CHAPTER 8

The Atiyah-Singer Index Theorem

The main reference for this chapter is the monograph [**BGV**]. We also refer to [**Sh**, §8] for the section on Fredholm operators. The proof of the Atiyah-Singer's local index formula given in Section 8.12 is taken from [**Po**].

We refer to the scans on the course's website for the notes of Sections 1–9 about Clifford algebras, spin structures, Dirac operators and Fredholm operators, and for the notes of the appendix on characteristic classes.

8.10. The Local Index Formula of Atiyah-Singer

Let (M^n, g) be an even dimensional compact oriented Riemannian manifold and let $E = E^+ \oplus E^-$ be a Clifford-module bundle equipped with a Clifford connection ∇^E. We shall denote by $D_E : C^\infty(M, E) \to C^\infty(M, E)$ the associated Dirac operator as defined in Section 8.

As we saw in Section 8, the operator D_E is elliptic and takes the form,

$$D_E = \begin{pmatrix} 0 & D_E^- \\ D_E^+ & 0 \end{pmatrix}, \qquad D_E^\pm : C^\infty(M, E^\pm) \to C^\infty(M, E^\pm).$$

Moreover, the selfadjointness of D_E implies that $(D_E^\pm)^* = D_E^\mp$.

As D_E is elliptic, the results of Section 9 show that D_E is Fredholm. However, as D_E is selfadjoint its index must be zero. However, the ellipticity of D_E implies that $D_E^\pm$ is elliptic, and hence is Fredholm. We then define the index of D_E to be

$$\operatorname{ind} D_E := \operatorname{ind} D_E^+ = \dim\ker D_E^+ - \dim\ker D_E^-,$$

where we have used the fact that D_E^- is the adjoint of D_E^+.

Let us denote by $\operatorname{Str}_{E/\$}$ the relative supertrace on the fibers of $\operatorname{Hom}_{\mathrm{Cl}_{\mathbb{C}}(M)}(E, E)$. Recall that

$$\operatorname{Str}_{E/\$}(T) = 2^{-\frac{n}{2}} \operatorname{Str}_E(\Gamma T) \qquad \forall T \in \mathbb{C}^\infty(M, \operatorname{Hom}_{\mathrm{Cl}_{\mathbb{C}}(M)}(E, E)),$$

where we have denoted by Γ the chirality operator (or, more precisely, the section $\operatorname{End} E$ given by the pointwise action of chirality operator on the fibers of E).

In the sequel, we denote by $F^{E/\$}$ the twisted curvature of the Clifford connection ∇^E (see Section 8 for its precise definition).

Definition 8.10.1. *The relative Chern form of the twisted curvature $F^{E/\$}$ is*

$$\operatorname{Ch}(F^{E/\$}) = \operatorname{Str}_{E/\$}\left[e^{-F^{E/\$}}\right] \in C^\infty(M, \Lambda^{ev} T^* M).$$

For instance, suppose that M is spin and let $W = W^+ \otimes W^-$ be a $\mathbb{Z}_2$-graded Hermitian vector bundle with Hermitian connection ∇^W preserving its $\mathbb{Z}_2$-grading.

Then we can we form the twisted Clifford-bundle module $E = \not{S} \otimes W$ and equipped with the twisted Clifford connection,

$$\nabla^E = \nabla^{\not{S}} \otimes 1_W + 1_{\not{S}} \otimes \nabla^W.$$

As alluded to in Section 8, the twisted curvature of this twisted connection is just $1_{\not{S}} \otimes F^W$. Therefore, its relative Chern form is equal to

$$\begin{aligned} \mathrm{Ch}(F^{E/\not{S}}) &= \mathrm{Str}_{E/\not{S}} \left[1_{\not{S}} \otimes e^{-F^W}\right] = \mathrm{Str}_W \left[e^{-F^W}\right] \\ &= \mathrm{Tr}_{W^+} \left[e^{-F^{W^+}}\right] - \mathrm{Tr}_{W^-} \left[e^{-F^{W^-}}\right] \\ &= \mathrm{Ch}(F^{W^+}) - \mathrm{Ch}(F^{W^-}), \end{aligned}$$

where $F^{W^\pm}$ is the curvature of the connection on $W^\pm$ induced by ∇^W and $\mathrm{Ch}(F^{W^\pm})$ is its usual Chern form.

In addition, as explained in the appendix on characteristic classes, the $\hat{A}$-form of the Riemann curvature R^M of M (i.e., the curvature of the Levi-Civita connection on TM) is

$$\hat{A}(R^M) := \det{}^{\frac{1}{2}} \left[\frac{R^M/2}{\sinh(R^M/2)}\right].$$

We are now in a position to state the index formula of Atiyah-Singer.

Theorem 8.10.2 (Atiyah-Singer). *We have*

$$\mathrm{ind}\, D_E = (2i\pi)^{-\frac{n}{2}} \int_M \left[\hat{A}(R^M) \wedge \mathrm{Ch}(F^{E/\not{S}}\right]^{(n)}, \tag{8.1}$$

where $\left[\hat{A}(R^M) \wedge \mathrm{Ch}(F^{E/\not{S}}\right]^{(n)}$ *denotes the n-th degree component of the even form* $\hat{A}(R^M) \wedge \mathrm{Ch}(F^{E/\not{S}})$.

Remark 8.10.3. The local index formula of Atiyah- Singer continue to hold even M is non-orientable. Notice that the integrand $\left[\hat{A}(R^M) \wedge \mathrm{Ch}(F^{E/\not{S}}\right]^{(n)}$ defines a density, namely,

$$\langle \left[\hat{A}(R^M) \wedge \mathrm{Ch}(F^{E/\not{S}}\right]^{(n)}, \omega_M \rangle v_g(x), \tag{8.2}$$

where ω_M is the volume form and $v_g(x)$ is the Riemannian density. The integral $\int_M \left[\hat{A}(R^M) \wedge \mathrm{Ch}(F^{E/\not{S}}\right]^{(n)}$ then is the integral of this density.

The above process allows us to identify n-the degree forms and densities, but it depends on a choice of orientation. Observe that the definition of the relative Chern form $\mathrm{Ch}(F^{E/\not{S}}$ depends on the definition of relative supertrace which involves the chirality operator, hence depends on the choice of the orientation (see Eq. (8.6). However, if at a point $x \in M$, we reverse the orientation of T_xM then it only affects by a change of sign the values at x of the chirality operator Γ and relative Chern form $\mathrm{Ch}(F^{E/\not{S}}$. Since this similarly affects the value at x of the volume form, we see that the value at x of the density (8.2) is actually independent of the orientation of T_xM. Therefore (8.2) defines a density even M is non-orientable. Then the index of D_E continue to be given by the formula (8.1), where the r.h.s. is interpreted as a multiple of the integral of the density (8.2).

The above index formula is often referred to as the *local index formula* of Atiyah-Singer. It is not a mere special case of the *full index theorem* of Atiyah-Singer [**AS**] for general elliptic ΨDOs. In many respects, the local index formula is equally important, if not even more important, than the full index theorem.

As it turns out, K-theoretic arguments show that the local index formula is actually equivalent to the full index theorem (see, e.g., [**ABP**]). More importantly, its only the case of Dirac operators on Clifford-module bundles that the Atiyah-Singer's index theorem reaches its true geometric contents. Therefore, the local index formula for Dirac operators is often confused with the full index theorem of Atiyah-Singer.

We shall describe some of the geometric applications of the Atiyah-Singer's local index formula in the next section. Before doing this let us briefly outline of the proof this formula.

As D_E^2 is a selfadjoint operator with non-negative spectrum, the Borel functional calculus allows us to define the heat semi-group $e^{-tD_E^2}$, $t \geq 0$, as a family of bounded operators on $L^2(M,E)$ (cf. Chapter II). Indeed, if $(\xi_k)_{k\geq 0}$ is an orthonormal basis of eigenvectors such that $D_E^2\xi_k = \lambda_k(D_E^2)\xi_k$ for all $k \geq 0$, then

$$e^{-tD_E^2}\xi_k = e^{-t\lambda_k(D_E^2)}\xi_k \qquad \forall k \geq 0.$$

In addition,we denote by Str the supertrace on the $\mathbb{Z}_2$-graded Hilbert space $L^2(M,E) = L^2(M,E^+) \oplus L^2(M,E^-)$, that is,

$$\operatorname{Str}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \operatorname{Trace}(A) - \operatorname{Tr}(B)$$

for any trace-class operator $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ on $L^2(M,E) = L^2(M,E^+) \oplus L^2(M,E^-)$.

Let $t > 0$. The ellipticity of D_E^2 ensures us that, for $t > 0$, the operator $e^{-tD_E^2}$ is smoothing, and hence has a smooth Schwartz kernel, i.e., a section $k_t(x,y)$ in $C^\infty(M \times M, E \boxtimes (E^* \otimes |\Lambda|(M)))$, which is called the *heat kernel* of D_E^2 (see Section 8.12 on this point). It then follows that $e^{-tD_E^2}$ is trace-class and its supertrace is given by

$$\operatorname{Str} e^{-tD_E^2} = \int_M \operatorname{Str}_E[k_t(x,x)],$$

where $\operatorname{Str}_E[k_t(x,x)]$ is defined as a density on M.

Theorem 8.10.4 (McKean-Singer Formula). *For all $t > 0$,*

$$\operatorname{ind} D_E = \operatorname{Str} e^{-tD_E^2} = \int_M \operatorname{Str}_E[k_t(x,x)].$$

Proof. We know that the Dirac operator D_E is of the form,

$$D_E = \begin{pmatrix} 0 & D_E^- \\ D_E^+ & 0 \end{pmatrix},$$

where $D_E^\pm : C^\infty(M,E^\pm) \to C^\infty(M,E^\pm)$ and $(D_E^\pm)^* = D_E^\mp$. Thus,

$$D_E^2 = \begin{pmatrix} D_E^- D_E^+ & 0 \\ 0 & D_E^+ D_E^- \end{pmatrix}.$$

Let $t > 0$. Then

$$e^{-tD_E^2} = \begin{pmatrix} e^{-tD_E^- D_E^+} & 0 \\ 0 & e^{-tD_E^+ D_E^-} \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \operatorname{Str} e^{-tD_E^2} &= \operatorname{Trace}\left[e^{-tD_E^- D_E^+}\right] - \operatorname{Trace}\left[e^{-tD_E^+ D_E^-}\right] \\ &= \sum_{\lambda \in \mathbb{C}} e^{-t\lambda} \dim E_\lambda^+ - \sum_{\lambda \in \mathbb{C}} e^{-t\lambda} \dim E_\lambda^- \\ &= \dim E_0^+ - \dim E_0^- + \sum_{\lambda \in \mathbb{C}\setminus 0} e^{-t\lambda}(\dim E_\lambda^+ - \dim E_\lambda^-), \end{aligned}$$

where, for all $\lambda \in \mathbb{C}$, we have set

$$E_\lambda^+ := \ker(D_E^- D_E^+ - \lambda) \quad \text{and} \quad E_\lambda^- := \ker(D_E^+ D_E^- - \lambda).$$

Notice that, as $(D_E^\pm)^* = D_E^\mp$, we have

$$E_0^\pm = \ker D_E^\mp D_E^\pm = \ker (D_E^\pm)^* D_E^\pm = \ker D_E^\pm.$$

Thus,

$$\dim E_0^+ - \dim E_0^- = \dim\ker D_E^+ - \dim\ker D_E^- = \operatorname{ind} D_E,$$

and hence

$$\operatorname{Str} e^{-tD_E^2} = \operatorname{ind} D_E + \sum_{\lambda \in \mathbb{C}\setminus 0} e^{-t\lambda}(\dim E_\lambda^+ - \dim E_\lambda^-). \tag{8.3}$$

Next, let $\lambda \in \mathbb{C}\setminus 0$ and let ξ be in $E_\lambda^+ = \ker(D_E^- D_E^+ - \lambda)$. Then

$$D_E^+ D_E^-(D_E^+\xi) = D_E^+(D_E^- D_E^+ \xi) = \lambda D_E^+ \xi,$$

that is, $D_E^+\xi$ is contained in $E_\lambda^- = \ker(D_E^+ D_E^- - \lambda)$. Thus D_E^+ induces a linear map from E_λ^+ to E_λ^-.

