Noncommutative Geometry
Chapter 7:
Noncommutative Residue,
and Zeta Functions of Elliptic Operators

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Noncommutative Residue

Additional References

- Slides of my 2022 online course.

Logarithmic Singularity of ΨDO Kernels

Setup

- $U \subseteq \mathbb{R}^n$ open set.
- $P \in \Psi^m(U)$, $m \in \mathbb{Z}$, with symbol $p(x,\xi) \sim \sum_{j\geq 0} p_{m-j}(x,\xi)$.
- $k_P(x,y) =$ Schwartz kernel of P.

Reminder

1. $k_P(x,y)$ is C^∞ for $x \neq y$.
2. If $m < -n$, then $k_P(x,y)$ is continuous on $U \times U$.

Question

If $m \geq -n$, then what is the behaviour of $k_P(x,y)$ near $x = y$?

Proposition

Let $P \in \Psi^m(U)$, $m \in \mathbb{Z}$, $m \geq -n$. Near $x = y$ its Schwartz kernel $k_P(x, y)$ has a behaviour of the form,

$$k_P(x,y) = \sum_{1 \leq k \leq m+n} a_k\Big(x, \frac{x-y}{|x-y|}\Big)|x-y|^{-k} - c_P(x)\log|x-y| + \mathrm{O}(1).$$

Here $a_k(x,\theta) \in C^\infty(U \times \mathbb{S}^{n-1})$, and $c_P(x) \in C^\infty(U)$ is given by

$$c_P(x) = (2\pi)^{-n}\int_{\mathbb{S}^{n-1}} p_{-n}(x,\xi)d\xi,$$

where $p_{-n}(x,\xi)$ is the symbol of degree $-n$ of P.

Remark

$a_k(x,\theta)$ only depends on the homogeneous symbol $p_{m-k}(x,\xi)$ of degree $m-k$ of P.

Logarithmic Singularity of ΨDO Kernels

Setup

- $U_1 \subseteq \mathbb{R}^n$ open set.
- $\phi : U_1 \to U$ is a C^∞-diffeomorphism.

Fact

$\phi^* P := (\phi^{-1})_* \in \Psi^m(U_1)$.

Proposition

We have

$$c_{\phi^* P}(x) = |\phi'(x)| c_P(\phi(x)) \qquad \forall x \in U_1.$$

Remark

In other words the logarithmic singularities $c_P(x)$ satisfy the transformation law of densities.

Setup

M is a smooth manifold of dimension n.

Reminder

If $P \in \Psi^m(M)$, then, $\kappa_*(P_{|U}) \in \Psi^m(V)$ for every chart $\kappa : U \to V$.

Proposition

Let $P \in \Psi^m(M)$, $m \in \mathbb{Z}$, $m \geq -n$. There is a unique smooth density c_P on M such that, for every chart $\kappa : U \to V$, we have

$$(c_P)_\kappa(x) = c_{\kappa_*(P_{|U})}(x) = (2\pi)^{-n} \int_{\mathbb{S}^{n-1}} p^\kappa_{-n}(x,\xi)d\xi,$$

where $p^\kappa_{-n}(x,\xi)$ is the symbol of degree $-n$ of $\kappa_*(P_{|U})$.

Logarithmic Singularity of ΨDO Kernels

Remark

If P is a differential operator on M, then $c_P(x) = 0$ everywhere, since P has no symbols of negative degree.

Remark

- In case $P \in \Psi^{-n}(M)$, i.e., P has order $m = -n$, then its symbol of degree $-n$ agrees with its principal symbol.
- It thus makes sense as a function $p_{-n}(x, \xi) \in C^\infty(T^*M \setminus 0)$.
- It then can be shown that, if $u \in C_c(M)$, then

$$\int_M u(x) c_P(x) = (2\pi)^{-n} \iint_{S^*M} f(x) p_{-n}(x, \xi) dx d\xi,$$

where $S^*M = (T^*M \setminus 0)/\mathbb{R}_+^*$ is the cosphere bundle and $dxd\xi$ is its Liouville measure.

