $\langle \xi_1, \xi_2 \rangle_E = 0$.

It then follows, that

$$\langle \xi, \eta \rangle_E = \frac{1}{|G|} \sum_{\xi \in G} \langle \xi_1, \xi_2 \rangle_E = 0.$$

This proves that E^+ and E^- are orthogonal subspaces with respect to the inner product $\langle \cdot, \cdot \rangle_E$.

As π is a $\ast$ -representation with respect to the inner product, we see that E is a selfadjoint $\mathbb{Z}_2$ -graded $C_0(\mathbb{R}^n)$ -module. The Lemma is proved. $\square$

Remark: As it follows from the proof of Prop. 4.5 and of the above Lemma, if E is a Hermitian vector space, then a representation $\pi: C_0(\mathbb{R}^n) \rightarrow \text{End}(E)$ is a $\ast$ -representation if and only if,

$$\pi(c(v))^* = -\pi(c(v)) \quad \forall v \in \mathbb{R}^n.$$

Moreover, if π is a $\ast$ -representation, then each endomorphism $\pi(c(v))$ is unitary and skew-adjoint.

In the sequel, for any given $k \in \mathbb{N}$, we endow $\mathbb{S}^k$ with the structure of selfadjoint $C_0(\mathbb{R}^n)$ -module given by the diagonal action of $C_0(\mathbb{R}^n)$, namely,

$$a \cdot (\sigma_1, \dots, \sigma_k) := (a\sigma_1, \dots, a\sigma_k) \quad \forall \sigma_i \in \mathbb{S} \quad \forall a \in C_0(\mathbb{R}^n).$$

Notice that $\mathbb{S}^k$ is a selfadjoint $C_0(\mathbb{R}^n)$ -module since for each $a \in C_0(\mathbb{R}^n)$,

$$(p(a) \oplus \dots \oplus p(a))^* = p(a)^* \oplus \dots \oplus p(a)^* = p(a^*) \oplus \dots \oplus p(a^*)$$

The structure of selfadjoint $C_0(\mathbb{R}^n)$ -module is equivalent to that of $\mathbb{S} \otimes \mathbb{S}^k$ under the unitary isomorphism $\kappa: \mathbb{S} \otimes \mathbb{S}^k \rightarrow \mathbb{S}^k$ defined by

$$\kappa(\sigma \otimes \lambda) = (\lambda_1 \sigma, \dots, \lambda_k \sigma) \quad \forall \sigma \in \mathbb{S} \quad \forall \lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k.$$

Lemma 52: Let E be a selfadjoint $C_0(\mathbb{R}^n)$ -module. Then for some $k \in \mathbb{N}$ there is a $C_0(\mathbb{R}^n)$ -equivariant isometric isomorphism $E \cong \mathbb{S}^k$.

Proof: Let $\pi: C_0(\mathbb{R}^n) \rightarrow \text{End}(E)$ be the representation that defines the module structure of E .

This is a $\ast$ -representation since E is selfadjoint $C_0(\mathbb{R}^n)$ -module. Therefore, we get a $\ast$ -homomorphism,

$$\pi \circ \rho': \text{End}(\mathbb{S}) \rightarrow \text{End}(E).$$

For $\sigma_1, \sigma_2 \in \mathcal{S}$ let us denote by $\sigma_2 \otimes \sigma_1^*$ the endomorphism of $\mathcal{S}$ defined by $(\sigma_1 \otimes \sigma_2^*)\sigma = \langle \sigma_2, \sigma \rangle \sigma_1 \quad \forall \sigma \in \mathcal{S}$.

Notice that

$$(\sigma_1 \otimes \sigma_2^*)(\sigma_3 \otimes \sigma_4^*) = \langle \sigma_2, \sigma_3 \rangle \sigma_1 \otimes \sigma_4^*,$$

$$(\sigma_1 \otimes \sigma_2^*)^* = \sigma_2 \otimes \sigma_1^*.$$

Therefore, if $\{\sigma^1, \dots, \sigma^p\}$ is an orthonormal basis of $\mathcal{S}$, then each endomorphism $\sigma_j \otimes \sigma_j^*$ is in fact the orthogonal projection onto $\mathbb{C}\sigma_j$, and hence

$$\sum_{j=1}^p \sigma_j \otimes \sigma_j^* = 1.$$

Thus,
$$\sum_{j=1}^p (\pi \circ \rho^j)(\sigma_j \otimes \sigma_j^*) = \pi \circ \rho^j(1) = 1.$$

This implies that there exists j_0 s.t. $\pi \circ \rho^{j_0}(\sigma_{j_0} \otimes \sigma_{j_0}^*) \neq 0$.

Set $\sigma_0 = \sigma_{j_0}$. As $(\sigma_0 \otimes \sigma_0^*)^* = (\sigma_0 \otimes \sigma_0^*) = \sigma_0 \otimes \sigma_0^*$ and $\pi \circ \rho^{j_0}$ is a $*$ -conjugation we see that the $\Pi := \pi \circ \rho^{j_0}(\sigma_0 \otimes \sigma_0^*)$ satisfies $\Pi^* = \Pi^2 = \Pi$, that is, Π is an orthogonal projection.

