Noncommutative Residue and Zeta Functions of Elliptic Operators

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Holomorphic Families of ΨDOs

Setup

- $U \subset \mathbb{R}^n$ open set.
- $\Omega \subset \mathbb{C}$ open set

Notation

$$S^{\bullet}(U \times \mathbb{R}^n) := \bigcup_{q \in \mathbb{C}} S^q(U \times \mathbb{R}^n), \quad \Psi^{\bullet}(U) := \bigcup_{q \in \mathbb{C}} \Psi^q(U).$$

Holomorphic Families of ΨDOs

Definition

A family $p(z)(x,\xi)$, $z \in \Omega$, in $S^{\bullet}(U \times \mathbb{R}^n)$ is called holomorphic if:

(i) The order $w(z)$ of $p(z)(x,\xi)$ is a holomorphic function on Ω.

(ii) For every $(x,\xi) \in U \times \mathbb{R}^n$, the function $z \to p(z)(x,\xi)$ is holomorphic on Ω.

(iii) We have an asymptotic expansion,

$$p(z)(x,\xi) \sim \sum p(z)_{w(z)-j}(x,\xi), \quad p(z)_{w(z)-j} \in S_{w(z)-j}(U \times \mathbb{R}^n),$$

whose bounds are uniform on compact sets of Ω.

Remark

The condition (iii) means that for all compacts $L \subset \Omega$ and $K \subset U$, we have

$$\left| D_x^\alpha D_\xi^\beta \left(p(z) - \sum_{j<N} p(z)_{w(z)-j} \right)(x,\xi) \right| \leq C_{LKN\alpha\beta} |\xi|^{\Re w(z)-j}$$

for all $z \in L$, $x \in K$ and $\xi \in \mathbb{R}^n$, $|\xi| \geq 1$.

Holomorphic Families of ΨDOs

Remark

A family $R(z)$, $z \in \Omega$, in $\Psi^{-\infty}(U)$ is holomorphic if it is of the form,

$$R(z)u(x) = \int k_R(z)(x,y)u(y)dy,$$

where $k_R(z)(x,y)$, $z \in \Omega$, is a holomorphic family in $C^\infty(U \times U)$.

Definition

A family $P(z)$, $z \in \Omega$, in $\Psi^\bullet(U)$ is holomorphic if it can be put in the form,

$$P(z) = p(z)(x,D) + R(z),$$

where $p(z)(x,\xi)$ and $R(z)$ are holomorphic families in $S^\bullet(U \times \mathbb{R}^n)$ and $\Psi^{-\infty}(U)$, respectively.

Holomorphic Families of ΨDOs

Definition

A family $P(z)$, $z \in \Omega$, in $\Psi^{\bullet}(U)$ is called properly supported if, for all compacts $L \subset \Omega$ and $K \subset U$, there are compact sets $K_1 \subset U$ and $K_2 \subset U$ such that

$$\operatorname{supp} u \subset K \implies \operatorname{supp} P(z)u \subset K_1 \quad \forall z \in L.$$

$$K_2 \cap \operatorname{supp} u = \emptyset \implies K \cap \operatorname{supp} P(z)u = \emptyset \quad \forall z \in L.$$

Holomorphic Families of ΨDOs

Proposition

Assume that

- $P_1(z)$, $z \in \Omega$, *is a holom. family in* $\Psi^{\bullet}(U)$ *of order* $w_1(z)$.
- $P_2(z)$, $z \in \Omega$, *is a holom. family in* $\Psi^{\bullet}(U)$ *of order* $w_2(z)$.
- *One those families is properly supported.*

Then $P_1(z)P_2(z)$, $z \in \Omega$, *is a holomorphic family in* $\Psi^{\bullet}(U)$ *of order* $w_1(z) + w_2(z)$.

Proposition

Suppose that $\phi : U \rightarrow V \subset \mathbb{R}^n$ *is a diffeomorphism. If* $P(z)$, $z \in \Omega$, *is a holomorphic family in* $\Psi^{\bullet}(U)$ *of order* $w(z)$, *then* $\phi_* P(z)$, $z \in \Omega$, *a holomorphic family in* $\Psi^{\bullet}(V)$ *of order* $w(z)$.

Holomorphic Families of ΨDOs

Setup

- M^n = smooth manifold.