Noncommutative Residue

Setup

M is a compact manifold of dimension n.

Definition (Noncommutative Residue)

If $P \in \Psi^m(M)$, $m \in \mathbb{Z}$, then its noncommutative residue is

$$\operatorname{Res}(P) := \int_M c_P(x).$$

Remarks

1. $\operatorname{Res}(P) = 0$ if P is a differential operator or if P has order $< -n$.
2. If P has order $-n$, then

$$\operatorname{Res}(P) = (2\pi)^{-n} \iint_{S^*M} p_{-n}(x,\xi)\,dx d\xi,$$

where $p_{-n}(x,\xi)$ is the principal symbol of P.

Noncommutative Residue

Notation

$\Psi^{\mathbb{Z}}(M) := \bigcup_{m\in\mathbb{Z}} \Psi^m(M)$.

Remark

As M is compact, $\Psi^{\mathbb{Z}}(M)$ is an algebra:

- If $P_1 \in \Psi^{m_1}(M)$ and $P_2 \in \Psi^{m_2}(M)$ with $m_1, m_2 \in \mathbb{Z}$, then $m_1 + m_2 \in \mathbb{Z}$, and so

$$P_1 P_2 \in \Psi^{m_1+m_2}(M) \subseteq \Psi^{\mathbb{Z}}(M).$$

- If $m_2 \geq m_1$, then $\Psi^{m_1}(M) \subseteq \Psi^{m_2}(M)$, and so we have

$$P_1 + P_2 \in \Psi^{m_1}(M) + \Psi^{m_2}(M) \subseteq \Psi^{m_2}(M) \subseteq \Psi^{\mathbb{Z}}(M).$$

Remark

We then can regard the noncommutative residue as a linear functional $\mathrm{Res} : \Psi^{\mathbb{Z}}(M) \to \mathbb{C}$.

Noncommutative Residue

Proposition

1. *If* $P_1 \in \Psi^{m_1}(M)$ *and* $P_2 \in \Psi^{m_2}(M)$ *are such that* $m_1 + m_2 \in \mathbb{Z}$, *then*

$$\mathrm{Res}(P_1 P_2) = \mathrm{Res}(P_2 P_1).$$

2. *In particular, the noncommutative residue* $\mathrm{Res} : \Psi^{\mathbb{Z}}(M) \to \mathbb{C}$ *is a linear trace on the algebra* $\Psi^{\mathbb{Z}}(M)$.

Theorem (Wodzicki, Guillemin)

Every linear trace on $\Psi^{\mathbb{Z}}(M)$ *is a constant multiple of the noncommutative residue.*

Example

Δ_g = Laplacian on M associated with some Riemannian metric g.

- The principal symbol of Δ_g is $p_2(x,\xi) = |\xi|_g^2$, where $|\xi|_g^2 = \sum g^{ij}\xi_i\xi_j$ is the Riemannian metric on T^*M.
- $\Delta_g^{-n/2}$ is in $\Psi^{-n}(M)$, and its principal symbol is $p_{-n}(x,\xi) = |\xi|_g^{-n}$.
- It then can be shown that

$$c_{\Delta_g^{-n/2}}(x) = (2\pi)^{-n}\int_{S_x^*M} |\xi|_g^{-n} d\xi = (2\pi)^{-n}|\mathbb{S}^{n-1}|\nu(g)(x),$$

  where $\nu(g)$ is the Riemannian density.
- Thus, if we set $c(n) := (2\pi)^{-n}|\mathbb{S}^{n-1}|$, then

$$\operatorname{Res}\left(\Delta_g^{-\frac{n}{2}}\right) = c(n)\int_M \nu(g) = c(n)\operatorname{Vol}_g(M).$$

Meromorphic Extension of the Trace

Setup

- M = compact manifold of dimension n.
- μ = positive smooth measure on M.
- $P \in \Psi^m(M)$, $m > 0$, is positive-elliptic.
- $p_m(x, \xi)$ = principal symbol of P.