Similarly, the operator D_E^- induces a linear map from E_λ^- and E_λ^+. As by the very definitions of E_λ^+ and E_λ^- we have

$$\left(D_E^- D_E^+\right)_{|E_\lambda^+} = \lambda\, \mathrm{id}_{E^+} \quad \text{and} \quad \left(D_E^+ D_E^-\right)_{|E_\lambda^-} = \lambda\, \mathrm{id}_{E^-},$$

we see that D_E^+ induces a linear isomorphism from E_λ^+ onto E_λ^-. Thus,

$$\dim E_\lambda^+ = \dim E_\lambda^- \qquad \forall \lambda \in \mathbb{C}\setminus 0.$$

Combining this with (8.3) yields the the McKean-Singer formula. □

Now, the local index formula of Atiyah-Singer is proved by combining the McKean-Singer formula with the following:

Theorem 8.10.5 (Patodi, Gilkey, Atiyah-Bott-Patodi). *In $C^\infty(M, |\Lambda|(M))$,*

$$\operatorname{Str}_E[k_t(x,y)] \longrightarrow \left[\hat{A}(R^M) \wedge \operatorname{Ch}(F^{E/\not{S}})\right]^{(n)} \qquad \text{as } t \to 0^+.$$

Theorem 8.10.5 is often referred to as the *local index theorem*. In the above general form, it was first proved by Atiyah-Bott-Patodi [**ABP**] by making use of Riemannian invariant theory (see also [**Gi**]). The first purely analytical proofs were produced by Getzler [**Ge1**] and Bismut [**Bi**] by means of different approaches. These proofs were further simplified by the celebrated"short proof" of Getzler [**Ge2**], where was introduced the use of the so-called "Getzler's rescaling". This proof paved the way for many generalizations of the local index formula of Atiyah-Singer'.

In Section 8.12, we reproduce the proof of the local index theorem given in [**Po**]. This proof is especially relevant for applications in noncommutative geometry (cf. Chapter 11).

Notice, that it can be shown that in $C^\infty(M, |\Lambda|(M) \otimes \operatorname{End} E)$,

$$k_t(x,x) \sim t^{-\frac{n}{2}} \sum_{j \geq 0} t^j a_j(D_E^2)(x) \qquad \text{as } t \to 0^+,$$

where $a_0(D_E^2)(x)$ is the positive multiple of the Riemannian density (cf. Section 8.12). Therefore, it is a very striking fact that under the supertrace all the divergent terms in the above asymptotics vanish and we obtain a convergent quantity.

It should also be stressed out that this phenomenon is really specific to Dirac operators coming from Clifford connections; in general the local index theorem does not occur for other operators.

8.11. Geometric Applications

Let (M^n, g) be an even-dimensional oriented compact Riemannian manifold. Depending on the existence of an additional structure (e.g.spin structure, complex structure) the Atiyah-Singer index formula has various striking consequences. We give here a brief overview of them, referring to the book of Berline-Getzler-Vergne for a more detailed treatment, including proofs of the results stated here.

8.11.1. Dirac Operators on Spin Manifolds. Assume that M is *spin* and let us denote by $\not{S}$ its spinor bundle. In addition, let $W = W^+ \otimes W^-$ be a Hermitian $\mathbb{Z}_2$-graded vector bundle equipped with a Hermitian connection ∇^W preserving the $\mathbb{Z}_2$-grading of W.

We denote by $\mathcal{D}_W : C^\infty(M, \not{S} \otimes W) \to C^\infty(M, \not{S} \otimes W)$ the associated twisted Dirac operator, that is, $\mathcal{D}_W$ is the Dirac operator associated to the twisted Clifford connection $\nabla^{\not{S} \otimes W} := \nabla^{\not{S}} \otimes 1 + 1 \otimes \nabla^W$, where $\nabla^{\not{S}}$ is the spin connection.

As shown before, the twisted curvature of $\nabla^{\not{S} \otimes W}$ is $1 \otimes F^W$, where F^W is the curvature of the connection ∇^W, and its relative Chern form agrees with the Chern form $\operatorname{Ch}(F^W)$ of F^W. Therefore, we the local index formula of Atiyah-Singer gives

Theorem 8.11.1 (Atiyah-Singer). *We have*

$$\operatorname{ind} \not{D}_W = (2i\pi)^{-\frac{n}{2}} \int_M \left[\hat{A}(R^M) \wedge \operatorname{Ch}(F^W) \right]^{(n)}.$$

In particular, in the case of the Dirac operator $\not{D}_M$ of M acting on spinors (i.e., when W is the flat trivial line bundle over M) we get

Theorem 8.11.2 (Atiyah-Singer). *We have*

$$\operatorname{ind} \not{D}_M = (2i\pi)^{-\frac{n}{2}} \int_M \hat{A}(R^M)^{(n)}.$$

As an immediate corollary we obtain:

Corollary 8.11.3. *If the $\hat{A}$-number of M, i.e.,*

$$\hat{A}(M) := \int_M \left[\hat{A}(R^M) \right]^{(n)},$$

is not an integer, then M does not admit a spin structure.

In the sequel, we denote by κ_M the scalar curvature of M.

THEOREM 8.11.4 (Lichnerowicz). *Suppose that $\kappa_M(x) > 0$ for all $x \in M$. Then the null space of $\not{D}_M$ is reduced to $\{0\}$.*

PROOF. This is a direct consequence of the Lichnerowicz's formula,

$$\not{D}_M^2 = (\nabla^{\not{S}})^* \nabla^{\not{S}} + \frac{1}{4}\kappa_M.$$

It implies that, for all $u \in C^\infty(M, \not{S})$,

$$\langle \not{D}_M u, u \rangle = \langle \nabla^{\not{S}} u, \nabla^{\not{S}} u \rangle + \frac{1}{4}\langle \kappa_M u, u \rangle \geq \frac{1}{4} \int \kappa_M(x) \|u(x)\|^2_{\not{S}_x} v_g(x) \geq c\|u\|^2_{L^2(M,\not{S})},$$

where $v_g(x)$ is the Riemannian density and we have set $c := \inf_{x \in M} \kappa_M(x)$.

As M is compact and κ_M is positive, the constant c is positive. Therefore, no non-zero smooth section of $\not{S}$ can be contained in $\ker \not{D}_M^2$. As $\not{D}_M$ is elliptic and selfadjoint, $\ker \not{D}_M^2 = \ker \not{D}_M \subset C^\infty(M, \not{S})$, so we see that $\ker \not{D}_M$ is trivial. □

If $\not{D}_M$ has a trivial null space, then, obviously, $\operatorname{ind} \not{D}_M = 0$. Therefore, combining Theorem 8.11.2 and Theorem 8.11.4 gives

THEOREM 8.11.5. *If M carries a metric of positive scalar curvature, then its $\hat{A}$-number $\hat{A}(M)$ vanishes.*

8.11.2. The Chern-Gauss-Bonnet Theorem. The (complexified) exterior-algebra bundle $\Lambda^*_{\mathbb{C}} T^*M$ carries the $\mathbb{Z}_2$-grading,

$$\Lambda^* T^* - C^\infty(M) = \Lambda^{\mathrm{ev}} T^*M \oplus \Lambda^{\mathrm{odd}} T^*M.$$

This is a Clifford-module bundle with respect to the natural action of $\mathrm{Cl}_{\mathbb{C}}(M)$ seen as a sub-bundle of $\mathrm{End}(\Lambda^*_{\mathbb{C}} T^*M)$. Because we can define another $\mathbb{Z}_2$-grading by means of the chirality operator (see next subsection), we shall denote by $\Lambda^{\mathrm{ev}/\mathrm{odd}} T^*_{\mathbb{C}} M$, or simply$\Lambda^{\mathrm{ev}/odd}$ the Clifford-module bundle defined by means of the above $\mathbb{Z}_2$-grading.

As it turns out, the Levi-Civita connection $\nabla^{\Lambda^*_{\mathbb{C}} T^*M}$ on $\Lambda^{\mathrm{ev}/odd}_{\mathbb{C}} T^*M$ is a Clifford connection and the associated Dirac operator agrees with the de Rham operator,

$$d + d^* : C^\infty(M, \Lambda^* T^*M) \longrightarrow C^\infty(M, \Lambda^* T^*M),$$

where d is the usual exterior differential and d^* is its adjoint (see [**BGV**]).

The index of $d + d^*$ has an important topological interpretation as follows. For $j = 0, \ldots, n$ let us denote by $H^k(M)$ the $(k+1)$-th de Rham cohomology group of M, that is,

$$H^k(M) = \ker d_k / \operatorname{im} d_{k-1},$$

where $d_k : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$ is the de Rham differential on forms of degree k. The *Euler characteristic* of M then is

$$\chi(M) := \sum_{k=0}^{n} (-1)^k \dim H^k(M).$$

In addition, let $\Delta := (d + d^*)^2 = dd^* + d^*d$ be the Laplace-Beltrami operator and let us denote by $\Delta_k := d_{k-1} d^*_{k+1} + d^*_k d_k$ its restriction to forms of degree k.

PROPOSITION 8.11.6 (Hodge; see [**BGV**]). *We have an orthogonal splitting,*

$$\ker d_k = \operatorname{im} d_{k-1} \oplus \ker \Delta_k,$$

and hence

$$H^k(M) \simeq \ker \Delta_k.$$

That is, any class in $H^k(M)$ can be represented by a unique harmornic form of degree k.

It follows from this that $\dim H^k(M) = \dim \ker \Delta_k$, and hence

$$\begin{aligned} \chi(M) &= \sum_{k=0}^{n} (-1)^k \dim \ker \Delta_k \\ &= \dim \ker \Delta_{|\Lambda^{\mathrm{ev}} T^*M} - \dim \ker \Delta_{|\Lambda^{\mathrm{odd}} T^*M} \\ &= \dim \ker (d + d^*)_{|\Lambda^{\mathrm{ev}} T^*M} - \dim \ker (d + d^*)_{|\Lambda^{\mathrm{odd}} T^*M} \\ &= \operatorname{ind}(d + d^*). \end{aligned} \tag{8.4}$$

In the sequel, we denote by $\mathrm{Eul}(R^M)$ the *Euler form* of the curvature R^M of M. Regarding R^M as a section of $(\Lambda^2 T^*M) \otimes \mathrm{End}(TM)$, the *Euler form* is the form of degree of n on M such that, for any local orthonormal frame $\{e_i\}$ of TM with dual coframe $\{e^i\}$, we have

$$\frac{1}{(n/2)!} \left(-\frac{1}{4} \sum_{i,j,k,l} R^M_{ijkl} (dx^i \wedge dx^j) \otimes (e^k \wedge e^l) \right)^{\frac{n}{2}} = \mathrm{Eul}(R^M) \otimes (e^1 \wedge \ldots \wedge e^n),$$

where we have set $R^M_{ijkl} := \langle R^M(\partial_i, \partial_j) e_k, e_l \rangle$.

LEMMA 8.11.7 (See [**BGV**]).