As $\Pi = \pi \circ \rho^{j_0}(\sigma_0 \otimes \sigma_0^*) \neq 0$ we can find a unit vector $\xi \in E$ contained in $\text{im } \Pi$, i.e., $\Pi \xi = \xi$. We then define a linear map $\varphi: \mathcal{S} \rightarrow E$ by

$$\varphi(\sigma) = \pi \circ \rho^{j_0}(\sigma \otimes \sigma_0^*) \xi.$$

Let $\alpha \in \mathcal{C}_0(\mathbb{R}^n)$. Then, as $T(\sigma \otimes \sigma_0^*) = (T\sigma) \otimes \sigma_0^* \quad \forall T \in \text{End}(\mathcal{S})$, we see that

$$\begin{aligned} \varphi(\alpha\sigma) &= \pi \circ \rho^{j_0}((\rho(\alpha)\sigma) \otimes \sigma_0^*) = \pi \circ \rho^{j_0}(\rho(\alpha)(\sigma \otimes \sigma_0^*)) \xi \\ &= \pi \circ \rho^{j_0}(\rho(\alpha)) \pi \circ \rho^{j_0}(\sigma \otimes \sigma_0^*) \xi \\ &= \pi(\alpha) \varphi(\sigma) \\ &= \alpha \varphi(\sigma). \end{aligned}$$

Thus φ is a $\mathcal{C}_0(\mathbb{R}^n)$ -equivariant map. Therefore, $\text{Ran } \varphi$ is a $\mathcal{C}_0(\mathbb{R}^n)$ -invariant subspace of E . Observe that $\text{Ran } \varphi \subseteq \mathcal{S}$, since

$$\varphi(\sigma_0) = \pi \circ \rho^{j_0}(\sigma_0 \otimes \sigma_0^*) \xi = \Pi \xi = \xi \neq 0.$$

As $\mathcal{S}$ is an irreducible module (cf. Thm. 4.2), we deduce that $\text{Ran } \varphi = \mathcal{S}$, i.e., φ is surjective.

Notice that $E := \text{im } \varphi$ is $\mathcal{C}_0(\mathbb{R}^n)$ -invariant subspace of E , and hence is a $\mathcal{C}_0(\mathbb{R}^n)$ -module. Thus, φ gives rise to an isomorphism of $\mathcal{C}_0(\mathbb{R}^n)$ -module from $\mathcal{S}$ onto E .

This isomorphism is isometric. Indeed, for $\sigma_1, \sigma_2 \in \mathcal{S}$,

$$\begin{aligned} \langle \varphi(\sigma_1), \varphi(\sigma_2) \rangle_E &= \langle \pi \circ \rho^{j_0}(\sigma_1 \otimes \sigma_0^*) \xi, \pi \circ \rho^{j_0}(\sigma_2 \otimes \sigma_0^*) \xi \rangle_E \\ &= \langle (\pi \circ \rho^{j_0}(\sigma_2 \otimes \sigma_0^*))^* \pi \circ \rho^{j_0}(\sigma_1 \otimes \sigma_0^*) \xi, \xi \rangle_E \end{aligned}$$

Observe that

$$\begin{aligned} (\pi \circ \tilde{\rho}'(\sigma_2 \otimes \sigma_0^*))^* \pi \circ \tilde{\rho}'(\sigma_1 \otimes \sigma_0^*) &= \pi \circ \tilde{\rho}'\{(\sigma_2 \otimes \sigma_0^*)^* \sigma_1 \otimes \sigma_0^*\} \\ &= \pi \circ \tilde{\rho}'\{(\sigma_0 \otimes \sigma_2^*)(\sigma_1 \otimes \sigma_0^*)\} \\ &= \pi \circ \tilde{\rho}'\{\langle \sigma_2, \sigma_1 \rangle_{\mathcal{H}} \sigma_0 \otimes \sigma_0^*\} \\ &= \langle \sigma_2, \sigma_1 \rangle_{\mathcal{H}} \pi. \end{aligned}$$

Then,

$$\langle \varphi(\sigma_1), \varphi(\sigma_2) \rangle_E = \langle \sigma_2, \sigma_1 \rangle_{\mathcal{H}} \langle \pi \xi, \xi \rangle_E = \langle \sigma_2, \sigma_1 \rangle_{\mathcal{H}} \langle \xi, \xi \rangle_E = \langle \sigma_2, \sigma_1 \rangle_{\mathcal{H}},$$

which shows that φ is an isometric isomorphism from $\mathcal{H}$ into E_1 .

If $\dim E_1 = \dim E$, then we are done; φ is a $C_0(\mathbb{R}^n)$ -equivariant isometric isomorphism from $\mathcal{H}$ into E .

Suppose now that $E_1 \subsetneq E$. Then $E_1^\perp$ is a non-trivial $C_0(\mathbb{R}^n)$ -invariant subspace of E . More precisely, if $\xi \in E_1^\perp$ and $a \in C_0(\mathbb{R}^n)$, then by the fact that π is a $*$ -representation we see that, for all $\eta \in E_1$,

$$\langle a\xi, \eta \rangle = \langle \pi(a)\xi, \eta \rangle = \langle \xi, \pi(a)^* \eta \rangle = \langle \xi, \pi(a^*) \eta \rangle = 0,$$

because $\pi(a^*)\eta \in E_1$. We thus get an orthogonal decomposition into non-trivial $C_0(\mathbb{R}^n)$ -invariant subspaces,

$$E = E_1 \oplus E_1^\perp,$$

in such a way that there is a $C_0(\mathbb{R}^n)$ -equivariant isometric isomorphism $\varphi_1: \mathcal{H} \rightarrow E_1$.

