Noncommutative Geometry
Chapter 8:
Connes' Trace Theorem
and Lower Dimensional Volumes

Sichuan University, Spring 2025

Additional References

- Slides of my 2022 online course.

Definition (Weak Schatten Classes $\mathcal{L}_{p,\infty}$)

Let $p > 0$.

1. The weak Schatten class $\mathcal{L}_{p,\infty}$ consists of all $T \in \mathcal{L}(\mathcal{H})$ such that
$$\mu_j(T) = \mathrm{O}\left(j^{-\frac{1}{p}}\right) \qquad \text{as } j \to \infty.$$

Reminder: Weak Schatten Classes

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$$\mu_j(T) = \mathrm{O}\left(j^{-\frac{1}{p}}\right) \qquad \text{as } j \to \infty.$$
2. For $T \in \mathcal{L}(\mathcal{H})$, we set
$$\|T\|_{p,\infty} := \sup_{j\geq 0}(n+1)^{\frac{1}{p}}\mu_j(T).$$

Reminder: Weak Schatten Classes

Proposition

1. $\mathcal{L}_{p,\infty}$ *is a two-sided ideal of* $\mathcal{L}(\mathcal{H})$.
2. $\|\cdot\|_{p,\infty}$ *is a quasi-norm which respect to which* $\mathcal{L}_{p,\infty}$ *is a quasi-Banach ideal.*

Reminder: Weak Schatten Classes

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1. *$\mathcal{L}_{p,\infty}$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$.*
2. *$\|\cdot\|_{p,\infty}$ is a quasi-norm which respect to which $\mathcal{L}_{p,\infty}$ is a quasi-Banach ideal.*

Remark

1. If $p > 1$, then $\|\cdot\|_{p,\infty}$ is equivalent to the norm,

$$\|T\|'_{p,\infty} = \sup_{N\geq 1}\left\{N^{-1+\frac{1}{p}}\sum_{j<N}\mu_j(T)\right\}, \qquad T \in \mathcal{L}_{p,\infty}.$$

2. In this case, $\mathcal{L}_{p,\infty}$ is a Banach ideal (w.r.t. that norm).

Reminder: Weak Schatten Classes

Notation

$(\mathcal{L}_{p,\infty})_0$ is the closure in $\mathcal{L}_{p,\infty}$ of finite-rank operators.

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We have

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Remark

For $0 < p < q$ we have strict inclusions,

$$\mathcal{L}_p \subsetneq (\mathcal{L}_{p,\infty})_0 \subsetneq \mathcal{L}_{p,\infty} \subsetneq \mathcal{L}_q.$$

Reminder: Quantized Calculus

Classical	**Quantum** (Connes)
Complex variable	Operator on Hilbert space $\mathcal{H}$
Real variable	Selfadjoint operator on $\mathcal{H}$
Infinitesimal variable	Compact operator on $\mathcal{H}$
Infinitesimal of order α	Compact operator s.t. $\mu_j(T) = \mathrm{O}(j^{-\alpha})$
Integral $\int f(x)dx$	NC integral $⨍ T$

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Remarks

- $\mu_0(T) \geq \mu_1(T) \geq \cdots$ are the singular values of T.
- T is infinitesimal of order α iff $T \in \mathcal{L}_{p,\infty}$, with $p = \alpha^{-1}$.

Reminder: NC Integral

Ansatz for a NC Integral (Connes)

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2. It should take non-negative values on positive operators.
3. It vanishes on infinitesimals of order > 1.
4. It vanishes on the commutator space,

$$\mathrm{Com}(\mathcal{L}_{1,\infty}) := \mathrm{Span}\left\{[A,T];\ A \in \mathcal{L}(\mathcal{H}),\ T \in \mathcal{L}_{1,\infty}\right\}.$$

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$$\mathrm{Com}(\mathcal{L}_{1,\infty}) := \mathrm{Span}\left\{[A,T];\ A \in \mathcal{L}(\mathcal{H}),\ T \in \mathcal{L}_{1,\infty}\right\}.$$

In other words, it should be a positive trace on $\mathcal{L}_{1,\infty}$ vanishing on infinitesimals of order > 1.

Reminder: NC Integral

Definition

If T is a compact operator, then by an eigenvalue sequence $\lambda(T) = \{\lambda_j(T)\}_{j\geq 0}$ it is meant any sequence such that:

- $\lambda_j(T)$ is an eigenvalue of T and is repeated according to (algebraic) multiplicity.
- $|\lambda_0(T)| \geq |\lambda_1(T)| \geq \cdots$.