(1) *The twisted curvature $F^{(\Lambda^{ev/odd})/\$}$ of the Levi-Civita connection on $\Lambda^{ev/odd}_{\mathbb{C}} T^*M$ is such that, for any local orthonormal frame $\{e_i\}$ of TM with dual coframe $\{e^i\}$, we have*

$$F^{(\Lambda^{ev/odd})/\$} = -\frac{1}{4} \sum_{i,j,k,l} R^M_{ijkl} (dx^i \wedge dx^j) \otimes \left(\varepsilon(e^k) + \iota(e^k) \right) \left(\varepsilon(e^l) + \iota(e^l) \right).$$

(2) *The relative Chern form of the twisted curvature $F^{(\Lambda^{ev/odd})/\$}$ is given by*

$$\mathrm{Ch}\left(F^{(\Lambda^{ev/odd})/\$} \right) = i^{\frac{n}{2}} \, \mathrm{Eul}(R^M). \tag{8.5}$$

As the zeroth-degree component of the even form $\hat{A}(R^M)$ is equal to 1, from (8.5) we get

$$\left[\hat{A}(R^M) \wedge \mathrm{Ch}(F^{E/\$}) \right]^{(n)} = \left[\hat{A}(R^M) \right]^{(0)} \mathrm{Eul}(R^M) = \mathrm{Eul}(R^M).$$

Therefore, by using (8.4) and applying the local index formula of Atiyah-Singer we obtain

THEOREM 8.11.8 (Chern-Gauss-Bonnet Theorem). *We have*

$$\chi(M) = \operatorname{ind}(d + d^*) = (2\pi)^{-n} \int_M \mathrm{Eul}(R^M).$$

In the above form, the Chern-Gauss-Bonnet theorem is originally due to S.S. Chern. This is the most classical index theorem. When $n = 2$ we recover the well-known Gauss-Bonnet theorem,

$$\chi(M) = \frac{1}{2\pi}\int_M \kappa_M(x) v_g(x),$$

where $v_g(x)$ is the Riemannian density.

8.11.3. Hirzebruch Signature Theorem. Recall that the chirality operator is defined by

$$\Gamma := i^{\frac{n}{2}} c(\omega_M) \in C^\infty(M, \mathrm{Cl}_{\mathbb{C}}(M)) \subset C^\infty(M, \mathrm{End}(\Lambda_{\mathbb{C}} T^*M)), \tag{8.6}$$

where $\omega_M = \sqrt{\det g(x)} dx^1 \wedge \cdots \wedge dx^n$ is the volume form of M. As $\Gamma^2 = 1$ it defines an alternative $\mathbb{Z}_2$-grading on $\Lambda^*_{\mathbb{C}} T^*M$ given by

$$\Lambda^*_{\mathbb{C}} T^*M = \Lambda^+ T^*_{\mathbb{C}} M \otimes \Lambda^- T^*_{\mathbb{C}} M, \qquad \Lambda^\pm T^*_{\mathbb{C}} M := \ker(\Gamma \mp 1). \tag{8.7}$$

This $\mathbb{Z}_2$-grading is preserved by the action of $\mathrm{Cl}_{\mathbb{C}}(M)$, and so we get a new Clifford-module bundle structure on $\Lambda^*_{\mathbb{C}} T^*M$. We shall denote by $\Lambda^{+/-}_{\mathbb{C}} T^*M$ the $\Lambda^*_{\mathbb{C}} T^*M$ equipped with this Clifford-module bundle structure.

The Levi-Civita connection on $\Lambda^{+/-}_{\mathbb{C}} T^*M$ is a Clifford connection (cf. [**BGV**]), and so the associated Dirac operator is again the de Rham operator $d + d^*$. Notice that, as we are using a different $\mathbb{Z}_2$-grading, the index differs from that in (8.4). As we shall now explain the index that we get is intimately related to the signature of the manifold.

Let $\star \in C^\infty(M, \mathrm{End}\, \Lambda^* T^*M)$ be the Hodge operator, i.e.,

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega_M \qquad \forall \alpha, \beta \in C^\infty(M, \Lambda^* T^*M),$$

Lemma 8.11.9 (See [**BGV**]). *We have*

$$\Gamma = (-1)^{\frac{k(k-1)}{2}} i^{\frac{n}{2}} \star \qquad \text{on } \Lambda^k_{\mathbb{C}} T^*M,$$
$$d^* = - \star d\star$$

It follows from this lemma that $d + d^* = d - \star d\star$. In the sequel, in order to distinguish with the index in (8.4), we shall denote by $\mathrm{ind}(d - \star d\star)$ the index of $d + d^*$ when using the $\mathbb{Z}_2$-grading defined by the chirality operator. That is,

$$\begin{aligned} \mathrm{ind}(d - \star d\star) &:= \dim\ker(d + d^*)_{|\Lambda^+ T^*_{\mathbb{C}} M} - \dim\ker(d + d^*)_{|\Lambda^- T^*_{\mathbb{C}} M} \\ &= \dim\ker(d - \star d\star)_{|\Lambda^+ T^*_{\mathbb{C}} M} - \dim\ker(d - \star d\star)_{|\Lambda^- T^*_{\mathbb{C}} M}. \end{aligned}$$

It also follows from Lemma 8.11.9 that the Hodge $\star$-operator descends to a linear map,

$$\star : H^k(M) \longrightarrow H^{n-k}(M), \tag{8.8}$$

which is an isomorphism since $\star^2 = (-1)^k$ on $\Lambda^k T^*M$. Using this isomorphism we can prove the nondegeneracy of the bilinear pairing,

$$H^k(M) \times H^{n-k}(M) \ni (\alpha, \beta) \longrightarrow \int_M \alpha \wedge \beta.$$

When n is divisible by 4 this pairing is symmetric on $H^{\frac{n}{2}}(M) \times H^{\frac{n}{2}}(M)$, and hence it gives rise to a nondegenerate quadratic form on $H^{\frac{n}{2}}(M)$. The signature of this quadratic form is called the *signature* of M and is denoted $\sigma(M)$. This is a topological invariant of M.

Lemma 8.11.10 (See [**BGV**]). *If n is divisible by 4, then*
$$\sigma(M) = \mathrm{ind}(d - \star d\star).$$

Because of this lemma, the operator $d - \star d \star$ is often called *signature operator*.

Next, the twisted curvature of the Levi-Civita on $\Lambda^{+/-}T^*_{\mathbb{C}}M$ is continue to be given by (1). As we shall now see, since we are using a different $\mathbb{Z}_2$-grading, we obtain a different relative Chern form, which we shall denote by $\mathrm{Ch}(F^{(\Lambda^{+/-})/\$})$.

Recall that L-form $L(R^M)$ is the of the curvature R^M is defined by
$$L(R^M) := \det{}^{\frac{1}{2}}\left(\frac{R^M/2}{\tanh(R^M/2)}\right).$$

Lemma 8.11.11 (See [**BGV**]). *We have*
$$\mathrm{Ch}(F^{E/\$}) = 2^{\frac{n}{2}} \det{}^{\frac{1}{2}}\left(\frac{\sinh(R^M/2)}{R^M/2}\right) \wedge L(R^M).$$

Observe that $\hat{A}(R^M) \wedge \det^{\frac{1}{2}}\left(\frac{\sinh(R^M/2)}{R^M/2}\right) = 1$. Thus,
$$\hat{A}(R^M) \wedge \mathrm{Ch}(F^{E/\$}) = 2^{\frac{n}{2}} L(R^M).$$

Therefore, using the Atiyah-Singer index formula we obtain:

Theorem 8.11.12 (Atiyah-Singer). *We have*
$$\mathrm{ind}(d - \star d\star) = (i\pi)^{-\frac{n}{2}} \int_M \left[L(R^M)\right]^{(n)}.$$

Combining this with Lemma 8.11.10, we immediately get

Theorem 8.11.13 (Hirzebruch Signature Theorem). *If $\dim M$ is divisible by 4, then*
$$\sigma(M) = \mathrm{ind}(d - \star d\star) = (i\pi)^{-\frac{n}{2}} \int_M \left[L(R^M)\right]^{(n)}.$$

Since $\left[L(R^M)\right]^{(4)} = -\frac{1}{96\pi^2}\mathrm{Tr}[(R^M)^2]$, we see that in dimension 4,
$$\sigma(M) = -\frac{1}{24\pi^2}\int_M \mathrm{Tr}[(R^M)^2].$$

More generally, let W be a Hermitian vector bundle equipped with a Hermitian connection ∇^W. Then we can form the twisted Clifford-module bundle $\Lambda^{+/-}T^*_{\mathbb{C}}M \otimes W$, where the fiberwise action of $\mathrm{Cl}_{\mathbb{C}}(M)$ is such that, above all $x \in M$,
$$a(\xi \otimes w) = (a\xi) \otimes w, \qquad \forall a \in \mathrm{Cl}_x(M) \quad \forall (\xi, w) \in \Lambda^* T^*_x M \times W_x.$$

The twisted connection $\nabla^{\Lambda^{+/-}T^*_{\mathbb{C}}M\otimes W} = \nabla^{\Lambda^* T^* M} \otimes 1 + 1 \otimes \nabla^W$ is a Clifford connection on $\Lambda^{+/-}T^*_{\mathbb{C}}M \otimes W$. We shall denote by $(d - \star d\star)_W$ the associated Dirac operator. Its index is denoted $\sigma(M, W)$ and is called the twisted signature with coefficients in W. It depends only on the topology of the manifold M and the vector bundle W.

The twisting curvature $F^{(\Lambda^{+/-}T^*_{\mathbb{C}}M\otimes W)/\$}$ is equal to
$$F^{(\Lambda^{+/-})/\$} \otimes 1 + 1 \otimes F^W,$$
where F^W is the curvature of ∇^W. Thus, its relative Chern form is given by
$$\mathrm{Ch}(F^{(\Lambda^{+/-}T^*_{\mathbb{C}}M\otimes W)/\$}) = \mathrm{Ch}(F^{E/\$}) \wedge \mathrm{Ch}(F^W).$$

Therefore, we obtain:

THEOREM 8.11.14 (Twisted Signature Theorem). *We have*

$$\sigma(M,W) = (d - \star d\star)_W = (i\pi)^{-\frac{n}{2}} \int_M \left[L(R^M) \wedge \mathrm{Ch}(F^W)\right]^{(n)}.$$

The twisted signature theorem was originally proved by Atiyah-Singer. It is important because it can be used to prove the *full* Atiyah-Singer index theorem, i.e., the index theorem for *general* elliptic ΨDOs on compact manifolds.

8.11.4. The Hirzebruch-Riemann-Roch Formula. In this subsection, we assume that M is a *complex* manifold of complex dimension n. This means that the tangent bundle TM is thus endowed with a complex structure $J \in C^\infty(M, \mathrm{End}_{\mathbb{R}} TM)$, $J^2 = -1$, so that the holomorphic tangent bundle $T_{1,0}M := \ker(J - i)$ is integrable in Fróbenius' sense, i.e., $[Z,W] \in C^\infty(M, T_{1,0}M)$ for all $Z, W \in C^\infty(M, T_{1,0}M)$.

In addition, we assume that the orientation of M is the orientation defined by its complex structure and the Riemannian metric on $g\ TM$ is the real part of a Hermitian metric h on $T_{\mathbb{C}}M$ with respect to which J is unitary.

The complexified tangent bundle $T_{\mathbb{C}}M := TM \otimes \mathbb{C}$ admits the orthogonal decomposition,

$$T_{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M, \qquad T_{0,1}M := \overline{T_{1,0}M}.$$

By duality this gives rise to orthogonal splittings,

$$T^*_{\mathbb{C}}M = \Lambda^{1,0}T^*M \oplus \Lambda^{0,1}T^*M, \qquad \Lambda^*_{\mathbb{C}}T^*M = \bigoplus_{q=0}^{n} \Lambda^{p,q}T^*M,$$

where $\Lambda^{p,q}T^*M := \left(\Lambda^{1,0}T^*M\right)^p \wedge \left(\Lambda^{0,1}T^*M\right)^q$ is the bundle of (p,q)-covectors.

If $\alpha \in C^\infty(M, \Lambda^{0,q})$, then

$$d\alpha = \partial\alpha + \overline{\partial}\alpha,$$

where $\partial\alpha$ is a $(1,q)$-form and $\overline{\partial}\alpha$ is a $(0,q+1)$-form. Furthermore, the integrability of $T_{1,0}$ implies that $\overline{\partial}^2 = 0$. We thus get a chain complex,

$$\overline{\partial} : C^\infty(M, \Lambda^{0,\bullet}T^*M) \longrightarrow C^\infty(M, \Lambda^{0,\bullet+1}T^*M).$$

This complex is called the *Dolbeault complex* of M and its cohomology groups are denoted $H^{0,q}(M)$, $q = 0, \ldots, n$.