By repeating the above process we can successively construct orthogonal $C_0(\mathbb{R}^n)$ -invariant subspaces $E_1, \dots, E_P, F_P$, such that:

- For each $j=1, \dots, P$ there is a $C_0(\mathbb{R}^n)$ -equivariant isometric isomorphism $\varphi_j: \mathcal{H} \rightarrow E_j$.
- We have an orthogonal decomposition:

$$E = E_1 \oplus \dots \oplus E_P \oplus F_P.$$

If we repeat this process enough times we can obtain that $\dim F_P \leq \dim \mathcal{H}$. Then, then

as then again we can construct a $C_0(\mathbb{R}^n)$ -equivariant isometric linear map $\varphi: \mathcal{H} \rightarrow F_P$.

As $\dim F_P \leq \dim \mathcal{H}$, we see that $\dim F_P = \dim \mathcal{H}$ and φ is an isomorphism. Setting $E_{P+1} = F_P$ and $\varphi_{P+1} = \varphi$ we have the orthogonal decomposition

$$E = E_1 \oplus \dots \oplus E_{P+1},$$

and we can define a linear map $\mathcal{H} \rightarrow E$ by

$$\varphi(\sigma) := \varphi_1(\sigma_1) + \dots + \varphi_{P+1}(\sigma_{P+1}) \quad \forall \sigma = (\sigma_j)_{1 \leq j \leq P+1} \in \mathcal{H}^{P+1}.$$

to find

This map is auto and $C_0(\mathbb{R}^n)$ -equivariant since so are all the φ_j . Moreover, as the decomposition $E = E_1 \oplus \dots \oplus E_{p+1}$ is an orthogonal decomposition and all the φ_j are isometric, we see that, for all $(\sigma_j) \in \mathcal{S}^p$,

$$\|\varphi(\sigma)\|_E^2 = \sum \|\varphi_j(\sigma_j)\|_E^2 = \sum \|\sigma_j\|_{\mathcal{S}}^2 = \|\sigma\|_{\mathcal{S}^p}^2,$$

so that φ is isometric. This implies that φ is one-to-one, so we deduce that φ is a $C_0(\mathbb{R}^n)$ -equivariant isometric isomorphism from $\mathcal{S}^p$ into E . The lemma is proved. $\square$

Prop. 5.3: Let $E = E^+ \oplus E^-$ be a selfadjoint $\mathbb{Z}_2$ -graded module over $C_0(\mathbb{R}^n)$. Let $\omega = \text{equi } W = \text{Hom}_{C_0(\mathbb{R}^n)}(\mathcal{S}, E)$ with the $\mathbb{Z}_2$ -grading defined in Section 1 and endow $\mathcal{S} \otimes W$ with the structure of $\mathbb{Z}_2$ -graded $C_0(\mathbb{R}^n)$ -module defined on page 29. Let $\beta: \mathcal{S} \otimes W \rightarrow E$ be the linear map defined by

$$\beta(\sigma \otimes w) := w(\sigma) \quad \forall (\sigma, w) \in \mathcal{S} \times W$$

Then β is a $\mathbb{Z}_2$ -graded $C_0(\mathbb{R}^n)$ -equivariant linear isomorphism.

Proof: By Lemma 5.2 for some $k \in \mathbb{N}$ we have a $C_0(\mathbb{R}^n)$ -equivariant isometric isomorphism $\varphi: E \rightarrow \mathcal{S}^k$

$$\varphi(\xi) = (\varphi_1(\xi), \dots, \varphi_k(\xi)) \quad \forall \xi \in E.$$

This naturally gives rise to the linear isomorphism $\varphi_\#: W = \text{Hom}_{C_0(\mathbb{R}^n)}(\mathcal{S}, E) \rightarrow \text{Hom}_{C_0(\mathbb{R}^n)}(\mathcal{S}, \mathcal{S}^k)$ given by

$$\varphi_\#(w) = (\varphi_1 \circ w, \dots, \varphi_k \circ w).$$

Let $T \in \text{Hom}_{C_0(\mathbb{R}^n)}(\mathcal{S}, \mathcal{S})$. The fact that T is $C_0(\mathbb{R}^n)$ -equivariant here means that

$$T\rho(a) = \rho(a)T \quad \forall a \in C_0(\mathbb{R}^n)$$

As $\rho(C_0(\mathbb{R}^n)) = \text{End}(\mathcal{S})$ this implies that T commutes with all the elements of $\text{End}(\mathcal{S})$, and hence must be a scalar multiple of the identity. Yes $\text{Hom}_{C_0(\mathbb{R}^n)}(\mathcal{S}, \mathcal{S}) = \mathbb{C} \cdot \text{id}_{\mathcal{S}}$ and, by fixing a unit element $\sigma_0 \in \mathcal{S}$, the map

$$\gamma: \text{Hom}_{C_0(\mathbb{R}^n)}(\mathcal{S}, \mathcal{S})^{\mathbb{Z}_2} \ni (T_j)_{1 \leq j \leq k} \longrightarrow (\langle \sigma_0, T_j \sigma_0 \rangle)_{1 \leq j \leq k} \in \mathbb{C}^k$$

is a linear isomorphism from $\text{Hom}_{C_0(\mathbb{R}^n)}(\mathcal{S}, \mathcal{S})^{\mathbb{Z}_2}$ onto $\mathbb{C}^k$. Composing it with $\varphi_\#$ yields the linear isomorphism,

$$\varphi_\# : W \longrightarrow \mathbb{C}^k$$

$$w \longmapsto (\langle \sigma_0, \varphi_j \circ w(\sigma_0) \rangle)_{1 \leq j \leq k}.$$