Definition

A family $P(z)$, $z \in \Omega$, in $\Psi^\bullet(M)$ is holomorphic if

- The order $w(z)$ of $P(z)$ is a holomorphic function on Ω.
- For every chart $\kappa : U \to V$, the family $\kappa_*(P(z)_{|U})$, $z \in \Omega$, is a holomorphic family in $\Psi^\bullet(V)$ of order $w(z)$.

Proposition

Assume that

- $P_1(z)$, $z \in \Omega$, *is a holom. family in* $\Psi^\bullet(M)$ *of order* $w_1(z)$.
- $P_2(z)$, $z \in \Omega$, *is a holom. family in* $\Psi^\bullet(M)$ *of order* $w_2(z)$.
- *One of those families is properly supported.*

Then $P_1(z)P_2(z)$, $z \in \Omega$, *is a holomorphic family in* $\Psi^\bullet(M)$ *of order* $w_1(z) + w_2(z)$.

In particular, if M is compact, then we get:

Proposition

Assume that M is compact, and

- $P_1(z)$, $z \in \Omega$, *is a holom. family in* $\Psi^{\bullet}(M,)$ *of order* $w_1(z)$.
- $P_2(z)$, $z \in \Omega$, *is a holom. family in* $\Psi^{\bullet}(M)$ *of order* $w_2(z)$.

Then $P_1(z)P_2(z)$, $z \in \Omega$, *is a holomorphic family in* $\Psi^{\bullet}(M)$ *of order* $w_1(z) + w_2(z)$.

Holomorphic Families of ΨDOs

Definition (Guillemin)

A holomorphic gauging for $P \in \Psi^m(M)$ is any holomorphic family $P(z)$, $z \in \mathbb{C}$, in $\Psi^\bullet(M)$ such that

$$P(0) = P, \qquad \operatorname{ord} P(z) = z + m.$$

Proposition

Every $P \in \Psi^m(M)$ admits a holomorphic gauging.

Proof.

We first show the result for $P = 1$ on an open set $V \subset \mathbb{R}^n$.

- Define
$$a(z)(x,\xi) = (1-\chi(\xi))|\xi|^z + \chi(\xi), \quad z \in \mathbb{C}, \ (x,\xi) \in V \times \mathbb{R}^n.$$
- $a(z)(x,\xi)$ is a holomorphic family in $S^\bullet(V \times \mathbb{R}^n)$ of order z with $a(z)(x,\xi) \sim |\xi|^z$.
- For $z = 0$ we have
$$a(0)(x,\xi) = (1-\chi(\xi)) + \chi(\xi) = 1.$$
- Thus, $a(z)(x,D)$, $z \in \mathbb{C}$, is a holomorphic family in $\Psi^\bullet(V)$ of order z with $a(0)(x,D)$, i.e., this is a holom. gauging for 1.

□

Holomorphic Families of ΨDOs

Proof.

- Let $\kappa : U \to V$ be a chart. Then $A(z) := \kappa^*(a(z)(x, D))$ is a holomorphic gauging for 1 on U.
- Let (φ_i) be a partition of unity subordinate to a covering (U_i) by domains of charts.
- For each i let $A_i(z)$, $z \in \mathbb{C}$, be a holomorphic gauging for 1 on U_i. Define

$$A(z) = \sum \varphi_i A(z) \psi_i, \quad z \in C,$$

  where $\psi_i \in C^\infty(U_i)$ is such that $\psi_i = 1$ near $\operatorname{supp} \varphi_i$.
- Then $A(z)$ is a holomorphic gauging for 1 on M.
- By construction the family $A(z)$, $z \in \mathbb{C}$, is properly supported.

□

Proof.

- Let $P \in \Psi^m(M)$, and set
$$P(z) = A(z)P, \qquad z \in \mathbb{C}.$$
- We regard P as a constant holomorphic family of ΨDOs of order m.
- The product of a holomorphic family of ΨDOs with a properly supported family of ΨDOs is a holomorphic family of ΨDOs.
- Therefore, $P(z)$, $z \in \mathbb{C}$, is a holomorphic family of ΨDOs of order $z+m$.
- As $P(0) = A(0)P = 1 \cdot P = P$, we get a holomorphic gauging for P.

□

Complex Powers of Elliptic ΨDOs

Assumptions

- M is compact with smooth measure μ.
- $P \in \Psi^m(M)$, $m > 0$, is elliptic, selfadjoint and ≥ 0.