Let W be a holomorphic vector bundle over M. This means that M can be covered by open subsets U_α over which there are local trivializations $\tau_\alpha : W_{|U_\alpha} \rightarrow U_\alpha \times \mathbb{C}^r$ in such way that the transition maps are holomorphic maps. Then, for $q = 0, \ldots, n$, there exists a unique operator

$$\overline{\partial}_W : C^\infty(M, (\Lambda^{0,q}T^*M) \otimes W) \longrightarrow C^\infty(M, (\Lambda^{0,q+1}T^*M) \otimes W)$$

such that, for all local holomorphic frames $e_1, \ldots, e_r$ of W and sections $\omega = \sum \omega_j \otimes e_j$ of $(\Lambda^{0,q}T^*M) \otimes W$,

$$\overline{\partial}_W\left(\sum \omega_j \otimes e_j\right) = \sum_j (\overline{\partial}\omega_j) \otimes e_j.$$

Since $(\overline{\partial}_W)^2 = 0$ this gives rise to a chain complex, called the *Dolbeault complex* of M *with coefficients in* W. Its cohomology groups are denoted $H^{0,q}(M,W)$, $q = 0, \ldots, n$.

Let us also endow W with a Hermitian metric $\langle .,. \rangle_W$. Together with the Hermitian metric on $\Lambda^{0,*}T^*M$ this defines a Hermitian metric on $(\Lambda^{0,*}T^*M) \otimes W$ and an inner inner products on $C^\infty(M,(\Lambda^{0,*}T^*M)\otimes W)$, using which we define the formal adjoint $\overline{\partial}_W^*: C^\infty(M,(\Lambda^{0,*}T^*M)\otimes W)\rightarrow C^\infty(M,(\Lambda^{0,*}T^*M)\otimes W)$. The *Dolbeault Laplacian* then is

$$\Box_W := (\overline{\partial}_W+\overline{\partial}_W^*)^2 = \overline{\partial}_W^*\overline{\partial}_W + \overline{\partial}_W\overline{\partial}_W^*.$$

Notice that $\Box_W$ maps $C^\infty(M,(\Lambda^{0,q}T^*M)\otimes W)$ to itself. We then denote by $\Box_W^{0,q}$ its restriction to $(\Lambda^{0,q}T^*M)\otimes W$.

In addition, by looking at its expression in local coordinates, it is not difficult to check that $\Box_W$ has same principal symbol as a Laplacian and hence is elliptic. As by assumption M is compact, it follows that $\ker\Box_W$ is a finite dimensional subspace of $C^\infty(M,(\Lambda^{0,*}T^*M)\otimes W)$. We then define the *holomorphic Euler-characteristic* of M with coefficients in W to be

$$\chi(M,W) := \sum_{q=0}^n (-1)^q \dim H^{0,q}(M,W).$$

This is a holomorphic invariant of M and W.

Proposition 8.11.15 (Hodge, see [**BGV**]). *For $q=0,\ldots,n$,*

$$H^{0,q}(M,W)\simeq \ker\Box_W^{0,q}.$$

Since the operators $\overline{\partial}_W+\overline{\partial}_W^*$ and $\Box_W$ have same kernel, it follows from Proposition 8.11.15 that $H^{0,q}(M,W)$ is isomorphic to the kernel of the Dolbeault operator $\overline{\partial}_W+\overline{\partial}_W^*$ on $(0,q)$-forms. Thus,

$$\chi(M,W) = \operatorname{ind}\left(\overline{\partial}_W+\overline{\partial}_W^*\right), \tag{8.9}$$

where the index of $\overline{\partial}_W+\overline{\partial}_W^*$ is defined to be

$$\dim\ker\left(\overline{\partial}_W+\overline{\partial}_W^*\right)_{|(\Lambda^{0,\mathrm{ev}}T^*M)\otimes W} - \dim\ker\left(\overline{\partial}_W+\overline{\partial}_W^*\right)_{|(\Lambda^{0,\mathrm{odd}}T^*M)\otimes W}.$$

Let $\nabla: C^\infty(M,W)\rightarrow C^\infty(M,T^*_{\mathbb{C}}M\otimes W)$ be a connection on W. Using the splitting $T^*_{\mathbb{C}}M=\Lambda^{1,0}T^*M\oplus\Lambda^{0,1}T^*M$ we can split ∇ as

$$\nabla=\nabla^{1,0}+\nabla^{0,1},$$

where $\nabla^{1,0}$ (resp., $\nabla^{0,1}$) maps to sections of $(\Lambda^{1,0}T^*M)\otimes W$ (resp., sections of $(\Lambda^{0,1}T^*M)\otimes W$). We then say that ∇ is a *holomorphic connection* when $\nabla^{0,1}=\overline{\partial}_W$. If ∇ is a holomorphic connection, then its curvature F^W is actually an $(1,1)$-form with values in $\operatorname{End}W$.

Proposition 8.11.16. *There is a unique holomorphic connection ∇^W on W that preserves its Hermitian metric, i.e.,*

$$\langle\nabla^W_X w_1,w_2\rangle_W+\langle w_1,\nabla^W_X w_2\rangle = X(\langle w_1,w_2\rangle_W)\quad \forall w_j\in C^\infty(M,W)\quad \forall X\in C^\infty(M,TM).$$

The connection ∇^W in Proposition 8.11.16 is called the *Chern connection* of W.

Let Ω be the imaginary part of the Hermitian metric h of TM. This is a 2-form on M since, for all $X,Y\in C^\infty(M,TM)$,

$$\Omega(Y,X)=\Im h(Y,X)=\Im\overline{h(X,Y)}=-\Im h(X,Y)=-\Omega(X,Y).$$

We then say that M is a *Kähler manifold* when Ω is a closed form, i.e., $d\Omega=0$.

Recall that by assumption the Riemannian metric of M is the real part of the Hermitian metric h.

Proposition 8.11.17 (See [**BGV**]). *The manifold M is Kähler if and only if the Levi-Civita connection ∇^{TM} on TM is a holomorphic connection, i.e., it agrees with the Chern connection of TM.*

From now on, we assume that M is a compact Kähler manifold. Then ∇^{TM} preserves the complex structure, and hence it can be lifted to a Hermitian connection on $\Lambda^{0,q}T^*M$ for each $q = 0, 1, \ldots, n$.

In addition, in the very same way as we constructed the spinor representation, we can endow the bundle $\Lambda^{0,*}T^*M$ with a Clifford-module bundle structure as follows.

The $\mathbb{Z}_2$-grading of $\Lambda^{0,*}T^*M$ is simply given by the splitting,

$$\Lambda^{0,*}T^*M = \Lambda^{0,\mathrm{ev}}T^*M \oplus \Lambda^{0,\mathrm{odd}}T^*M.$$

The action of $\mathrm{Cl}_{\mathbb{C}}(M)$ is such that, above all $x \in M$,

$$c(\xi).\omega = \sqrt{2}\left(\varepsilon(\xi^{1,0}) - \imath(\xi^{1,0})\right)\omega \qquad \forall \xi \in T^*_{\mathbb{C},x}M \quad \forall \omega \in \Lambda^{0,*}T^*M,$$

where $\varepsilon(\xi^{1,0})$ is the exterior by the component $\xi^{0,1}$ of ξ in $\Lambda^{0,1}T^*_xM$ and $\imath(\xi^{1,0})$ is the interior product by its component $\xi^{1,0}$ in $\Lambda^{1,0}T^*_xM$.

This allows us to realize the spinor bundle of M as the bundle $\Lambda^{0,*}T^*M$ equipped with the above Clifford-module structure. We then endow the bundle $(\Lambda^{0,*}T^*M) \otimes W$ with its twisted Clifford-module bundle structure. Namely, its Z_2-grading is given by

$$(\Lambda^{0,*}T^*M) \otimes W = \Big((\Lambda^{0,\mathrm{ev}}T^*M) \otimes W\Big) \oplus \Big((\Lambda^{0,*}T^*M) \otimes W\Big),$$

and the action of $\mathrm{Cl}_{\mathbb{C}}(M)$ is such that, for all $x \in M$,

$$a.(\omega \otimes w) = (a.\omega) \otimes w \qquad \forall a \in \mathrm{Cl}_x(M) \quad \forall (\omega, w) \in \Lambda^{0,*}T^*_xM \times W_x.$$

In addition, we equipped $(\Lambda^{0,*}T^*M) \otimes W$ with the twisted connection,

$$\nabla^{(\Lambda^{0,*}T^*M)\otimes W} := \nabla^{\Lambda^{0,*}T^*M} \otimes 1 + 1 \otimes \nabla^W,$$

where ∇^W is the Chern connection of W.

Lemma 8.11.18 (See [**BGV**]). *If M is Kähler, then $\nabla^{(\Lambda^{0,*}T^*M)\otimes W}$ is a Clifford connection on the Clifford-module bundle $(\Lambda^{0,*}T^*M) \otimes W$ and its associated Dirac operator is equal to $\sqrt{2}(\overline{\partial}_W + \overline{\partial}_W^*)$.*

In the sequel, we denote by $R^{T^{1,0}M}$ the curvature of the Levi-Civita connection of $T^{1,0}M$.

Lemma 8.11.19 (See [**BGV**]).

(1) *The twisted curvature of $\nabla^{(\Lambda^{0,*}T^*M)\otimes W}$ is given by*

$$F^{(\Lambda^{0,*}T^*M)\otimes W/\$} = \frac{1}{2}\operatorname{Tr}\left[R^{T^{1,0}M}\right] + 1 \otimes F^W. \tag{8.10}$$

(2) *We have*

$$\hat{A}(R^M) = \det\left[\frac{R^{T^{1,0}M}}{e^{\frac{1}{2}R^{T^{1,0}M}} - e^{-\frac{1}{2}R^{T^{1,0}M}}}\right]. \tag{8.11}$$

It follows from (8.10) that the relative Chern form of $F^{(\Lambda^{0,*}T^*M)\otimes W/\$}$ equals

$$\exp\left(-\frac{1}{2}\operatorname{Tr}\left[R^{T^{1,0}M}\right]\right)\wedge \operatorname{Ch}(F^W) = \det\left[e^{-\frac{1}{2}R^{T^{1,0}M}}\right]\wedge \operatorname{Ch}(F^W).$$

Combining this with (8.11) shows that $\hat{A}(R^M)\wedge \operatorname{Ch}(F^{(\Lambda^{0,*}T^*M)\otimes W/\$})$ is equal to

$$\det\left[\frac{R^{T^{1,0}M}}{e^{\frac{1}{2}R^{T^{1,0}M}}-e^{-\frac{1}{2}R^{T^{1,0}M}}}\right]\wedge\det\left[e^{-\frac{1}{2}R^{T^{1,0}M}}\right]\wedge\operatorname{Ch}(F^W) = \operatorname{Td}(R^{T^{1,0}M})\wedge\operatorname{Ch}(F^W),$$

where $\operatorname{Td}(R^{T^{1,0}M})$ is the Todd form of $R^{T^{1,0}M}$, i.e.,

$$\operatorname{Td}(R^{T^{1,0}M}) = \det\left[\frac{R^{T^{1,0}M}}{e^{R^{T^{1,0}M}}-1}\right].$$

Therefore, by applying the local index formula for Atiyah-Singer and using (8.9) we obtain

THEOREM 8.11.20 (Hirzebruch-Riemann-Roch). *If M is Kähler, then*

$$\chi(M,W) = \operatorname{ind}(\overline{\partial}_W+\overline{\partial}_W^*) = (2i\pi)^{-\frac{n}{2}}\int_M\left[\operatorname{Td}(R^{T^{1,0}M})\wedge\operatorname{Ch}(F^W)\right]^{(n)}.$$

8.12. Proof of the Local Index Theorem

In this section we reproduce the proof of the local index theorem (i.e., Theorem 8.10.5) given in [**Po**]. As mentioned in Section 8.10, this theorem yields the local index formula of Atiyah-Singer. The proof here is given for twisted Dirac operators on spin manifolds, but the argument can be extended to more general Dirac operators on Clifford-module bundle coming from a Clifford connection.