Denote by $\alpha: \mathcal{S} \otimes \mathbb{C}^k \rightarrow \mathcal{S}^k$ the linear isomorphism,

$$\alpha(\sigma \otimes \lambda) := (\lambda_1 \sigma, \dots, \lambda_k \sigma) \quad \forall \sigma \in \mathcal{S} \quad \forall \lambda = (\lambda_j) \in \mathbb{C}^k$$

Let $(\sigma, w) \in \mathcal{S} \otimes W$. Then

$$\varphi \circ \beta(\sigma \otimes w) = \varphi(w(\sigma)) = (\varphi_1(w(\sigma)), \dots, \varphi_k(w(\sigma)))$$

As $\phi_{\beta} \omega$ is contained in $\text{Hom}_{\mathcal{O}_E(V)}(\mathcal{S}, \mathcal{E})$ it agrees with $\langle \sigma_0, \phi_{\beta} \omega(\sigma_0) \rangle \text{id}_{\mathcal{E}}$, and hence $\phi(\phi_{\beta} \omega(\sigma)) = \langle \sigma_0, \phi_{\beta} \omega(\sigma_0) \rangle \sigma$.

Thus,

$$\phi \circ \beta = (\langle \sigma_0, \phi_{\beta} \omega(\sigma_0) \rangle \sigma, \dots, \langle \sigma_0, \phi_{\beta} \omega(\sigma_0) \rangle \sigma)$$

On the other hand,

$$\alpha \circ (1 \otimes \hat{\phi}_{\#})(\sigma \otimes \omega) = \alpha(\sigma \otimes (\langle \sigma_0, \phi_{\beta} \omega(\sigma_0) \rangle))_{1 \leq i \leq s}$$

$$= (\langle \sigma_0, \phi_{\beta} \omega(\sigma_0) \rangle \sigma)_{1 \leq i \leq s}$$

Therefore we see that we have a commutative diagram,

$$\begin{array}{ccc} \mathcal{S} \otimes W & \xrightarrow{\beta} & E \\ 1 \otimes \hat{\phi} \downarrow & & \downarrow \phi \\ \mathcal{S} \otimes \mathcal{E}^2 & \xrightarrow{\alpha} & \mathcal{E}^2 \end{array}$$

As the vertical maps and the bottom horizontal maps are linear isomorphisms we see that β is a linear isomorphism.

Moreover, β is $\mathcal{O}_E(K^*)$ -equivariant, since for $a \in \mathcal{O}_E(K^*)$ and $(\sigma, \omega) \in \mathcal{S} \otimes W$, we have

$$\beta(a(\sigma \otimes \omega)) = \beta((a\sigma) \otimes \omega) = \omega(a\sigma) = a\omega(\sigma),$$

since ω is $\mathcal{O}_E(K^*)$ -equivariant.

In addition, recall that by definition of the $\mathbb{Z}$ -grading of $\mathcal{S} \otimes W$, and $W = \text{Hom}_{\mathcal{O}_E(V)}(\mathcal{S}, E)$,

$$(\mathcal{S} \otimes W)^+ = (\mathcal{S}^+ \otimes W^+) \oplus (\mathcal{S}^- \otimes W^-),$$

$$(\mathcal{S} \otimes W)^- = (\mathcal{S}^+ \otimes W^-) \oplus (\mathcal{S}^- \otimes W^+),$$

and W^+ (resp., W^-) consists of $T \in \text{Hom}_{\mathcal{O}_E(V)}(\mathcal{S}, E)$ mapping $\mathcal{S}^{\pm}$ to $E^{\pm}$ (resp., $E^{\mp}$).

Now, if $\sigma \in \mathcal{S}^{\pm}$ and ω is contained in W^+ (resp., W^-), then $\beta(\sigma \otimes \omega) = \omega(\sigma)$ is contained in $E^{\pm}$ (resp., $E^{\mp}$). This means that β maps $(\mathcal{S} \otimes W)^{\pm}$ to $E^{\pm}$, that is, β is $\mathbb{Z}$ -graded.

It remains to check that β is isometric. Let

Let us endow $W = \text{Hom}_{\mathcal{O}_E(V)}(\mathcal{S}, E)$ with the Hermitian inner product defined by

$$\langle \omega_1, \omega_2 \rangle_W := 2^{\frac{1}{2}} \text{Tr}_{\mathcal{S}}(\omega_1^* \omega_2) \quad \forall \omega_i \in W$$

Notice that if $\omega_1 \in W^+$ and $\omega_2 \in W^-$, then $\omega_2^* \omega_1$ maps $E_+^+ \subseteq E^+$, and hence has a vanishing trace. Thus W^+ and W^- are orthogonal subspaces. Therefore, if we endow $\mathbb{S} \otimes W$ with the Hermitian inner product defined on page 29, then $\mathbb{S} \otimes W$ becomes a selfadjoint $\mathbb{Z}$ -graded module.

Prop. 5.4: The isomorphism $\beta: \mathbb{S} \otimes W \rightarrow E$ of Prop. 5.3 is coisometric.

Proof: Let us first observe that in the diagram (5.1) both φ and χ are coisometric isomorphisms. Therefore, we need only to show that $\varphi_\#$ is an coisometric isomorphism from W into $\mathbb{C}^2$.