Reminder

- $P^z \in \Psi^{mz}(M)$ for all $z \in \mathbb{C}$.
- $P^z|_{z=0} = 1 - \Pi_0(P)$, where $\Pi_0(P)$ is the orthogonal projection onto $\ker P$.
- $P^{z_1} P^{z_2} = P^{z_1+z_2}$, $z_i \in \mathbb{C}$.
- $\Pi_0(P)$ is a smoothing operator.

Theorem (Seeely, Shubin)

P^z, $z \in \mathbb{C}$, is a holomorphic family in $\Psi^\bullet(M)$ of order mz.

Corollary

If $Q \in \Psi^m(M)$, $\Re m > 0$, is elliptic, then $|Q|^z$, $z \in \mathbb{C}$, is a holomorphic family in $\Psi^\bullet(M)$ of order $z\Re m$.

Corollary

Let $Q \in \Psi^q(M)$, $q \in \mathbb{C}$, and set

$$Q(z) = Q\left(P^{\frac{z}{m}} + \Pi_0(P)\right), \qquad z \in \mathbb{C}.$$

Then $Q(z)$ is a holomorphic gauging for Q.

Proof.

- We have $Q(z) = QA(z)$, with $A(z) = P^{z/m} + \Pi_0(P)$.
- Here $P^{z/m}$, $z \in \mathbb{C}$, is a holomorphic family in $\Psi^\bullet(M)$ of order $m(z/m) = z$.
- $\Pi_0(P)$ is a smoothing operator, and so it can be regarded as a constant (holomorphic) family in $\Psi^{-\infty}(M) \subset \Psi^\bullet(M)$.
- Thus, $A(z) = P^{z/m} + \Pi_0(P)$, $z \in \mathbb{C}$, is a holomorphic family in $\Psi^\bullet(M)$ of order z.
- We have

$$A(0) = P^{z/m}\big|_{z=0} + \Pi_0(P) = \big(1 - \Pi_0(P)\big) + \Pi_0(P) = 1.$$

- Thus, $A(z)$, $z \in \mathbb{C}$, is a holomorphic gauging for 1.
- It then follows that $Q(z) = QA(z)$ is a holomorphic gauging for Q.

□

Analytic Extension of the Trace

Definition

$S^q(\mathbb{R}^n)$, $q \in \mathbb{C}$, consists of symbols $p(\xi) \in C^\infty(\mathbb{R}^n)$ such that

$$p(\xi) \sim \sum p_{q-j}(\xi), \quad p(\lambda\xi) = \lambda^{q-j} p(\xi).$$

Remarks

1. In other words $S^q(\mathbb{R}^n)$ consists of classical symbols of order q that do not depend on x.
2. If $p(x,\xi) \in S^q(U \times \mathbb{R}^n)$, then $p(x,\cdot) \in S^q(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$.

Analytic Extension of the Trace

Notation

For any subset $A \subset \mathbb{C}$ we set

$$S^A(\mathbb{R}^n) = \bigcup_{q \in A} S^q(\mathbb{R}^n), \qquad \Psi^A(M) = \bigcup_{q \in A} \Psi^q(M).$$

Remark

For $A = \{\Re z < a\}$ we use instead the notation $S^{<a}(\mathbb{R}^n)$ and $\Psi^{<a}(M)$.

Remark

We use similar notations for symbols on $U \times \mathbb{R}^n$.

Analytic Extension of the Trace

Definition

Assume $A \subset \mathbb{C}$ is open and X is a topological vector space. We say that a linear map $\Phi : S^A(\mathbb{R}^n) \to X$ is holomorphic, if $\Phi(p(z))$, $z \in \Omega$, is a holomorphic family in X for every holomorphic family $p(z)$, $z \in \Omega$, in $S^A(\mathbb{R}^n)$.

Remark

We similarly define holomorphic linear maps from $\Psi^A(M)$ or $S^A(U \times \mathbb{R}^n)$ to X.

Analytic Extension of the Trace

Reminder

1. If $P \in \Psi^{<-n}(M)$, then $k_P(x,x)$ is a C^∞ density on M.
2. If M is compact, then P is trace-class and
$$\mathrm{Tr}[P] = \int_M k_P(x,x).$$
3. If $p(x,\xi) \in S^{<-n}(U \times \mathbb{R}^n)$ and $P = p(x,D)$, then
$$k_P(x,x) = \check{p}_{\xi \to y}(x,0) = \int p(x,\xi) đ\xi.$$

Main Goal

1. Get an analytic extension to $\Psi^{\mathbb{C}\setminus\mathbb{Z}}(M)$ of the map $P \to \mathrm{Tr}[P]$.
2. The noncommutative residue will then appear as the "residual" functional on $\Psi^{\mathbb{Z}}(M)$ induced by this extension.