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Facts

1. If $T \in \mathcal{L}_{1,\infty}$, and $\{\lambda_j(T)\}_{j\geq 0}$ is any eigenvalue sequence, then
$$\frac{1}{\log N} \sum_{j<N} \lambda_j(T) = \mathrm{O}(1).$$

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$$\frac{1}{\log N}\sum_{j<N}\lambda_j(T) = \mathrm{O}(1).$$
2. If $T \in (\mathcal{L}_{1,\infty})_0$, then we further have
$$\frac{1}{\log N}\sum_{j<N}\lambda_j(T) \longrightarrow 0.$$

Reminder: NC Integral

Notation

- $\ell_\infty = C^*$-algebra of bounded complex-valued sequences.
- $\mathfrak{c}_0 =$ closed ideal of sequences converging to 0.

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Definition

An extended limit is any positive linear map $\lim_\omega : \ell_\infty \to \mathbb{C}$ s.t.:

(i) $\lim_\omega 1 = 1$.

(ii) $\lim_\omega a_j = 0$ if $(a_j) \in \mathfrak{c}_0$.

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(ii) $\lim_\omega a_j = 0$ if $(a_j) \in \mathfrak{c}_0$.

Remark

If $(a_j) \in \ell_\infty$, then

$$\big(\lim a_j = L\big) \Longleftrightarrow \big(\lim_\omega a_j = L \quad \forall \lim_\omega\big).$$

Reminder: NC Integral

Definition

If $\lim_\omega$ is an extended limit, then $\mathrm{Tr}_\omega : \mathcal{L}_{1,\infty} \to \mathbb{C}$ is given by

$$\mathrm{Tr}_\omega(T) := \lim_\omega \left\{ \frac{1}{\log N} \sum_{j<N} \lambda_j(T) \right\}, \qquad T \in \mathcal{L}_{1,\infty}.$$

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Proposition (Dixmier)

1. Tr_ω *is a positive linear trace on* $\mathcal{L}_{1,\infty}$.
2. *It is annihilated by* $(\mathcal{L}_{1,\infty})_0$, *and hence it vanishes on infinitesimals of order* > 1.

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Definition

Tr_ω is called the Dixmier trace associated with $\lim_\omega$.

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Definition (Connes)

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2. We denote by $\mathcal{M}$ the class of measurable operators.
3. If $T \in \mathcal{M}$, we define its NC integral by

$$\fint T := \mathrm{Tr}_\omega(A),$$

where Tr_ω is any Dixmier trace.

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where Tr_ω is any Dixmier trace.

Proposition (Connes, Lord-Sukochev-Zanin)

Let $T \in \mathcal{L}_{1,\infty}$. Then TFAE:

1. *T is measurable and $⨍ T = L$.*
2. *We have*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \lambda_j(T) = L.$$

Reminder: NC Integral

Proposition

1. *$\mathcal{M}$ is a closed subspace of $\mathcal{L}_{1,\infty}$ that contains $\mathrm{Com}(\mathcal{L}_{1,\infty})$ and $(\mathcal{L}_{1,\infty})_0$.*

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Proposition

1. $\mathcal{M}$ *is a closed subspace of* $\mathcal{L}_{1,\infty}$ *that contains* $\mathrm{Com}(\mathcal{L}_{1,\infty})$ *and* $(\mathcal{L}_{1,\infty})_0$.
2. $\varphi : \mathcal{M} \to \mathbb{C}$ *is a positive linear functional that vanishes on* $\mathrm{Com}(\mathcal{L}_{1,\infty})$ *and* $(\mathcal{L}_{1,\infty})_0$.

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2. *$⨍ : \mathcal{M} \to \mathbb{C}$ is a positive linear functional that vanishes on $\mathrm{Com}(\mathcal{L}_{1,\infty})$ and $(\mathcal{L}_{1,\infty})_0$.*
3. *In particular, this is a positive trace that annihilates infinitesimals of order > 1.*

Strong Measurability

Definition

A trace $\varphi : \mathcal{L}_{1,\infty} \to \mathbb{C}$ is called normalized if

$$\big(T \geq 0 \text{ and } \lambda_j(T) = (j+1)^{-1}\big) \implies \varphi(T) = 1.$$

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Definition

An operator $T \in \mathcal{L}_{1,\infty}$ is called strongly measurable (or positively measurable) if $\varphi(T)$ takes the same value as φ ranges over all normalized positive traces.

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2. *It contains $\mathrm{Com}(\mathcal{L}_{1,\infty})$ and $(\mathcal{L}_{1,\infty})_0$. In particular, it contains all infinitesimals of order > 1.*
3. *It does not depend on the inner product of $\mathcal{L}(\mathcal{H})$.*

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Proposition

Let $T \in \mathcal{L}_{1,\infty}$, $T \geq 0$, be such that

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Then T is strongly measurable and $⨍ T = L$.