The argument is based on combining the rescaling of Getzler [**Ge2**] with the approach to the heat kernel asymptotics of Greiner [**Gr**].

8.12.1. Greiner's approach of the heat kernel asymptotics. In this section we recall Greiner's approach of the heat kernel asymptotics as in [**Gr**] and [**BGS**].

Let E be a Hermitian vector bundle over M and let $\Delta: C^\infty(M,E)\to C^\infty(M,E)$ be a selfadjoint elliptic second order differential operator with positive principal symbol.

If we regard Δ as an unbounded operator on $L^2(M,E)$ with domain the Sobolev space $L^2_2(M,E)$, then Δ is selfadjoint and bounded from below. Therefore, by standard Borel functional calculus we can define the heat semi-group $e^{-t\Delta}$, $t\geq 0$, as a family of selfadjoint bounded operators on $L^2(M,E)$.

Furthermore, the ellipticity of Δ implies that, for all $t>0$, the operator $e^{-t\Delta}$ is smoothing. That is, its Schwartz kernel $k_t(x,y)$ is contained in $C^\infty(M,E)\hat{\otimes}C^\infty(M,E^*\otimes|\Lambda|(M))$, where $|\Lambda|(M)$ is the density bundle of M. The kernel $k_t(x,y)$ is called the *heat kernel* of Δ.

Recall that the heat semigroup allows us to invert the heat equation. Namely, consider the operator $Q_0: C_c^\infty(M\times\mathbb{R},E)\to\mathcal{D}'(M\times\mathbb{R},E)$ defined by

$$Q_0u(x,s) := \int_0^\infty e^{-s\Delta}u(x,t-s)dt \qquad \forall u\in C_c^\infty(M\times\mathbb{R},E). \tag{8.12}$$

Then Q_0 maps continuously $C_c^\infty(M\times\mathbb{R},E)$ to $C^0(\mathbb{R},L^2(M,E))\subset\mathcal{D}'(M\times\mathbb{R},E)$ and satisfies

$$(\Delta+\partial_t)Q_0u=Q_0(\Delta+\partial_t)u=u\qquad\forall u\in C_c^\infty(M\times\mathbb{R},E).$$

Notice that the operator Q_0 has the *Volterra property* in the sense of [**Pi**], i.e., it has a Schwartz kernel of the form $K_{Q_0}(x,y,t-s)$, where $K_{Q_0}(x,y,t)$ vanishes on the region $t<0$. In fact,

$$K_{Q_0}(x,y,t)=\begin{cases}k_t(x,y) & \text{if } t>0,\\ 0 & \text{if } t<0.\end{cases}$$

These equalities are the main motivation for using pseudodifferential techniques to study the heat kernel $k_t(x,y)$. The idea is to consider a class of ΨDOs, the Volterra ΨDOs([**Gr**], [**Pi**], [**BGS**]), taking into account

(i) The aforementioned Volterra property.
(ii) The parabolic homogeneity of the heat operator $\Delta+\partial_t$, i.e., the homogeneity with respect to the dilations $\lambda.(\xi,\tau)=(\lambda\xi,\lambda^2\tau)$, $(\xi,\tau)\in\mathbb{R}^{n+1}$, $\lambda\neq 0$.

In the sequel, for $g\in\mathcal{S}'(\mathbb{R}^{n+1})$ and $\lambda\neq 0$, we denote by g_λ the element of $\mathcal{S}'(\mathbb{R}^{n+1})$ defined by

$$\langle g_\lambda(\xi,\tau),u(\xi,\tau)\rangle:=|\lambda|^{-(n+2)}\langle g(\xi,\tau),u(\lambda^{-1}\xi,\lambda^{-2}\tau)\rangle\quad\forall u\in\mathcal{S}(\mathbb{R}^{n+1}).\tag{8.13}$$

Definition 8.12.1. *A distribution $g\in\mathcal{S}'(\mathbb{R}^{n+1})$ is (parabolic) homogeneous of degree m, $m\in\mathbb{Z}$, when*

$$g_\lambda=\lambda^m g\qquad\forall\lambda\in\mathbb{R}\setminus 0.$$

In the sequel, we denote by $\mathbb{C}_-$ the complex halfplane $\{\Im\tau>0\}$ with closure $\overline{\mathbb{C}_-}$. Then:

Lemma 8.12.2 ([**BGS**, Prop. 1.9]). *Let $q(\xi,\tau)\in C^\infty((\mathbb{R}^n\times\mathbb{R})\setminus 0)$ be a parabolic homogeneous symbol of degree m such that:*

(i) *$q(x,\xi,\tau)$ extends to a continuous function on $(\mathbb{R}^n\times\overline{\mathbb{C}_-})\setminus 0$ in such way to be holomorphic in the last variable when the latter is restricted to $\mathbb{C}_-$.*

Then there is a unique $g\in\mathcal{S}'(\mathbb{R}^{n+1})$ agreeing with q on $\mathbb{R}^{n+1}\setminus 0$ so that:

(ii) *g is homogeneous of degree m.*
(iii) *The inverse Fourier transform $\check{g}(x,t)$ vanishes for $t<0$.*

Remark 8.12.3. If $m\leq -(n+2)$, then (ii) need not hold for symbols that do not satisfy (i).

Let U be an open subset of $\mathbb{R}^n$. We define Volterra symbols and Volterra ΨDOson $U\times\mathbb{R}^{n+1}\setminus 0$ as follows.

Definition 8.12.4. *$S_{\mathrm{v}}^m(U\times\mathbb{R}^{n+1})$, $m\in\mathbb{Z}$, consists of smooth functions $q(x,\xi,\tau)$ on $U\times\mathbb{R}^n\times\mathbb{R}$ with an asymptotic expansion $q(x,\xi,\tau)\sim\sum_{j\geq 0}q_{m-j}(x,\xi,\tau)$, where*

- *$q_l(x,\xi,\tau)\in C^\infty(U\times[(\mathbb{R}^n\times\mathbb{R})\setminus 0])$ is a homogeneous Volterra symbol of degree l, i.e. q_l is parabolic homogeneous of degree l and satisfies the property (i) in Lemma 8.12.2 with respect to the last $n+1$ variables.*

- *The sign $\sim$ means that, for all compacts $K \subset U$, integers N and k and multi-orders α and β, there is a constant $C_{NK\alpha\beta k} > 0$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k (q - \sum_{j<N} q_{m-j})(x,\xi,\tau)| \leq C_{NK\alpha\beta k}(|\xi| + |\tau|^{1/2})^{m-N-|\beta|-2k}, \tag{8.14}$$

for all $(x,\xi,\tau) \in K \times \mathbb{R}^n \times \mathbb{R}$ with $|\xi| + |\tau|^{\frac{1}{2}} \geq 1$.

In the sequel, for any symbol $q(x,\xi,\tau) \in S_{\mathrm{v}}^m(U \times \mathbb{R})$ we denote by $q(x, D_x, D_t)$ the operator from $C_c^\infty(U \times \mathbb{R})$ to $C^\infty(U \times \mathbb{R})$ defined by

$$q(x, D_x, D_t)u(x,t) := (2\pi)^{-(n+1)} \iint e^{i(x\cdot\xi + t\tau)} q(x,\xi,\tau)\hat{u}(\xi,\tau) d\xi d\tau \quad \forall u \in C_c^\infty(U \times \mathbb{R}).$$

Definition 8.12.5. *$\Psi_{\mathrm{v}}^m(U \times \mathbb{R})$, $m \in \mathbb{Z}$, consists of continuous linear operators $Q : C_c^\infty(U_x \times \mathbb{R}_t) \to C^\infty(U_x \times \mathbb{R}_t)$ such that*

(i) *Q has the Volterra property.*
(ii) *Q can be put in the form,*

$$Q = q(x, D_x, D_t) + R,$$

for some symbol $q(x,\xi,\tau) \in S_{\mathrm{v}}^m(U \times \mathbb{R})$ and some smoothing operator R.

If Q is a Volterra ΨDO we shall denote by $K_Q(x,y,t-s)$ its distribution kernel, so that the distribution $K_Q(x,y,t)$ vanishes for $t < 0$.

Example 8.12.6. Let P be a differential operator of order 2 on U and let $p_2(x,\xi)$ denote the principal symbol of P. Then the heat operator $P + \partial_t$ is a Volterra ΨDO of order 2 with principal symbol $p_2(x,\xi) + i\tau$.

Other examples of Volterra ΨDOsare given by the homogeneous operators defined below.

Definition 8.12.7. *Let $q_m(x,\xi,\tau) \in C^\infty(U \times (\mathbb{R}^{n+1} \setminus 0))$ be a homogeneous Volterra symbol of order m and let $g_m \in C^\infty(U)\hat{\otimes}\mathcal{S}'(\mathbb{R}^{n+1})$ denote its unique homogeneous extension given by Lemma 8.12.2. Then:*

- *$\check{q}_m(x,y,t)$ is the inverse Fourier transform of $g_m(x,\xi,\tau)$ in the last $n+1$ variables.*
- *$q_m(x, D_x, D_t)$ is the operator with kernel $\check{q}_m(x, y-x, t)$.*

Proposition 8.12.8 ([**Gr**], [**Pi**], [**BGS**]). *The following hold.*

(1) Composition. *Let $Q_j \in \Psi_{\mathrm{v}}^{m_j}(U \times \mathbb{R})$, $j = 1, 2$, have symbol q_j and suppose that Q_1 or Q_2 is properly supported. Then Q_1Q_2 lies in $\Psi_{\mathrm{v}}^{m_1+m_2}(U \times \mathbb{R})$ and has symbol $q_1 \# q_2 \sim \sum \frac{1}{\alpha!}\partial_\xi^\alpha q_1 D_\xi^\alpha q_2$.*
(2) Parametrices. *An operator $Q \in \Psi_{\mathrm{v}}^m(U \times \mathbb{R})$ admits a parametrix in $\Psi_{\mathrm{v}}^{-m}(U \times \mathbb{R})$ if, and only if, its principal symbol is nowhere vanishing on $U \times [(\mathbb{R}^n \times \overline{\mathbb{C}_-} \setminus 0)]$.*
(3) Invariance. *Let $\phi : U \to V$ be a diffeomorphism onto another open subset V of $\mathbb{R}^n$ and let Q be a Volterra Ψ on $U \times \mathbb{R}$ of order m. Then $Q = (\phi \oplus \mathrm{id}_{\mathbb{R}})_* Q$ is a Volterra Ψ on $V \times \mathbb{R}$ of order m.*

In addition, the following property shows the relevance of Volterra ΨDOs for deriving small times asymptotics.

LEMMA 8.12.9 ([**Gr**, Chap. I], [**BGS**, Thm. 4.5]). *Let $Q \in \Psi_{\mathrm{v}}^{m}(U \times \mathbb{R})$ have symbol $q \sim \sum q_{m-j}$. Then ,in $C^\infty(U)$,*

$$K_Q(x,y,t) \sim t^{-(\frac{n}{2}+[\frac{m}{2}]+1)} \sum_{l\geq 0} t^l \check{q}_{2[\frac{m}{2}]-2l}(x,0,1) \quad \textit{as } t \to 0^+, \tag{8.15}$$

where the notation $\check{q}_k$ has the same meaning as in Definition 8.12.7.