We endow $V = \text{Hom}_{\mathbb{C}}(W, \mathbb{S})$ with the Hermitian inner product defined by

$$\langle S, T \rangle_V := 2^{n/2} \text{Tr}_{\mathbb{S}}(\mathbb{S}^* T). \quad \forall S, T \in \text{Hom}_{\mathbb{C}}(W, \mathbb{S}).$$

In fact, as $S = \langle \sigma_0, S \sigma_0 \rangle \text{id}_{\mathbb{S}}$ and $T = \langle \sigma_0, T \sigma_0 \rangle \text{id}_{\mathbb{S}}$, we see that

$$\begin{aligned} \langle S, T \rangle_V &:= 2^{n/2} \text{Tr}(\overline{\langle \sigma_0, S \sigma_0 \rangle} \langle \sigma_0, T \sigma_0 \rangle \text{id}_{\mathbb{S}}) \\ &= \overline{\langle \sigma_0, S \sigma_0 \rangle} \langle \sigma_0, T \sigma_0 \rangle \end{aligned}$$

It then follows that χ is an coisometric isomorphism from $\text{Hom}_{\mathbb{C}}(\mathbb{S}, \mathbb{S})^{\otimes 2}$ into $\mathbb{C}^2$.

Let $\omega_1, \omega_2 \in W$. Then

$$\begin{aligned} 2^{n/2} \langle \varphi_\#(\omega_1), \varphi_\#(\omega_2) \rangle_{\mathbb{C}^2} &= \sum_{j=1}^2 \sum_{\ell=1}^2 \langle \varphi_j \circ \omega_1, \varphi_j \circ \omega_2 \rangle_V \\ &= \sum_{j=1}^2 \text{Tr}_{\mathbb{S}}(\omega_1^* \varphi_j^* \varphi_j \omega_2) \\ &= \text{Tr}_{\mathbb{S}}(\omega_1^* T \omega_2), \end{aligned}$$

where we have $T = \sum_{j=1}^2 \varphi_j^* \varphi_j \in \text{Hom}_{\mathbb{C}}(W, W)$. In fact, as φ is coisometric, $\sum_{j=1}^2 \varphi_j^* \varphi_j = \text{id}_W$.

$$\begin{aligned} \langle \xi, \xi \rangle_E &= \langle \varphi(\xi), \varphi(\xi) \rangle_{\mathbb{S}^{\otimes 2}} = \sum_j \langle \varphi_j(\xi), \varphi_j(\xi) \rangle_{\mathbb{S}^{\otimes 2}} \\ &= \sum_j \langle \varphi_j^* \varphi_j(\xi), \xi \rangle_E = \langle T\xi, \xi \rangle. \end{aligned}$$

Since T is selfadjoint, it follows that $T = \text{id}_W$ and hence

$$\langle \varphi_\#(\omega_1), \varphi_\#(\omega_2) \rangle_{\mathbb{C}^2} = 2^{n/2} \text{Tr}_{\mathbb{S}}(\omega_1^* \omega_2) = \langle \omega_1, \omega_2 \rangle_W.$$

This shows that $\varphi_\#$ is an coisometric isomorphism. Thus $1 \otimes \varphi_\# = 1 \otimes (\chi \circ \varphi_\#)$ is an coisometric isomorphism. This completes the proof. $\square$

Finally, let $E = \mathbb{S} \otimes W$ be a twisted $\mathbb{Z}_2$ -graded module over $\mathcal{C}_0(\mathbb{R}^n)$, where $W \cong W \otimes W$ is a superspace. Let $T \in \text{End}(W)$. Then

$$\text{Str}_{\mathbb{S} \otimes W}(\Gamma T) = \text{Str}_{\mathbb{S}}(\rho(\Gamma)) \text{Str}_W(T).$$

As shown in the proof of Prop. 4.8, $\text{Str}_{\mathbb{S}}(\rho(\Gamma)) = 2^{n/2}$, so we see that

$$\text{Str}_{\mathbb{S} \otimes W}(\Gamma T) = 2^{n/2} \text{Str}_W(T),$$

that is,

$$\text{Str}_W(T) = 2^{-n/2} \text{Str}_{\mathbb{S} \otimes W}(\Gamma T).$$

This formula motivates the following definition.

Def. (Relative Supertrace): Let E be a $\mathbb{Z}_2$ -graded $\mathcal{C}_0(\mathbb{R}^n)$ -module. The relative supertrace on E is the linear form

$$\text{Str}_{E/\mathbb{S}}: \text{Hom}_{\mathcal{C}_0(\mathbb{R}^n)}(E, E) \rightarrow \mathbb{C}$$

defined by

$$\text{Str}_{E/\mathbb{S}}(T) = 2^{n/2} \text{Str}_E(\Gamma T) \quad \forall T \in \text{Hom}_{\mathcal{C}_0(\mathbb{R}^n)}(E, E).$$

Rule: Let $S, T \in \text{Hom}_{\mathcal{C}_0(\mathbb{R}^n)}(E, E)$. As Γ commutes with S and T ,

$$\Gamma[S, T]' = [\Gamma S, T]',$$

and hence

$$\text{Str}_{E/\mathbb{S}}[S, T]' = 2^{n/2} \text{Str}_E\{\Gamma[S, T]'\} = 2^{n/2} \text{Str}_E[\Gamma S, T]' = 0.$$

Thus $\text{Str}_{E/\mathbb{S}}$ is a supertrace on $\text{Hom}_{\mathcal{C}_0(\mathbb{R}^n)}(E, E)$, in the sense that it vanishes on super-commutators.

6. The Spin Group $\text{Spin}(n)$:

Define

$$\mathcal{C}P^1(\mathbb{R}^n) = \mathcal{C}(\mathbb{R}^n) \quad \text{and} \quad \mathcal{C}P^2(\mathbb{R}^n) = \mathcal{C}(\wedge^2 \mathbb{R}^n).$$

Lemma 6.1: (i) $[a, c(v)] \in \mathcal{C}P^1(\mathbb{R}^n) \quad \forall a \in \mathcal{C}P^2(\mathbb{R}^n) \quad \forall v \in \mathbb{R}^n$.

