Noncommutative Geometry
Chapter 9:
Spectral Triples and Dirac Operators

Sichuan University, Spring 2025

What is a Noncommutative Manifold?

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  - $(D + i)^{-1}$ is a *compact* operator.

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Proposition

The following is a spectral triple,

$$\left(C^\infty(M), L^2(M, \Lambda^* T^* M), d + d^* \right),$$

with $L^2(M, \Lambda^* T^* M) = L^2(M, \Lambda^{\text{even}} T^* M) \oplus L^2(M, \Lambda^{\text{odd}} T^* M)$.

Definition

The *Fredholm index* of the operator $d + d^*$ is

$$\text{ind}(d + d^*) := \dim\ker\left[(d + d^*)_{|\Lambda^{\text{even}}}\right] - \dim\ker\left[(d + d^*)_{|\Lambda^{\text{odd}}}\right].$$

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Definition (Euler Characteristic $\chi(M)$)

$$\chi(M) := \sum_{k=0}^{n} (-1)^k \dim H^k(M),$$

where $H^k(M)$ is the de Rham cohomology of M.

The Chern-Gauss-Bonnet Theorem

Theorem (Chern-Gauss-Bonnet)

$$\chi(M) = \mathrm{ind}(d + d^*) = \int_M \mathrm{Pf}\left(R^M\right),$$

where $\mathrm{Pf}\left(R^M\right)$ *is the Pfaffian form of the curvature* R^M *of* M*.*

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Definition (Hodge Operator)

The operator $\star : \Lambda^k T^*M \to \Lambda^{n-k} T^*M$ is defined by

$$\star\alpha \wedge \beta = \langle \alpha, \beta \rangle \, \mathrm{Vol}_g(x) \quad \forall \alpha, \beta \in \Lambda^k T_x^* M,$$

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Remark

As $\star^2 = 1$, there is a splitting

$$\Lambda^* T^* M = \Lambda^+ \oplus \Lambda^-, \quad \text{with } \Lambda^\pm := \{\alpha;\ \star\alpha = \pm\alpha\}.$$

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The following is a spectral triple,

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Definition (Signature $\sigma(M)$)

If $n = 4p$, then $\sigma(M)$ of M is the signature of the bilinear form,

$$H^{\frac{n}{2}}(M) \times H^{\frac{n}{2}}(M) \ni (\alpha, \beta) \to \int_M \alpha \wedge \beta.$$

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Theorem (Hirzebruch)

$$\sigma(M) = \operatorname{ind}(d - \star d \star) \quad \textit{if } n = 4p,$$
$$= 2^{\frac{n}{2}} \int_M L\left(R^M\right),$$

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where $L\left(R^M\right) := \det^{\frac{1}{2}}\left[\frac{R^M/2}{\tanh\left(R^M/2\right)}\right]$ *is called the* L*-form of the curvature* R^M.

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- $\overline{\partial} : C^\infty(M, \Lambda^{0,q} T^* M) \to C^\infty(M, \Lambda^{0,q+1} T^* M)$ is the Dolbeault differential with adjoint $\overline{\partial}^*$.

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The following is a spectral triple,

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with $L^2\left(M, \Lambda^{0,*}T^*M\right) = L^2\left(M, \Lambda^{0,\text{even}}T^*M\right) \oplus L^2\left(M, \Lambda^{0,\text{odd}}T^*M\right)$.

The Hirzebruch-Riemann-Roch Theorem

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The *Fredholm index* of the operator $\overline{\partial} + \overline{\partial}^*$ is

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Definition (Holomorphic Euler Characteristic)

$$\chi(M) := \sum_{q=0}^{n}(-1)^q \dim H^{0,q}(M),$$

where $H^{0,q}(M)$ is the Dolbeault cohomology of M.

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where $\mathrm{Td}\left(R^{1,0}\right) := \det\left[\frac{R^{1,0}}{e^{R^{1,0}}-1}\right]$ *is called the Todd form of the holomorphic curvature* $R^{1,0}$ *of* M.

Fact

On $\mathbb{R}^n$ the square root $\sqrt{\Delta}$ is a ΨDO, but not a differential operator.

The Dirac Operator

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Dirac's Idea

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Dirac's Idea

Seek for a square root of Δ as a differential operator with *matrix* coefficients,

$$\not{D} = \sum c^j \partial_j .$$

Definition

The Clifford algebra of $\mathbb{R}^n$ is the $\mathbb{C}$-algebra $\mathrm{Cl}(\mathbb{R}^n)$ generated by the canonical basis vectors $e^1, \cdots, e^n$ of $\mathbb{R}^n$ with relations,

$$e^i e^j + e^j e^i = -2\delta^{ij}.$$

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Remark

Any Euclidean space $(V, \langle \cdot, \cdot \rangle)$ defines a Clifford algebra.

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Remark

This is not an isomorphism of algebras, e.g., for all $\xi, \eta \in \mathbb{R}^n$, we have $c^{-1}(c(\xi)c(\eta)) = \xi \wedge \eta - \langle \xi, \eta \rangle$.

Corollary

There is a $\mathbb{Z}_2$-grading,

$$\mathrm{Cl}(\mathbb{R}^n) = \mathrm{Cl}^+(\mathbb{R}^n) \oplus \mathrm{Cl}^-(\mathbb{R}^n), \quad \mathrm{Cl}^{\pm}(\mathbb{R}^n) := c(\Lambda_{\mathbb{C}}^{\mathrm{even}/odd}\mathbb{R}^n).$$

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1. $\mathrm{Cl}(\mathbb{R}^n)$ *has a unique irreducible representation,*

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3. *If* n *is even, the spinor representation gives rise to an isomorphism,*
$$\mathrm{Cl}(\mathbb{R}^n) \simeq \mathrm{End}\,\$_n.$$

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The spinor representation splits into the half-spin representations,

$$\rho_\pm : \mathrm{Spin}(n) \longrightarrow \mathrm{End}(\not{S}_n^\pm).$$

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where $\mathrm{Cl}(T_x^* M)$ is the Clifford algebra of $(T_x^* M, g^{-1})$.

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- Here $\mathrm{Cl}^{+}(M)$ is a sub-bundle of algebras of $\mathrm{Cl}(M)$.

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3. *The Levi-Civita connection lifts to a connection* $\nabla^{\$}$ *on* $\$$ *preserving its* $\mathbb{Z}_2$*-grading and Hermitian metric.*

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The following is a spectral triple,

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with $L^2(M, \not{S}) = L^2(M, \not{S}^+) \oplus L^2(M, \not{S}^-)$.