The Functional L

Definition

The functional $L : S^{<-n}(\mathbb{R}^n) \to \mathbb{C}$ is defined by

$$L(p) = \int_{\mathbb{R}^n} p(\xi) đ\xi, \quad p(\xi) \in S^{<-n}(\mathbb{R}^n).$$

Remark

If $p(x,\xi) \in S^{<-n}(U \times \mathbb{R}^n)$ and $P = p(x,D)$, then

$$k_P(x,x) = L\left(p(x,\cdot)\right).$$

The Functional L

Lemma

The linear functional $L : S^{<-n}(\mathbb{R}^n) \to \mathbb{C}$ is holomorphic.

Proof.

If $p(z)(\xi)$, $z \in \Omega$, is a holomorphic family in $S^{<-n}(\mathbb{R}^n)$, then this is a holomorphic family in $L^1(\mathbb{R}^n)$. $\square$

Lemma

The linear functional $L : S^{<-n}(\mathbb{R}^n) \to \mathbb{C}$ has a unique holomorphic extension $\tilde{L} : S^{\mathbb{C}\setminus\mathbb{Z}}(\mathbb{R}^n) \to \mathbb{C}$.

The Functional L

Proof of Uniqueness.

- Let $\tilde{L}_j : S^{\mathbb{C}\setminus\mathbb{Z}}(\mathbb{R}^n) \to \mathbb{C}$, $j = 1, 2$, be holom. extensions of L.
- Let $p(\xi) \in S^q(\mathbb{R}^n)$, $q \in \mathbb{C}\setminus\mathbb{Z}$, and $p(z)(\xi)$, $z \in \mathbb{C}$, a holomorphic gauging for $p(\xi)$.
- As $p(z)(\xi)$ has order $q + z$, for $\Re(z + q) < -n$ we have

$$\tilde{L}_1 (p(z)) = L (p(z)) = \tilde{L}_2 (p(z)) .$$

- Here $\tilde{L}_j(p(z))$, $z + q \notin \mathbb{Z}$, are holomorphic functions.
- By the analytic continuation principle we then have

$$\tilde{L}_1 (p(z)) = \tilde{L}_2 (p(z)) \quad \forall z \in \mathbb{C},\ z + q \notin \mathbb{Z}.$$

- In particular, for $z = 0$ we get

$$\tilde{L}_1 (p) = \tilde{L}_1 (p(0)) = \tilde{L}_2 (p(0)) = \tilde{L}_2 (p) .$$

- This shows that $\tilde{L}_1$ and $\tilde{L}_2$ agree.

□

Proof of Existence.

- Let $N \geq 1$. For $p(\xi) \in S^{<-n+N}(\mathbb{R}^n)$, $p(\xi) \sim \sum p_{q-j}(\xi)$, set

$$L'_N(p) = \int_{|\xi|\geq 1} \Big(p(\xi) - \sum_{j<N} p_{q-j}(\xi)\Big) đ\xi.$$

- This is well defined, since with $\epsilon := N - (\Re q + n) > 0$, we have

$$p(\xi) - \sum_{j<N} p_{q-j}(\xi) = \mathrm{O}\left(|\xi|^{\Re q - N}\right) = \mathrm{O}\left(|\xi|^{-n-\epsilon}\right).$$

Fact

The linear functional $L'_N : S^{<-n+N}(\mathbb{R}^n) \to \mathbb{C}$ is holomorphic.

□

Proof of Existence.

- For $p(\xi) \in S^{\bullet}(\mathbb{R}^n)$, $p(\xi) \sim \sum p_{q-j}(\xi)$, we also set

$$L''(p) = \int_{|\xi|\leq 1} p(\xi) đ\xi, \qquad \gamma_j(p) = \int_{|\xi|=1} p_{q-j}(\xi) đ\xi, \quad j \geq 0.$$

Fact

The functionals $L'' : S^{\bullet}(\mathbb{R}^n) \to \mathbb{C}$ and $\gamma_j : S^{\bullet}(\mathbb{R}^n) \to \mathbb{C}$, $j \geq 0$, are holomorphic.

□

Proof of Existence.