Remark

The positive-elliptic assumption means that

- $p_m(x, \xi) > 0$ for all $(x, \xi) \in T^*M \setminus 0$.
- $P^* = P$ and $\mathrm{Sp}(P) \subseteq [0, \infty)$.

Meromorphic Extension of the Trace

Reminder

- Each operator P^{-z}, $z \in \mathbb{C}$, is in $\Psi^{-mz}(M)$.
- Its principal symbol is $p_m(x,\xi)^{-z}$.

Facts

Let $A \in \Psi^a(M)$, $a \in \mathbb{R}$.

- The operator AP^{-z} is in $\Psi^{a-mz}(M)$.
- In particular AP^{-z} is trace-class for $a - m\Re z > -n$, i.e., $\Re z > m^{-1}(n+a)$.
- Thus, $\mathrm{Tr}[AP^{-z}]$ is well defined for $\Re z > m^{-1}(n+a)$.

Meromorphic Extension of the Trace

Proposition (Wodzkicki, Guillemin)

Let $A \in \Psi^a(M)$, $a \in \mathbb{R}$, *and set* $\Sigma := \{m^{-1}(n+a-j);\ j \in \mathbb{N}_0\}$.

1. *The function* $z \to \operatorname{Tr}[AP^{-z}]$ *has a meromorphic extension to* $\mathbb{C}$ *with at worst simple pole singularities on* Σ.
2. *If* $\sigma \in \Sigma$, *then*
$$m \operatorname{Res}_{z=\sigma} \operatorname{Tr}\left[AP^{-z}\right] = \operatorname{Res}\left[AP^{-\sigma}\right].$$
3. *In particular, if* $a \in \mathbb{Z}$, $a \geq -n$, *then for* $\sigma = 0$ *we get*
$$m \operatorname{Res}_{z=0} \operatorname{Tr}\left[AP^{-z}\right] = \operatorname{Res}(A).$$

Remark

$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.

Reminder

- $P^{-z} \in \Psi^{-mz}(M)$.
- Thus, P^{-z} is trace-class for $-m\Re z < -n$, i.e., $\Re z > m^{-1}n$.

Definition

The zeta function of P is defined by

$$\zeta_P(z) = \operatorname{Tr}\left[P^{-z}\right], \qquad \Re z > m^{-1}n.$$

Applying the previous result to $A = 1$ and $a = 0$ gives:

Theorem

Set $\Sigma := \{m^{-1}(n-j);\ j \in \mathbb{N}_0\}$.

1. *The zeta function* $\zeta_P(z) = \mathrm{Tr}[P^{-z}]$ *has a meromorphic extension to* $\mathbb{C}$ *with at worst simple pole singularities on* $\Sigma \setminus 0$.
2. *If* $\sigma \in \Sigma \setminus 0$, *then*

$$m\,\mathrm{Res}_{z=\sigma}\,\zeta_P(z) = \mathrm{Res}\left[P^{-\sigma}\right].$$

3. *In particular,* $\zeta_P(z)$ *always has a regular value at* $z = 0$.

Remark

There is no residue at $z = 0$, because for $A = 1$ we get

$$m\,\mathrm{Res}_{z=0}\,\mathrm{Tr}\left[P^{-z}\right] = \mathrm{Res}(1) = 0.$$

Theorem (Ikehara's Tauberian Theorem; see Shubin)

Assume $N(t)$, $t \geq 0$, is a non-decreasing function s.t.

(i) $N(t) = 0$ *near* $t = 0$.

(ii) *The integral* $\zeta(z) := \int t^{-z} dN(t)$ *converges for* $\Re z > \sigma > 0$.