PROOF. As the Fourier transform relates the decay at infinity to the behavior at the origin of the Fourier transform the distribution $\check{q} - \sum_{j\leq J} \check{q}_{m-j}$ lies in $C^N(U_x \times \mathbb{R}^n_y \times \mathbb{R}_t)$ as soon as J is large enough. Since $Q - q(x, D_x, D_t)$ is smoothing it follows that $R_J(x,t) := K_Q(x,x,t) - \sum_{j\leq J} \check{q}_{m-j}(x,0,t)$ is of class C^N. As $R_J(x,y,t) = 0$ for $t < 0$ we see that

$$\partial_t^l R_J(x,0) = 0 \qquad \text{for } l = 0,1,\ldots,N.$$

Thus $R_J(.,t) = \mathrm{O}(t^N)$ in $C^N(U)$ as $t \to 0^+$. This proves that, in $C^\infty(U)$,

$$K_Q(x,x,t) \sim \sum \check{q}_{m-j}(x,0,t) \quad \text{as } t \to 0^+. \tag{8.16}$$

Let $j \in \mathbb{N}_0$. Observe that, for all $\lambda \neq 0$,

$$(\check{q}_{m-j})_\lambda = |\lambda|^{-(n+2)} (q_{m-j,\lambda^{-1}})^\vee = |\lambda|^{-(n+2)} \lambda^{j-m} \check{q}_{m-j}.$$

Thus, setting $\lambda = \sqrt{t}$ with $t > 0$, we get

$$\check{q}_{m-j}(x,0,t) = t^{\frac{j-n-m}{2}-1} \check{q}_{m-j}(x,0,1).$$

Furthermore, if we take $\lambda = -1$ and $m - j$ is odd, then

$$\check{q}_{m-j}(x,0,1) = (-1)^{m-j} q_{m-j}(x,0,1) = -q_{m-j}(x,0,1) = 0.$$

Combining all this with (8.16) shows that, in $C^\infty(U)$,

$$K_Q(x,x,t) \sim_{t\to 0^+} \sum_{m-j \text{ even}} t^{\frac{j-n-m}{2}-1} \check{q}_{m-j}(x,0,1), \tag{8.17}$$

that is, (8.15) holds. The lemma is thus proved. $\square$

The invariance property in Proposition 8.12.8 allows us to define Volterra ΨDOs on $M \times \mathbb{R}$ acting on the sections of the vector bundle E. Then all the preceding properties hold *verbatim* in this context. In particular, the heat operator $\Delta + \partial_t$ has a parametrix Q in $\Psi_{\mathrm{v}}^{-2}(M, \times\mathbb{R}, E)$.

In fact, comparing the operator (8.12) with any Volterra parametrix for $\Delta + \partial_t$ allows us to prove

THEOREM 8.12.10 ([**Gr**], [**Pi**], [**BGS**, pp. 363-362]). *The differential operator $\Delta + \partial_t$ is invertible and its inverse $(\Delta + \partial_t)^{-1}$ is a Volterra ΨDO of order -2.*

Combining this with Lemma 8.12.9 gives the heat kernel asymptotics below.

THEOREM 8.12.11 ([**Gr**, Thm. 1.6.1]). *In $C^\infty(M, |\Lambda|(M) \otimes \operatorname{End} E)$ we have*

$$k_t(x,x) \sim_{t\to 0^+} t^{\frac{-n}{2}} \sum_{l\geq 0} t^l a_l(\Delta)(x), \qquad a_l(\Delta)(x) = \check{q}_{-2-2l}(x,0,1), \tag{8.18}$$

where the equality on the right-hand side shows how to compute the densities $a_l(\Delta)(x)$ in local trivializing coordinates by means of the symbol $q(x,\xi,\tau) \sim \sum q_{-2-j}(x,\xi,\tau)$ of any Volterra parametrix for $\Delta + \partial_t$.

This approach to the heat kernel asymptotics present several advantages. First, as Theorem 8.12.11 is a purely local statement we can easily localize the heat kernel asymptotics. In fact, given a Volterra parametrix Q for $\Delta + \partial_t$ in some local trivializing coordinates around $x_0 \in M$, comparing the asymptotics (8.15) and (8.18) we get

$$k_t(x_0, x_0) = K_Q(x_0, x_0, t) + \mathrm{O}(t^\infty) \qquad \text{as } t \to 0^+. \tag{8.19}$$

Therefore in order to determine the heat kernel asymptotics (8.18) at x_0 we only need a Volterra parametrix for $\Delta + \partial_t$ near x_0.

Second, we have a genuine asymptotics with respect to the C^∞-topology and it can be differentiated as follows.

Proposition 8.12.12. *Let $P : C^\infty(M, E) \to C^\infty(M, E)$ be a differential operator of order m and let us denote by $h_t(x, y)$ the Schwartz kernel of $Pe^{-t\Delta}$. Then, in $C^\infty(M, |\Lambda| \otimes \operatorname{End} E)$,*

$$h_t(x, x) \sim_{t\to 0^+} t^{[\frac{m}{2}]-\frac{n}{2}} \sum_{l\geq 0} t^l b_l(x), \qquad b_l(x) = \check{r}_{2[\frac{m}{2}]-2-2l}(x, 0, 1), \tag{8.20}$$

where the equality on the right-hand side gives a formula for computing the densities $b_l(x)$ in local trivializing coordinates by means of the symbol $r \sim \sum r_{m-2-j}$ of $R = P(\Delta + \partial_t)^{-1}$ (or of $R = PQ$, where Q is any Volterra parametrix for $\Delta + \partial_t$).

Proof. Observe that

$$h_t(x, y) = P_x k_t(x, y) = P_x K_{(\Delta+\partial_t)^{-1}}(x, y, t) = K_{P(\Delta+\partial_t)^{-1}}(x, y, t).$$

Therefore, the result follows by applying Lemma 8.12.9 to $P(\Delta + \partial_t)^{-1}$ (or to PQ, where Q is any Volterra parametrix for $\Delta + \partial_t$). $\square$

Finally, in local trivializing coordinates the densities $a_j(\Delta)(x)$'s can be explicitly computed in terms of the symbol $p = p_2 + p_1 + p_0$ of Δ. To see this let $q \sim \sum q_{-2-j}$ be the symbol of a Volterra parametrix Q for $\Delta + \partial_t$. As $q\#p \sim q(p + i\tau) + \sum \frac{1}{\alpha!}\partial_\xi^\alpha q D_x^\alpha p \sim 1$ we get $q_{-2} = (p_2 + i\tau)^{-1}$ and

$$q_{-2-j} = -\Big(\sum_{k+l+|\alpha|=j} \frac{1}{\alpha!}\partial_\xi^\alpha q_{-2-k} D_x^\alpha p_{2-l}\Big)(p_2 + i\tau)^{-1}, \quad j \geq 1. \tag{8.21}$$

Therefore, combining with (8.18) we deduce that, as in [**Gi**], the densities $a_j(\Delta)(x)$'s are universal polynomials in the the jets at x_0 of the symbol of Δ with coefficients depending smoothly on its principal symbol.

Similarly, in local trivializing coordinates the densities $b_l(x)$'s in (8.20) can be expressed universal polynomials in the the jets at x_0 of the symbols of Δ and P with coefficients depending smoothly on the principal symbol of Δ.

8.12.2. Proof of the Local Index Theorem. In this subsection, we shall prove the local index theorem. We shall use the same notation as in Section 8.10. In particular, E is a Clifford-module bundle with Clifford connection ∇^E and associated Dirac operator D_E. In addition, we denote by $k_t(x, y)$ the heat kernel of D_E^2.

It follows from Theorem 8.12.11 that, in $C^\infty(M, |\Lambda|(M))$,

$$\operatorname{Str}_E[k_t(x, x)] \sim t^{-\frac{n}{2}} \sum_{j\geq 0} t^j \operatorname{Str}_E[a_j(D_E^2)(x)] \qquad \text{as } t \to 0^+.$$

We thus have an asymptotics in $C^\infty(M,|\Lambda|(M))$. Therefore, in order to prove the local index theorem it is enough to show that, near any given point x_0, in some local coordinates centered at x_0 and local trivialization of E near $x_0 E$,

$$\mathrm{Str}_E[k_t(0,0)] = (2i\pi)^{-\frac{n}{2}} \left[\hat{A}(R^M)\wedge \mathrm{Ch}(F^{E/\$})\right]^{(n)}(0) + \mathrm{O}(t). \tag{8.22}$$

Furthermore, if follows from (8.19) that in order to so we may replace the heat kernel $k_t(x,x)$ by the kernel $K_Q(x,y,t)$ of any Volterra-ΨDO parametrix for $D_E^2+\partial_t$ defined near $x=0$. Indeed, we then have

$$k_t(0,0) = K_Q(0,0,t) + \mathrm{O}(t^\infty) \qquad \text{as } t\rightarrow 0^+. \tag{8.23}$$

This enables use to reduce to the case where M is a neighborhood of the origin in $\mathbb{R}^n$ and E is a trivial vector bundle.

As we have total freedom on the choice of the local coordinates and the trivializations of E, we may choose to use local Riemannian coordinates and introduce a synchronous orthonormal tangent frame $\{e_1,\cdots,e_n\}$ such that $e_j=\partial_j$. Using this frame to construct the spinor representation, near $x=0$ we may realize E as a trivial twisted Clifford bundle with fiber $\S_n\otimes W$, where $\$_n$ is the spinor space of $\mathbb{R}^n$ and W is a $\mathbb{Z}_2$-graded vector space. Notice that the Clifford connection ∇^E then is a twisted Clifford connection $\nabla^{\$_n}\otimes 1 + 1\otimes\nabla^W$, where $\nabla^{\$_n}$ is the spin connection on the trivial vector bundle $M\times\S_n$ and ∇^W is a Hermitian connection on $M\times W$. Incidentally, if we denote by F^W the curvature of ∇^W, then the twisted curvature $F^{E/\$}$ and its relative Chern form are given by

$$F^{E/\$} = 1\otimes F^W \qquad \text{and} \qquad \mathrm{Ch}(F^{E/\$}) = \mathrm{Ch}(F^W).$$

In addition, as we are using normal coordinates, near $x=0$ the coefficients $g_{ij}=g(\partial_i,\partial_j)$ of the metric g are such that

$$g_{ij}(x) = \delta_{ij} + \mathrm{O}(|x|^2), \qquad \omega_{ikl}(x) = -\frac{1}{2}R^M_{ijkl}(0)x^j + \mathrm{O}(|x|^2), \tag{8.24}$$

A proof of this asymptotics can be found for instance in [**BGV**].

Let us also introduce the coefficients of the Levi-Civita connection ∇^{TM} and the curvature tensor R^M defined in terms of the synchronous orthonormal tangent frame $\{e_i\}$ by

$$\omega_{ikl} = \langle \nabla^{TM}_i e_k, e_l\rangle \quad \text{and} \quad R^M_{ijkl} = \langle R^M(\partial_i,\partial_j)e_k,e_l\rangle.$$

Then, near $x=0$,

$$\omega_{ikl}(x) = -\frac{1}{2}R^M_{ijkl}(0)x^j + \mathrm{O}(|x|^2). \tag{8.25}$$

See, e.g., [**BGV**] for a proof of this asymptotics.

Recall that the quantification and symbol maps are linear isomorphisms,

$$\Lambda^*_{\mathbb{C}}\mathbb{R}^n \xrightarrow{c} \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^n) \quad \text{and} \quad \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^n) \xrightarrow{\sigma} \Lambda^*_{\mathbb{C}}\mathbb{R}^n. \tag{8.26}$$

As n is even the spinor representation $\rho:\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^n)\rightarrow \mathrm{End}\,\$_n$ is an algebra isomorphism which allows us to identify $\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^n)$ with $\mathrm{End}\,\$_n$. Bearing in mind this identification, we shall also denote by c and σ the linear isomorphisms,

$$\Lambda^*_{\mathbb{C}}\mathbb{R}^n \xrightarrow{c} \mathrm{End}\,\$_n \quad \text{and} \quad \mathrm{End}\,\$_n \xrightarrow{\sigma} \Lambda^*_{\mathbb{C}}\mathbb{R}^n, \tag{8.27}$$

which are obtained by composing the linear isomorphisms (8.26) with ρ or its inverse.