(ii) $[a, b] \in \mathcal{C}P^2(\mathbb{R}^n) \quad \forall a, b \in \mathcal{C}P^2(\mathbb{R}^n)$.

Proof: Let $\alpha \in \wedge^2(\mathbb{R}^n)$ and $v \in \mathbb{R}^n$. Then by Lemma 3.6,

$$[c(\alpha), c(v)] = [c(\alpha), c(v)] = 2c(i(v)\alpha) \in \mathcal{C}P^1(\mathbb{R}^n).$$

This proves the first part of the Lemma.

Let $\{e^1, \dots, e^u\}$ be an orthonormal basis of $\mathbb{R}^u$. Then, for $i < j$,

$$\begin{aligned} [c(\eta), c(e^i \wedge e^j)] &= [c(\eta), c(e^i)c(e^j)] \\ &= c(e^i) [c(\eta), c(e^j)] + [c(\eta), c(e^i)] c(e^j) \\ &= 2 c(e^i) c(i(e^j)\eta) + 2 c(i(e^i)\eta) c(e^j). \end{aligned}$$

Combining this with Lemma 3.3 gives

$$[c(\eta), c(e^i \wedge e^j)] = 2 \{ c(e^i \wedge i(e^j)\eta) - \langle e^i, i(e^j)\eta \rangle + c(i(e^i)\eta \wedge e^j) - \langle i(e^i)\eta, e^j \rangle \}$$

As Lemma 2.3 tells us that $i(e^i)^* = E(e^i)$ and $i(e^j)^* = E(e^j)$, we see that

$$\langle e^i, i(e^j)\eta \rangle = \langle i(e^j)^* e^i, \eta \rangle = \langle E(e^j) e^i, \eta \rangle = \langle e^j \wedge e^i, \eta \rangle$$

Similarly,

$$\langle i(e^i)\eta, e^j \rangle = \langle \eta, e^i \wedge e^j \rangle = - \langle e^i, i(e^j)\eta \rangle.$$

Thus,

$$[c(\eta), c(e^i \wedge e^j)] = 2 c(e^i \wedge i(e^j)\eta) + c(i(e^i)\eta \wedge e^j) \in \mathcal{CP}^2(\mathbb{R}^u).$$

As the $c(e^i \wedge e^j), i < j$, span $\mathcal{CP}^2(\mathbb{R}^u)$ the 2nd part of the Lemma follows. The Lemma is proved. $\square$

It follows from the 2nd part of Lemma 6.1 that, when equipped with the commutator bracket, $\mathcal{CP}^2(\mathbb{R}^u)$ is a Lie subalgebra of $\mathcal{P}(\mathbb{R}^u)$. In addition, using the first part of Lemma 6.1, we see that we can define a linear map $\tau: \mathcal{CP}^2(\mathbb{R}^u) \rightarrow M_u(\mathbb{R})$ by

$$c(\tau(a).v) = [a, c(v)] \quad \forall a \in \mathcal{CP}^2(\mathbb{R}^u) \quad \forall v \in \mathbb{R}^u.$$

Consider the Lie algebra of $\mathfrak{so}(u)$, i.e.,

$$\mathfrak{so}(u) = \{ A = (a_{ij}); a_{ji} = -a_{ij} \quad \forall i, j \in \{1, \dots, u\} \}.$$

Prop. 6.2 gives rise to a Lie algebra isomorphism,

$$\tau: \mathcal{CP}^2(\mathbb{R}^u) \rightarrow \mathfrak{so}(u).$$

Moreover, its inverse map is given by

$$\tau^{-1}(A) := \frac{1}{2} \sum_{i < j} \langle A e^i, e^j \rangle c(e^i) c(e^j) \quad \forall A \in \mathfrak{so}(u),$$

where $\{e^i\}$ is any orthonormal basis of $\mathbb{R}^u$.

Proof: (Let us first show that τ maps to $\mathfrak{so}(u)$). Let $a \in \mathcal{CP}^2(\mathbb{R}^u)$ and $v, w \in \mathbb{R}^u$. Then

$$\begin{aligned} -2 \langle \tau(a).v, w \rangle &= c(\tau(a).v) c(w) + c(w) c(\tau(a).v) \\ &= [a, c(v)] c(w) + c(w) [a, c(v)]. \end{aligned}$$

Similarly,

$$-2 \langle v, \tau(a)w \rangle = c(w)c(\tau(a)w) + c(\tau(a)w)c(w) \\ = c(w)[a, c(w)] + [a, c(w)]c(w).$$

Thus,

$$-2 \{ \langle \tau(a)v, w \rangle + \langle v, \tau(a)w \rangle \} = [a, c(w)]c(w) + c(w)[a, c(w)] + c(w)[a, c(v)] + [a, c(v)]c(w) \\ = [a, c(w)c(w)] + [a, c(w)c(w)] \\ = [a, c(w)c(w) + c(w)c(v)] \\ = [a, -2\langle v, w \rangle] \\ = 0.$$

Therefore, for all $v, w \in \mathbb{R}^n$,

$$\langle \tau(a)v, w \rangle + \langle v, \tau(a)w \rangle = 0.$$

This shows that $\tau(a) \in \mathfrak{so}(n)$.