(iii) $\zeta(z)$ *admits a meromorphic extension to some half-plane* $\Re z > \sigma - \epsilon$, $\epsilon > 0$, *with only a simple pole at* $z = \sigma$ *s.t.*

$$\mathrm{Res}_{z=\sigma}\, \zeta(z) = c > 0.$$

Then, we have

$$N(t) \sim \frac{1}{\sigma} c t^{\sigma} \quad \text{as } t \to \infty.$$

Weyl's Law for Elliptic Operators

Definition

The counting function of P is

$$N_P(t) := \#\{j;\ 0 < \lambda_j(P) \leq t\}, \quad t > 0,$$

where $0 \leq \lambda_0(P) \leq \lambda_1(P) \leq \cdots$ are the eigenvalues of P.

Remark

- We have
$$dN_P(t) = \sum_{\lambda_j(P)>0} \delta_{\lambda_j(P)}(t),$$
where $\delta_{\lambda_j(P)}(t)$ is the Dirac measure at $\lambda_j(P)$.
- Thus, for $\Re z > m^{-1}n$ we have
$$\zeta_P(z) = \sum_{\lambda_j(P)>0} \lambda_j(P)^{-z} = \int t^{-z} dN_P(t).$$

Weyl's Law for Elliptic Operators

Theorem (Weyl's Law; 1st Version)

As $t \to \infty$, we have

$$N_P(t) \sim \frac{(2\pi)^{-n}}{n} \left(\iint_{S^*M} p_m(x,\xi)^{-\frac{n}{m}} dx d\xi \right) t^{\frac{n}{m}}.$$

Proof.

- For $\Re s > m^{-1}m$, we have $\int t^{-z} dN_P(t) = \zeta_P(z)$.
- The function $\zeta_P(z)$ is holomorphic for $\Re z > m^{-1}n$.
- It has a meromorphic continuation to the half-plane $\Re z > m^{-1}(n-1)$ with only a simple pole at $z = m^{-1}n$ s.t.

$$\mathrm{Res}_{z=m^{-1}n}\, \zeta_P(z) = \frac{1}{m} \mathrm{Res}\left[P^{-\frac{n}{m}} \right].$$

□

Proof.

- Here $P^{-n/m}$ has order $-n$.
- Its principal symbol is $p_m(x,\xi)^{-\frac{n}{m}} > 0$.
- Thus,

$$\operatorname{Res}\left[P^{-\frac{n}{m}}\right] = (2\pi)^{-n} \iint_{S^*M} p_m(x,\xi)^{-\frac{n}{m}}\, dx d\xi$$

- Therefore, we may apply Ikehara's Tauberian theorem to get

$$\begin{aligned} N_P(t) &= \frac{m}{n} \cdot \frac{1}{m} \operatorname{Res}\left[P^{-\frac{n}{m}}\right] t^{\frac{n}{m}} + \mathrm{o}\left(t^{\frac{n}{m}}\right) \\ &= \frac{(2\pi)^{-n}}{n} \left(\iint_{S^*M} p_m(x,\xi)^{-\frac{n}{m}}\, dx d\xi \right) t^{\frac{n}{m}} + \mathrm{o}\left(t^{\frac{n}{m}}\right). \end{aligned}$$

This gives the result. □

Weyl's Law for Elliptic Operators

Lemma

We always have

$$\limsup_{j\to\infty} j\lambda_j(P)^{-\frac{m}{n}} = \limsup_{t\to\infty} t^{-\frac{n}{m}} N_P(t),$$
$$\liminf_{j\to\infty} j\lambda_j(P)^{-\frac{m}{n}} = \liminf_{t\to\infty} t^{-\frac{n}{m}} N_P(t).$$

Therefore, we obtain:

Theorem (Weyl's Law; 2nd Version)

As $j \to \infty$ *we have*

$$\lambda_j(P) \sim j^{\frac{m}{n}} \left(\frac{(2\pi)^{-n}}{n} \iint_{S^*M} p_m(x,\xi)^{-\frac{n}{m}} dx d\xi \right)^{-\frac{m}{n}}.$$