As E is a trivial bundle with fiber $\$_n \otimes W$, we can regard the Volterra ΨDOs on $M \times \mathbb{R}$ acting on the sections of E as elements of $\Psi^*_{\mathrm{v}}(M\times\mathbb{R})\otimes(\operatorname{End}\$_n)\otimes(\operatorname{End}W)$. Using the linear maps (8.27), we then get linear maps,

$$\Psi^*_{\mathrm{v}}(M\times\mathbb{R})\otimes(\Lambda^*_{\mathbb{C}}\mathbb{R}^n)\otimes(\operatorname{End}W)\xrightarrow{c}\Psi^*_{\mathrm{v}}(M\times\mathbb{R},E),$$

$$\Psi^*_{\mathrm{v}}(M\times\mathbb{R},E)\xrightarrow{\sigma}\Psi^*_{\mathrm{v}}(M\times\mathbb{R})\otimes(\Lambda^*_{\mathbb{C}}\mathbb{R}^n)\otimes(\operatorname{End}W).$$

We get similar linear maps at the level of symbols and Schwartz kernels.

As E is the trivial $\mathbb{Z}_2$-graded bundle with fiber $\$_n\otimes W$, its supertrace is just $\operatorname{Str}_{\$_n}\otimes\operatorname{Str}_W$, where $\operatorname{Str}_{\$_n}$ is the supertrace on the trivial bundle $M\times\$_n$ and Str_W is the supertrace on W. Notice that, as the metric on M varies on the fibers of TM, so does the supertrace on the fibers of $M\times\operatorname{End}(\$_n)$. However, thanks (8.24) at $x=0$ the metric $g(x)$ agrees with the standard Euclidean metric, and hence we may use Proposition 4.8 to get

$$\operatorname{Str}_{\$_n}[T(0)]=(-2i)^{\frac{n}{2}}\sigma[T(0)]^{(n)}\qquad\forall T\in C^\infty(M,\operatorname{End}\$_n),$$

where $\sigma[T(0)]^{(n)}$ is the n-the degree component of $\sigma[T(0)]\in\Lambda^*_{\mathbb{C}}\mathbb{R}^n$.

It follows from all this that, for all $P\in\Psi^*_{\mathrm{v}}(M\times\mathbb{R},E)$,

$$\operatorname{Str}_E[K_P(0,0,t)]=(-2i)^{\frac{n}{2}}(\sigma\otimes\operatorname{Str}_W)[K_P(0,0,t)]^{(n)}\qquad\forall t>0.$$

Combining this with (8.23) then shows that

$$k_t(0,0)=(-2i)^{\frac{n}{2}}(\sigma\otimes\operatorname{Str}_W)[K_P(0,0,t)]^{(n)}+\mathrm{O}(t^\infty)\qquad\text{as } t\to 0^+. \tag{8.28}$$

Therefore, the proof of (8.22) boils down to showing the existence of a small-time limit of $(\sigma\otimes\operatorname{Str}_W)[K_P(0,0,t)]^{(n)}$ and identifying it.

In order to study the small-time behavior of $(\sigma\otimes\operatorname{Str}_W)[K_P(0,0,t)]^{(n)}$ we shall implement the rescaling of Getzler [**Ge2**] as a filtration on $\Psi^*_{\mathrm{v}}(M\times\mathbb{R},E)$. Roughly speaking this rescaling aims at assigning the following degrees:

$$\deg\partial_j=\frac{1}{2}\deg\partial_t=\deg c(dx^j)=-\deg x^j=1, \tag{8.29}$$

while $\deg B=0$ for any $B\in\operatorname{End}W$. We then obtain a filtration on $\Psi^*_{\mathrm{v}}(M\times\mathbb{R},E)$ as follows.

Let $Q\in\Psi^*_{\mathrm{v}}(M\times\mathbb{R},E)$ have symbol $q(x,\xi,\tau)\sim\sum_{k\leq m'}q_k(x,\xi,\tau)$. Then taking components in each subspace $\Lambda^jT^*_{\mathbb{C}}\mathbb{R}^n(n)$ and then using Taylor expansions at $x=0$ gives formal expansions

$$\sigma[q(x,\xi,\tau)]\sim\sum_{j,k}\sigma[q_k(x,\xi,\tau)]^{(j)}\sim\sum_{j,k,\alpha}\frac{x^\alpha}{\alpha!}\sigma[\partial_x^\alpha q_k(0,\xi,\tau)]^{(j)}. \tag{8.30}$$

According to (8.29) the symbol $\frac{x^\alpha}{\alpha!}\partial_x^\alpha\sigma[q_k(0,\xi,\tau)]^{(j)}$ is Getzler homogeneous of degree $k+j-|\alpha|$. Therefore, we can expand $\sigma[q(x,\xi,\tau)]$ as

$$\sigma[q(x,\xi,\tau)]\sim\sum_{j\geq 0}q_{(m-j)}(x,\xi,\tau),\qquad q_{(m)}\neq 0, \tag{8.31}$$

where $q_{(m-j)}$ is a Getzler homogeneous symbol of degree $m-j$.

Definition 8.12.13. *Using (8.31) we make the following definitions:*

- *The integer m is the Getzler order of Q.*
- *The symbol $q_{(m)}$ on $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ is the principal Getzler homogeneous symbol of Q.*

- *The operator* $Q_{(m)} = q_{(m)}(x, D_x, D_t)$ *on* $\mathbb{R}^n \times \mathbb{R}$ *is the model operator of* Q.

Remark 8.12.14. The model operator $Q_{(m)}$ is well defined according to definition 8.12.7.

Remark 8.12.15. By construction we always have Getzler order $\leq$ order $+\, n$, but this is not an equality in general.

Example 8.12.16. Let $A = A_i dx^i$ be the connection one-form on W. Then by (8.25) the covariant derivative $\nabla_i^E = \partial_i + \frac{1}{4}\omega_{ikl}(x)c(e^k)c(e^l) + A_i$ on $\$_n \otimes W$ has Getzler order 1 and model operator

$$\nabla^E_{i(1)} = \partial_i - \frac{1}{4}R^M_{ij}(0)x^j, \qquad R^M_{ij}(0) := R^M_{ijkl}(0)dx^k \wedge dx^l. \tag{8.32}$$

The interest to introduce Getzler orders stems from the following.

Lemma 8.12.17. *Let* $Q \in \Psi^*_{\mathrm{v}}(M \times \mathbb{R}, M)$ *have Getzler order* m *and model operator* $Q_{(m)}$. *Then, as* $t \to 0^+$,

$$\sigma[K_Q(0,0,t)]^{(j)} = \begin{cases} \mathrm{O}(t^{\frac{j-m-n-1}{2}}) & \textit{if } m-j \textit{ is odd,} \\ t^{\frac{j-m-n}{2}-1}K_{Q_{(m)}}(0,0,1)^{(j)} + \mathrm{O}(t^{\frac{j-m-n}{2}}) & \textit{if } m-j \textit{ is even.} \end{cases}$$

In particular, if $m=-2$, *then*

$$\sigma[K_Q(0,0,t)]^{(n)} = K_{Q_{(-2)}}(0,0,1)^{(n)} + \mathrm{O}(t). \tag{8.33}$$

Proof. Let $q(x,\xi,\tau) \sim \sum q_k(x,\xi,\tau)$ be the symbol of Q and let $q_{(m)}(x,\xi,\tau)$ be its principal Getzler-homogeneous symbol. By Lemma 8.12.9 we have

$$\sigma[K_Q(0,0,t)]^{(j)} \sim_{t\to 0^+} \sum t^{-\frac{n+2+m-j}{2}}\sigma[\check{q}_k(0,0,1)]^{(j)}, \tag{8.34}$$

and we know that $\check{q}_k(0,0,1) = 0$ if k is odd. Moreover, the symbol $\sigma[q_k(0,\xi,\tau)]^{(j)}$ is Getzler homogeneous of degree $k+j$, and so it must be zero if $k+j > m$ since otherwise Q would not have Getzler order m. Therefore,

$$\sigma[K_Q(0,0,t)]^{(j)} = \begin{cases} \mathrm{O}(t^{\frac{j-m-n-1}{2}}) & \text{if } m-j \text{ is odd,} \\ t^{\frac{j-m-n}{2}-1}\sigma[\check{q}_{m-j}(0,0,1)]^{(j)} + \mathrm{O}(t^{\frac{j-m-n}{2}}) & \text{if } m-j \text{ is even.} \end{cases} \tag{8.35}$$

In addition, the symbol $\sigma[q_{(m)}(0,\xi,\tau)]^{(j)}$ is equal to

$$\sum_{k+j-|\alpha|=m} \left(\frac{x^\alpha}{\alpha!}\partial_x^\alpha \sigma[q_k(0,\xi,\tau)]^{(j)}\right)_{x=0} = \sigma[q_{m-j}(0,\xi,\tau)]^{(j)}. \tag{8.36}$$

As $K_{Q_{(m)}}(x,y,t) = (q_{(m)})^\vee(x,y,t)$, we deduce that

$$\sigma[\check{q}_{m-j}(0,0,1)]^{(j)} = (q_{(m)})^\vee(0,0,1)^{(j)} = K_{Q_{(m)}}(0,0,1)^{(j)}.$$

Combining this with (8.35) proves the lemma. □

In the sequel, we say that a symbol or a ΨDO is $\mathrm{O_G}(m)$ if it has Getzler order $\leq m$.

Lemma 8.12.18. *For* $j=1,2$ *let* $Q_j \in \Psi^*_{\mathrm{v}}(M\times\mathbb{R}, E)$ *have Getzler order* m_j *and model operator* $Q_{(m_j)}$. *In addition, assume that* Q_1 *or* Q_2 *is properly supported. Then*

$$Q_1Q_2 = c\left[Q_{(m_1)}Q_{(m_2)}\right] + \mathrm{O}_G(m_1+m_2-1). \tag{8.37}$$

PROOF. Let q_j be the symbol of Q_j and let $q_{(m_j)}$ be its principal Getzler homogeneous symbol. By Proposition 8.12.8 the operator Q_1Q_2 has symbol $q_1\#q_2$.

Moreover, for N large enough, the symbol $q_1\#q_2 - \sum_{|\alpha|\leq N}\frac{1}{\alpha!}\partial_\xi^\alpha q_1 D_x^\alpha q_2$ has order $< m_1+m_2-n$, and hence has Getzler order $< m_1+m_2$. As $\partial_\xi^\alpha q_1.D_x^\alpha q_2 - c[\partial_\xi^\alpha q_{(m_1)}\wedge D_x^\alpha f_{(m_2)}]$ has Getzler order $\leq m_1+m_2-|\alpha|-1$ it follows that, for N large enough,

$$q_1\#q_2 = \sum_{|\alpha|\leq N}\frac{1}{\alpha!}c(\partial_\xi^\alpha q_{m_1}\wedge D_x^\alpha q_{m_2}) + \mathrm{O}_G(m_1+m_2-1). \tag{8.38}$$

Observe that $\sum\frac{1}{\alpha!}\partial_\xi^\alpha q_{(m_1)}\wedge D_x^\alpha q_{(m_2)}$ is exactly the symbol of $Q_{(m_1)}Q_{(m_2)}$ since $q_{(m_2)}(x,\xi,\tau)$ is polynomial in x and thus the sum is finite. Therefore, taking N large enough in (8.38) shows that the symbols of $\sigma[Q_1Q_2]$ and $Q_{(m_1)}Q_{(m_2)}$ coincide modulo a symbol of Getzler order $\leq m_1+m_2-1$. This proves the lemma. $\square$

Recall that by the Lichnerowicz formula,

$$D_E^2 = (\nabla^E)^*\nabla^E + \mathcal{F}^{E/\$} + \frac{1}{4}\kappa_M,$$

where

$$\mathcal{F}^{E/\$} = \frac{1}{2}\sum F^{E/\mathcal{S}}(e_i,e_j)c(e^i)c(e^j) = \frac{1}{2}\sum F^W(e_i,e_j)c(e^i)c(e^j).$$

Moreover, as explained in [**BGV**, p. 66],

$$(\nabla^E)^*\nabla^E = -g^{ij}(\nabla_i^E\nabla_j^E - \Gamma_{ij}^{\,k}\nabla_k),$$

where the g^{ij} are the coefficients of the inverse metric g^{-1} and the Γ_{ij}^k are the Christoffel symbols of g so that $\nabla_i^{TM}\partial_k = \Gamma_{ik}^{\,l}\partial_l$.