Setup

- $M =$ compact manifold with positive smooth density μ.

Schatten Properties of ΨDOs

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- $p_m(x, \xi) =$ principal symbol of P.

Remark

Positive-ellipticity means that $p_m(x, \xi) > 0$ & P is selfadjoint with non-negative spectrum.

Schatten Properties of ΨDOs

Reminder

- The spectrum of P can be arranged as a sequence,

$$0 \leq \lambda_0(P) \leq \lambda_1(P) \leq \lambda_2(P) \leq \cdots ,$$

where each eigenvalue is repeated according to multiplicity.

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- The spectrum of P can be arranged as a sequence,
$$0 \leq \lambda_0(P) \leq \lambda_1(P) \leq \lambda_2(P) \leq \cdots ,$$
where each eigenvalue is repeated according to multiplicity.
- By Weyl's law, as $j \to \infty$ we have
$$\lambda_j(P) \sim \left(\frac{j}{\gamma(P)}\right)^{\frac{m}{n}},$$
where
$$\gamma(P) = \frac{(2\pi)^{-n}}{n} \iint_{S^*M} p_m(x,\xi)^{-\frac{n}{m}} dx d\xi = \frac{1}{n} \operatorname{Res}\left[P^{-\frac{n}{m}}\right].$$

Schatten Properties of ΨDOs

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That is, $P^{-1} \in \mathcal{L}_{m^{-1}n,\infty}$.
- In particular, if g is a Riemannian metric, then
$$(1+\Delta_g)^{-\frac{m}{2}} \in \mathcal{L}_{m^{-1}n,\infty} \qquad \forall m > 0.$$

Proposition

If $P \in \Psi^{-m}(M)$, $m > 0$, then P is in the weak Schatten class $\mathcal{L}_{nm^{-1},\infty}$.

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In particular, for $m = n$ we obtain:

Corollary

If $P \in \Psi^{-n}(M)$, then $P \in \mathcal{L}_{1,\infty}$, i.e., P is an infinitesimal of order 1.

Proof of the Proposition.

Let $m > 0$ and $P \in \Psi^{-m}(M)$.

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Let $m > 0$ and $P \in \Psi^{-m}(M)$.

- Pick a Riemannian metric g on M, and write

$$P = P(1 + \Delta_g)^{\frac{m}{2}} \cdot (1 + \Delta_g)^{-\frac{m}{2}}$$

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- Pick a Riemannian metric g on M, and write
$$P=P\,(1+\Delta_g)^{\frac{m}{2}}\cdot(1+\Delta_g)^{-\frac{m}{2}}$$
- We already know that $(1+\Delta_g)^{-m/2}\in\mathcal{L}_{nm^{-1},\infty}$.

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- We already know that $(1+\Delta_g)^{-m/2}\in\mathcal{L}_{nm^{-1},\infty}$.
- As $P\in\Psi^{-m}(M)$ and $(1+\Delta_g)^{m/2}\in\Psi^{-m}(M)$, we see that $A(1+\Delta_g)^{m/2}\in\Psi^{0}(M)$.

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- This ensures that $P(1+\Delta_g)^{m/2}$ is bounded, i.e., it is contained in $\mathcal{L}(L^2(M))$.

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- This ensures that $P(1+\Delta_g)^{m/2}$ is bounded, i.e., it is contained in $\mathcal{L}(L^2(M))$.
- As $\mathcal{L}_{nm^{-1},\infty}$ is an ideal of $\mathcal{L}(L^2(M))$, it follows that
$$P = P(1+\Delta_g)^{\frac{m}{2}} \cdot (1+\Delta_g)^{-\frac{m}{2}} \in \mathcal{L}_{nm^{-1},\infty}.$$

The proof is complete. □

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Let $P \in \Psi^n(M)$ be positive-elliptic with $\ker P = \{0\}$ (e.g., $P = (1 + \Delta_g)^{n/2}$).

Connes' Trace Theorem

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- In this case the Weyl's law gives

$$\lambda_j(P^{-1}) = \lambda_j(P)^{-1} \sim \left(\frac{j}{\gamma(P)}\right)^{-1},$$

where $\gamma(P) = \frac{1}{n}\operatorname{Res}(P^{-1})$.

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  where $\gamma(P) = \frac{1}{n}\operatorname{Res}(P^{-1})$.

- It then follows that P^{-1} is strongly measurable, and we have

$$⨍ P^{-1} = \gamma(P) = \frac{1}{n}\operatorname{Res}(P).$$

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Theorem (Connes' Trace Theorem)

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Remark

1. Connes (CMP '88) established measurability and derived the trace formula.

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Remark

1. Connes (CMP '88) established measurability and derived the trace formula.
2. Lord-Potapov-Sukochev (Adv. Math. '13) obtained strong measurability.