Next, let $a, b \in \mathcal{C}^2(\mathbb{R}^n)$ and $v \in \mathbb{R}^n$. Then

$$c(\tau([a, b])v) = [[a, b], c(v)] = [a b, c(v)] - [b a, c(v)] \\ = a[b, c(v)] + [a, c(v)]b - b[a, c(v)] - [b, c(v)]a \\ = [a, [b, c(v)]] - [b, [a, c(v)]] \\ = [a, c(\tau(b)v)] - [b, c(\tau(a)v)] \\ = c(\tau(a).(\tau(b)v)) - c(\tau(b).(\tau(a)v)) \\ = c([\tau(a), \tau(b)]v).$$

Thus,

$$\tau([a, b]) = [\tau(a), \tau(b)] \quad \forall a, b \in \mathcal{C}^2(\mathbb{R}^n),$$

that is, τ is a morphism of Lie algebras.

Let $\{e^1, \dots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$ and for $i, j = 1, \dots, n$ let us denote by E_{ij} the elementary matrices s.t., for all $k = 1, \dots, n$,

$$E_{ij}e^k = \delta^{ik}e^j = \langle e^i, e^k \rangle e^j.$$

Then $\mathfrak{so}(n)$ has basis $\{E_{ij} - E_{ji}; i < j\}$. Indeed, for any $A \in \mathfrak{so}(n)$,

$$A = \sum_{i < j} \langle A e^i, e^j \rangle E_{ij} = \sum_{i < j} \langle A e^i, e^j \rangle (E_{ij} - E_{ji}).$$

Now, as τ is a morphism of Lemma 6.1:

$$[c(e^i)c(e^j), c(e^k)] = [c(e^i \wedge e^j), e^k] = -2c(L(e^k)(e^i \wedge e^j)) \\ = -2\langle e^k, e^i \rangle c(e^j) + 2\langle e^k, e^j \rangle c(e^i).$$

Thus,

$$\tau(c(e^i)c(e^j))e^k = \sigma([c(e^i)c(e^j), c(e^k)]) = -2\langle e^k, e^i \rangle e^j + 2\langle e^k, e^j \rangle e^i \\ = 2E_{ik}e^j - 2E_{jk}e^i.$$

Therefore, we see that

$$(6.1) \quad \tau(c(e^i)c(e^j)) = 2(E_{ij} - E_{ji}).$$

It then follows that τ maps into $\mathfrak{so}(n)$. As $\{c(e^i)c(e^j); i < j\}$ and $\{E_{ij} - E_{ji}; i < j\}$ are respective bases of $\mathcal{P}^2(\mathbb{R}^n)$ and $\mathfrak{so}(n)$ and have same cardinality, we see that $\mathcal{P}^2(\mathbb{R}^n)$ and $\mathfrak{so}(n)$ have same dimension. Therefore, the fact that τ maps into $\mathfrak{so}(n)$ actually implies that τ is an isomorphism from $\mathcal{P}^2(\mathbb{R}^n)$ onto $\mathfrak{so}(n)$.

Finally, (6.1) shows that, for $i < j$,

$$\tau^{-1}(E_{ij} - E_{ji}) = \frac{1}{2} c(e^i)c(e^j).$$

Thus, for all $A \in \mathfrak{so}(n)$,

$$\tau(A) = \sum_{i < j} \langle A e^i, e^j \rangle \tau^{-1}(E_{ij} - E_{ji})$$

$$= \frac{1}{2} \sum_{i < j} \langle A e^i, e^j \rangle c(e^i)c(e^j).$$

The proof is complete. $\square$

Def. The spin group of $\mathbb{R}^n$, denoted $\text{Spin}(n)$, is the Lie group obtained by exponentiating the Lie algebra $\mathcal{P}^2(\mathbb{R}^n)$ inside $\mathcal{P}(\mathbb{R}^n)$.

Ex: Let v and w be orthogonal unit vectors in $\mathbb{R}^n$. Then

$$c(v)^2 = c(w)^2 = -1 \quad \text{and} \quad c(v)c(w) = -c(w)c(v).$$

Thus,

$$(c(v)c(w))^2 = c(v)c(w)c(v)c(w) = -c(w)c(v)^2c(w) = c(w)^2 = -1.$$

An easy induction then shows that, for all $k \in \mathbb{N}_0$,

$$(c(v)c(w))^{2k} = (-1)^k \quad \text{and} \quad (c(v)c(w))^{2k+1} = (-1)^k c(v)c(w).$$

Therefore, for all $t \in \mathbb{R}$,

$$\begin{aligned} \exp(t c(v)c(w)) &= \sum_{j=0}^{\infty} \frac{t^j}{j!} (c(v)c(w))^j \\ &= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} (c(v)c(w))^{2k} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} (c(v)c(w))^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} (-1)^k + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} (-1)^k c(v)c(w). \end{aligned}$$

Recognizing the Maclaurin series of $\cos t$ and $\sin t$ we see that

$$\exp(t c(v) c(w)) = \cos t + \sin t c(w) c(v).$$

In particular, for $t = \pi$ we see that -1 is contained in $\text{Spin}(n)$.

The Lie algebra isomorphism $\tau: \mathcal{P}^2(\mathbb{R}^n) \rightarrow \mathfrak{so}(n)$ descends to a morphism of Lie groups,

$$\tau: \text{Spin}(n) \rightarrow \text{SO}(n),$$

such that $\tau(\exp(a)) = \exp(\tau(a)) \quad \forall a \in \mathcal{P}^2(\mathbb{R}^n)$.