Combining this with Lemma 8.12.18 and the equations (8.24), (8.25) and (8.32) shows that D_E^2 has Getzler order 2 and model operator,

$$\begin{aligned} D^2_{E(2)} &= -\delta_{ij}\nabla_{i(1)}\nabla_{j(1)} + \frac{1}{2}F^W(\partial_i,\partial_j)(0)dx^i\wedge dx^j \\ &= H_R + F^W(0), \qquad H_R := -\sum_{i=1}^n\left(\partial_i - \frac{1}{4}R_{ij}^M(0)x^j\right)^2. \end{aligned} \tag{8.39}$$

LEMMA 8.12.19. *Let Q be a Volterra parametrix for $D_E^2+\partial_t$. Then*

(1) *Q has Getzler order -2 and its model operator is $(H_R+F^W(0)+\partial_t)^{-1}$.*
(2) *We have*

$$K_{(H_R+F^W(0)+\partial_t)^{-1}}(0,x,t) = G_R(x,t)\wedge e^{-tF^W(0)}, \tag{8.40}$$

where $G_R(x,t)$ is the fundamental solution of $H_R+\partial_t$, i.e., the solution of the equation, $(H_R+F^W(0)+\partial_t)G_R(x,t) = \delta(x,t)$, where $\delta(x,t)$ is the Dirac function on $\mathbb{R}^n\times\mathbb{R}$.

(3) *As $t\to 0^+$,*

$$\sigma[K_Q(0,0,t)]^{(2j)} = t^{j-\frac{n}{2}}[G_R(0,1)\wedge e^{-F^W(0)}]^{(2j)} + \mathrm{O}(t^{j-\frac{n}{2}+1}). \tag{8.41}$$

PROOF. Note that (3) follows by combining (1) and (2) with Lemma 8.12.17. Therefore, we only need to prove the first two assertions.

Let $p(x,\xi)=\sum p_j(x,\xi)$ be the symbol of $\not{D}^2$ and let $q\sim\sum q_{-2-j}$ be the symbol of Q. As $\not{D}^2$ is elliptic and has Getzler order 2 we have $p_{(2)}(0,\xi)^{(0)} = p_2(0,\xi)\neq 0$.

Hence $q_{-2} = (p_2 + i\tau)^{-1}$ has Getzler order -2. It then follows from (8.21) that each symbol q_{-2-j} has Getzler order ≤ -2, and hence Q has Getzler order -2.

Notice also that, as $(\not{D}^2 + \partial_t)Q - 1$ is smoothing, it follows from Lemma 8.12.18 the operator $(H_R + F^W(0) + \partial_t)Q_{(-2)} - 1$ has Getzler order ≤ -1. As it is Getzler-homogeneous of degree 0 it must be zero, and hence $Q_{(-2)} = (H_R + F^W(0) + \partial_t)^{-1}$. It is not difficult to check that, at the level of Schwartz kernels, the equality $Q_{(-2)}(H_R + F^W(0) + \partial_t) = 1$ means that

$$(H_{R,y} + F^W(0) + \partial_s)^T \left[K_{Q_{(-2)}}(x, y, t - s)\right] = \delta(x - y, t - s),$$

where $(H_{R,y} + F^W(0) + \partial_s)^T = H_{R,y} + F^W(0) - \partial_s$ is the transpose of $H_{R,y} + F^W(0) + \partial_s$. This implies that

$$(H_{R,y} + F^W(0) + \partial_s) \left[K_{Q_{(-2)}}(0, y, s)\right] = \delta(y, s),$$

that is, $K_{Q_{(-2)}}(0, x, t)$ is the fundamental solution of $H_R + F^W(0) + \partial_t$.

Observe that if we denote by $G_R(x, t)$ be the fundamental solution of $H_R + \partial_t$, then $G_{R,F}(x, t) := G_R(x, t) \wedge e^{-tF^W(0)}$ too is the fundamental solution of $H_R + F^W(0) + \partial_t$. Thus $K_{Q_{(-2)}}(0, x, t) = G_R(x, t) \wedge e^{-tF^W(0)}$. The proof is complete. □

At this stage observe that H_R is the harmonic oscillator associated to the skew-symmetric matrix $R^M(0) = (R^M_{ij}(0))$. We shall now make use of a version of the Melher's formula to determine the fundamental solution of $H_R + \partial_t$ (compare [**BGV**]).

Lemma 8.12.20 (Melher Formula). *Let $a > 0$ and consider the harmonic oscillator $H_a := -\frac{d}{dx^2} + \frac{1}{4}a^2x^2$ on $\mathbb{R}$. Then the fundamental solution of $H_a + \partial_t$ is*

$$G_a(x, t) := \chi(t)(4\pi t)^{-\frac{1}{2}} \left(\frac{at}{\sinh at}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{4t}x^2 \frac{at}{\tanh at}\right),$$

where $\chi(t)$ is the characteristic function of the interval $(0, \infty)$.

Proof. For $(x, t) \in \mathbb{R} \times (0, \infty)$ define

$$S_a(x, t) := (4\pi t)^{-\frac{1}{2}} \left(\frac{at}{\sinh at}\right)^{\frac{1}{2}} \exp(-\frac{1}{4t}x^2 \frac{at}{\tanh at}), \quad t > 0. \tag{8.42}$$

Observe that $S_a(x, t)$ is integrable on all products $\mathbb{R} \times (0, c)$ with $c > 0$ and

$$(H_a + \partial_t)S_a(x, t) = 0 \qquad \forall (x, t) \in \mathbb{R} \times (0, \infty).$$

Notice further that, as $t \to 0^+$,

$$\hat{S}_{x \to \xi}(\xi, t) = \cosh^{-\frac{1}{2}}(at) \exp(-\xi^2 t \frac{\tanh at}{at}) \longrightarrow 1,$$

uniformly on compact sets of $\mathbb{R}$. Thus, as $t \to 0^+$,

$$S(x, t) \longrightarrow \delta(x) \qquad \text{in } \mathcal{S}'(\mathbb{R}).$$

It follows from this that, for all u and v in $C_c^\infty(\mathbb{R})$,

$$\begin{aligned}\langle (H_a+\partial_t)G_a(x,t), u(x)v(t)\rangle &= \int_0^\infty\int_{-\infty}^\infty S_a(x,t)(H_a-\partial_t)\left(u(x)v(t)\right)dxdt\\ &= \int_0^\infty\int_{-\infty}^\infty \left((H_a+\partial_t)S_a(x,t)\right)u(x)v(t)dxdt\\ &\qquad + \lim_{t\to 0^+}\left\{v(t)\int_{-\infty}^\infty S_a(x,t)u(x)dx\right\} = v(0)u(0).\end{aligned}$$

This shows that $(H_a+\partial_t)G_a(x,t) = \delta(x,t)$, that is, $G_a(x,t)$ is the fundamental solution of $H_a+\partial_t$. The lemma is thus proved. $\square$

LEMMA 8.12.21. *The fundamental solution $H_R+\partial_t$ is given by*

$$G_R(x,t) = \chi(t)(4\pi t)^{-\frac{n}{2}}\det{}^{\frac12}\left(\frac{tR^M(0)/2}{\sinh(tR^M(0)/2)}\right)\exp\left(-\frac{1}{4t}\langle\frac{tR^M(0)/2}{\tanh(tR^M(0)/2)}x,x\rangle\right).$$

PROOF. Let $A\in M_n(\mathbb{R})$ be a skew-symmetric matrix and set $B=-A^2$. Then an elaboration of Lemma 8.12.20 shows that the fundamental solution of the heat operator $-\sum_j\partial_j^2+\frac14\langle Bx,x\rangle+\partial_t$ on $\mathbb{R}^n\times\mathbb{R}$ is given by

$$G_A(x,t) = \chi(t)(4\pi t)^{-\frac{n}{2}}\det{}^{\frac12}\left(\frac{iAt}{\sinh(iAt)}\right)\exp\left(-\frac{1}{4t}\langle\frac{iAt}{\tanh(iAt)}x,x\rangle\right). \tag{8.43}$$

To see this we can notice that the $O(n)$-invariance of $G_A(x,t)$ enables us to reduce to the case where A is in normal form, i.e., it is a block-diagonal matrix of 2×2-matrices,

$$\begin{pmatrix}0 & -a_j\\ a_j & 0\end{pmatrix},\qquad a_j>0,$$

so that the eigenvalues of A are $\pm ia_j$ and B is a diagonal matrix with the a_j^2 as entries.

A further consequence of the $O(n)$-invariance of $G_A(x,t)$ is its invariance under rotations in the (x^j,x^k)-planes $j<k$. As the infinitesimal generator of the 1-parameter group of rotations in the (x^j,x^k)-plane is $x^j\partial_k - x^k\partial_j$, it follows that

$$(x^j\partial_k - x^k\partial_j)G_A(x,t) = 0. \tag{8.44}$$

Consider the harmonic oscillator on $\mathbb{R}^n\times R$ associated to A, i.e.,

$$H_A := -\sum_j\left(\partial_j - \sum_k\frac{i}{2}A_{jk}x^k\right)^2,$$

and observe that

$$\begin{aligned}H_A &= -\sum_j\partial_j^2 - i\sum_{j,k}A_{jk}x^k\partial_j + \frac14\sum_{jkl}A_{jk}A_{jl}x^kx^l\\ &= -\sum_j\partial_j^2 + i\sum_{j<k}A_{jk}(x^j\partial_k - x^k\partial_j) + \frac14\langle Bx,x\rangle.\end{aligned}$$

Therefore, in view of (8.44) we that $G_A(x,t)$ is also the fundamental solution of $H_A+\partial_t$. This means that, for all $u(x,t)\in C_c^\infty(\mathbb{R}^n\times\mathbb{R})$,

$$u(0,0) = \langle (H_A+\partial_t)G_A(x,t), u(x,t)\rangle\rangle = \langle G_A(x,t),(H_A-\partial_t)u(x,t)\rangle.$$

Observe that $\langle G_A(x,t), (H_A - \partial_t)u(x,t)\rangle$ is an analytic function of A. Therefore, substituting $-iR^M(0)/2$ for A in the formula (8.43) yields the fundamental solution of $H_R + \partial_t$. The proof is complete. $\square$

Combining Lemma 8.12.19 and Lemma 8.12.21 we get

$$\begin{aligned}\sigma\,[K_Q(0,0,t)]^{(n)} &= \left[G_R(0,1)\wedge e^{-F^W(0)}\right]^{(n)} + \mathrm{O}(t)\\ &= (4\pi)^{-\frac{n}{2}}\left[\det{}^{\frac{1}{2}}\Big(\frac{R^M(0)/2}{\sinh(R^M(0)/2)}\Big)\wedge e^{-F^W(0)}\right]^{(n)} + \mathrm{O}(t)\\ &= (4\pi)^{-\frac{n}{2}}\left[\hat{A}(R^M(0))\wedge e^{-F^W(0)}\right]^{(n)} + \mathrm{O}(t).\end{aligned}$$

Combining this with (8.28) and observing that $F^W(0) = F^{E/\$}(0)$, we deduce that

$$\mathrm{Str}\, k_t(0,0) = (2i\pi)^{-\frac{n}{2}}[\hat{A}(R^M(0))\wedge \mathrm{Ch}(F^{E/\$}(0))]^{(n)} + \mathrm{O}(t) \quad \text{as } t\to 0^+. \tag{8.45}$$

This proves (8.23) and completes the proofs of the local index theorem and local index formula of Atiyah-Singer.

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