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Consequence

- The NC integral makes sense for ΨDOs of order $\leq -n$

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Remark

Positive-ellipticity means that $p_m(x,\xi) > 0$ and $P^* = P \geq 0$.

Schatten Properties of ΨDOs

Reminder

- The spectrum of P can be arranged as a sequence,

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- By Weyl's law, as $j \rightarrow \infty$ we have
$$\lambda_j(P) \sim \left(\frac{j}{\gamma(P)}\right)^{\frac{m}{n}},$$
where
$$\gamma(P) = \frac{(2\pi)^{-n}}{n} \iint_{S^*M} p_m(x,\xi)^{-\frac{n}{m}} dx d\xi = \frac{1}{n} \operatorname{Res}\left[P^{-\frac{n}{m}}\right].$$

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That is, $P^{-1} \in \mathcal{L}_{m^{-1}n,\infty}$.
- In particular, if g is a Riemannian metric, then
$$(1+\Delta_g)^{-\frac{m}{2}} \in \mathcal{L}_{m^{-1}n,\infty} \qquad \forall m > 0.$$

Proposition

If $P \in \Psi^{-m}(M)$, $m > 0$, then P is in the weak Schatten class $\mathcal{L}_{nm^{-1},\infty}$. That is, P is an infinitesimal of order mn^{-1}.

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In particular, for $m = n$ we obtain:

Corollary

If $P \in \Psi^{-n}(M)$, then $P \in \mathcal{L}_{1,\infty}$, i.e., P is an infinitesimal of order 1.

Proof of the Proposition.

Let $m > 0$ and $P \in \Psi^{-m}(M)$.

- Pick a Riemannian metric g on M, and write

$$P = P\left(1+\Delta_g\right)^{\frac{m}{2}} \cdot \left(1+\Delta_g\right)^{-\frac{m}{2}}$$

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- As $\mathcal{L}_{nm^{-1},\infty}$ is an ideal of $\mathcal{L}(L^2(M))$, it follows that
$$P = P\,(1+\Delta_g)^{\frac{m}{2}} \cdot (1+\Delta_g)^{-\frac{m}{2}} \in \mathcal{L}_{nm^{-1},\infty}.$$

The proof is complete. □

Connes' Trace Theorem

Facts

Let $P \in \Psi^n(M)$ be positive-elliptic with $\ker P = \{0\}$ (e.g., $P = (1+\Delta_g)^{n/2}$).

- In this case the Weyl's law gives

$$\lambda_j(P^{-1}) = \lambda_j(P)^{-1} \sim \left(\frac{j}{\gamma(P)}\right)^{-1},$$

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- It then follows that P^{-1} is strongly measurable, and we have
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- Connes' integration formula shows that the NC integral recaptures the Riemannian volume density. Namely,

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- Thus, ds is the n-th root of the volume element.

Lower Dimensional Volumes

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The NC length element of (M^n, g) is the operator,

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- The NC integral has been extended to all ΨDOs.
- This enables us to define k-dimensional volumes for all $k = 1, \ldots, n-1$.

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Definition

For $k = 1, \ldots, n$, the k-th dimensional volume of (M^n, g) is

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In particular, the length and area of (M^n, g) are

$$\mathrm{Length}_g(M) := \fint ds = c(n)^{-\frac{1}{n}} \fint \Delta_g^{-\frac{1}{2}},$$

$$\mathrm{Area}_g(M) := \fint ds^2 = c(n)^{-\frac{2}{n}} \fint \Delta_g^{-1}.$$

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where $\kappa_g(x)$ is the scalar curvature of (M,g).

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3. *In general, we have*
$$\mathrm{Vol}_g^{(n-k)}(M)=c(n,k)\int_M I_g^{(k)}(x)\nu(g)(x),$$
where $I_g^{(k)}(x)$ *is a universal polynomial in the curvature tensor and its covariant derivatives.*

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- However, the formulas in the previous slide provide purely differential-geometric expressions for the k-th dimensional volumes.

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- We have

$$\begin{aligned}\int_M \kappa_g \nu(g) &= c(n,2)^{-1} ⨍ \Delta_g^{-n+2} \\ &= \frac{1}{n} c(n,2)^{-1} \operatorname{Res}\left(\Delta_g^{-n+2}\right) \\ &= \frac{2}{n} c(n,2)^{-1} \operatorname{Res}_{s=\frac{n}{2}-1} \operatorname{Tr}\left[\Delta_g^{-s}\right].\end{aligned}$$

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- This yields a spectral theoretic interpretation of the Einstein-Hilbert action.
- This an important ingredient in the spectral action formalism of Connes-Chamseddine.