Lemma 6.3: Let $g \in \text{Spin}(n)$ and let us regard g and $\bar{g}$ as elements of $\mathcal{P}(\mathbb{R}^n)$. Then

$$g c(v) \bar{g}' = c(\tau(g)v) \quad \forall v \in \mathbb{R}^n.$$

Proof: Let us write $g = \exp(a)$ with $a \in \mathcal{P}^2(\mathbb{R}^n)$. Then

$$\frac{d}{dt} \left\{ \exp(ta) c(v) \exp(-ta) \right\} = \exp(ta) [a, c(v)] \exp(-ta) = \exp(ta) c(\tau(a)v) \exp(-ta).$$

A simple induction then shows that, for all $k \in \mathbb{N}$,

$$\frac{d^k}{dt^k} \left\{ \exp(ta) c(v) \exp(-ta) \right\} = \exp(ta) c(\tau(a)^k v) \exp(-ta),$$

and hence

$$\left. \frac{d^k}{dt^k} \left\{ \exp(ta) c(v) \exp(-ta) \right\} \right|_{t=0} = c(\tau(a)^k v) = \left. \frac{d}{dt} c(\exp(t \tau(a)) v) \right|_{t=0}$$

$$= \left. \frac{d}{dt} c(\tau(\exp(ta)) v) \right|_{t=0}.$$

As both $\exp(ta) c(v) \exp(-ta)$ and $c(\tau(\exp(ta))v)$ are analytic functions of t we deduce that they agree everywhere. In particular, for $t=1$ we get

$$\exp(a) c(v) \exp(-a) = c(\tau(\exp(a))v),$$

that is,

$$g c(v) \bar{g}' = c(\tau(g)v).$$

The lemma is thus proved. $\square$

Prop. 6.4: The Lie group morphism $\tau: \text{Spin}(n) \rightarrow \text{SO}(n)$ is a double covering of $\text{SO}(n)$,

that is, τ is onto and $\ker \tau = \{\pm 1\}$.

Proof: As $\tau: \mathcal{P}^2(\mathbb{R}^n) \rightarrow \mathfrak{so}(n)$ is an isomorphism and $\exp: \mathfrak{so}(n) \rightarrow \text{SO}(n)$ we see that τ maps

$\text{Spin}(n)$ onto $\text{SO}(n)$. Moreover, as shown in the previous example, -1 is contained in $\text{Spin}(n)$ and

by Lemma 6.3, for all $v \in \mathbb{R}^n$,

$$c(\tau(-1)v) = (-1) c(v) (-1)^{-1} = c(v),$$

i.e., $\tau(-1) = 1$. The $\{\pm 1\}$ is contained in $\ker \tau$.

Conversely, let $g \in \ker \tau$. Then, by Lemma 6.3, for all $v \in \mathbb{R}^n$,

$$c(v) = c(\tau(g)v) = g c(v) g^{-1},$$

which implies that $c(v)g = g c(v)$. Observe that, in $\mathbb{C}P(\mathbb{R}^n)$, the exponential map leaves $\mathbb{C}P^+(\mathbb{R}^n)$

invariant, so $\text{Spin}(n)$ is contained in $\mathbb{C}P^+(\mathbb{R}^n)$. Therefore, using Lemma 3.6, we see that, for all $v \in \mathbb{R}^n$,

$$0 = [c(v), g] = [c(v), g]' = -2i(c(v)\sigma(g)).$$

Thus $i(c(v)\sigma(g)) = 0 \quad \forall v \in \mathbb{R}^n$, and hence $\sigma(g)$ is scalar, i.e., g too is scalar.

Next, let $\{e^1, \dots, e^n\}$ be an orthonormal basis of $\mathbb{R}^n$. Then by using Lemma 3.1 we see that

$$(c(e^i)c(e^j))^* = c(e^j)^*c(e^i)^* = (-c(e^j))(\tau(e^i)) = c(e^j)c(e^i) = -c(e^i)c(e^j).$$

It then follows that, for all $a \in \mathbb{C}P^+(\mathbb{R}^n)$,

$$a^* = -a,$$

and hence

$$\exp(a)^* = \exp(a^*) = \exp(-a) = \exp(a)^{-1}.$$

This shows that all the elements of $\text{Spin}(n)$ are unitary elements of $\mathbb{C}P(\mathbb{R}^n)$. As g is a (real) scalar, we deduce that $g = \pm 1$. This proves that $\ker \tau = \{\pm 1\}$ and completes the proof. $\square$

Using τ we can lift any representation of $\text{SO}(n)$ to a representation of $\text{Spin}(n)$. Conversely, it follows from Proposition 6.4 that a representation π of $\text{Spin}(n)$ in some vector space E descends to a representation of $\text{SO}(n)$ if and only if $\pi(-1) = -1_E$.

Next, any representation of $\mathbb{C}P(\mathbb{R}^n)$ restricts to a representation of $\text{Spin}(n)$. In particular, the spinor representation gives rise to a representation:

$$\rho: \text{Spin}(n) \longrightarrow \text{End}(\mathbb{S}).$$

As $\rho(-1) = -\text{id}_{\mathbb{S}}$, we see that this representation does not descend to a representation of $\text{SO}(n)$.

Furthermore, as pointed out in the proof of Proposition 6.4, the group $\text{Spin}(n)$ is contained in $\mathbb{C}P^+(\mathbb{R}^n)$, so its action preserves both $\mathbb{S}^+$ and $\mathbb{S}^-$. Therefore, we actually have two representations,

$$\rho_{\pm}: \text{Spin}(n) \longrightarrow \text{End}(\mathbb{S}^{\pm}).$$

These representations are called half-spinor representations.