- For $N \geq 1$, set

$$A_N = \{\Re z < -n + N\} \cap (\mathbb{C} \backslash \mathbb{Z}).$$

- For $p(\xi) \in S^{A_N}(\mathbb{R}^n)$, $p(\xi) \sim \sum p_{q-j}(\xi)$, define

$$\tilde{L}_N(p) = L'_N(p) + L''(p) - \sum_{j<N} \frac{1}{q+N-j} \gamma_j(p). \qquad (1)$$

- This is well defined, since $q + N - j \neq 0$ if $q \in A_N$ and $j < N$.

Fact

The functional $\tilde{L}_N : S^{A_N}(\mathbb{R}^n) \to \mathbb{C}$ is holomorphic.

□

Proof of Existence.

- Let $p(\xi) \in S^{<-n}(\mathbb{R}^n)$. We have

$$L(p) = \int_{\mathbb{R}^n} p(\xi) đ\xi = \int_{|\xi|\geq 1} \Big(p(\xi) - \sum_{j<N} p_{q-j}(\xi)\Big) đ\xi + \int_{|\xi|\leq 1} p(\xi) đ\xi + \sum_{j<N} \int_{|\xi|\geq 1} p_{q-j}(\xi) đ\xi.$$

- As $p_{q-j}(t\xi) = t^{q-j} p(\xi)$, by using polar coordinates we get

$$\begin{aligned} \int_{|\xi|\geq 1} p_{q-j}(\xi) đ\xi &= \int_1^{\infty} \int_{\mathbb{S}^{n-1}} t^{q-j} p_{q-j}(\xi) t^{n-1} dt đ\xi \\ &= \frac{-1}{n+q-j} \int_{\mathbb{S}^{n-1}} p_{q-j}(\xi) đ\xi \\ &= \frac{-1}{n+q-j} \gamma_j(p). \end{aligned}$$

□

Proof of Existence.

- Thus,

$$\begin{aligned} L(p) = &\int_{|\xi|\geq 1} \Big(p(\xi) - \sum_{j<N} p_{q-j}(\xi)\Big) đ\xi \\ &+ \int_{|\xi|\leq 1} p(\xi) đ\xi - \sum_{j<N} \frac{1}{n+q-j} \gamma_j(p) \\ =& L'_N(p) + L''(p) - \sum_{j<N} \frac{1}{n+q-j} \gamma_j(p) \\ =& \tilde{L}_N(p). \end{aligned}$$

- It follows that $\tilde{L}_N : S^{A_N}(\mathbb{R}^n) \to \mathbb{C}$ is the unique holomorphic extension of L to $S^{A_N}(\mathbb{R}^n)$.

□

The Functional L

Proof of Existence.

- The uniqueness implies that

$$\tilde{L}_{N+1} = \tilde{L}_N \quad \text{on } S^{A_N}(\mathbb{R}^n).$$

- Thus, we define a functional $\tilde{L} : S^{\mathbb{C}\setminus\mathbb{Z}}(\mathbb{R}^n) \to \mathbb{C}$ by letting

$$\tilde{L}(p) = \tilde{L}_N(p), \quad p \in S^q(\mathbb{R}^n),\ q \in \mathbb{C}\setminus\mathbb{Z},\ \Re(q) + N < -n,$$

  where the value of N is irrelevant.

- Then $\tilde{L} = L$ on $S^{<-n}(\mathbb{R}^n)$ and is holomorphic, and so this is the unique holomorphic extension of L to $S^{\mathbb{C}\setminus\mathbb{Z}}(\mathbb{R}^n)$.

The proof is complete. □

The Functional L

Lemma

Let $p(\xi) \in S^m(\mathbb{R}^n)$, $m \in \mathbb{Z}$, $m \geq -n$, $p(\xi) \sim \sum p_{m-j}(\xi)$. *Given any holomorphic gauging* $p(z)(\xi)$, $z \in \mathbb{C}$, *the function* $\tilde{L}(p(z))$ *has near* $z = 0$ *at worst a simple pole such that*

$$\operatorname{Res}_{z=0} \tilde{L}(p(z)) = -\gamma_{m+n}(p) = -\int_{\mathbb{S}^{n-1}} p_{-n}(\xi) đ\xi.$$

The Functional L

Proof.

- As $p(z)(\xi)$ has order $z+m$, for $\Re z<1$ we may take $N=m+n+1$ in the definition of $\tilde{L}(p(z))$.
- We get

$$\tilde{L}\left(p(z)\right)=\tilde{L}_{m+n+1}\left(p(z)\right)$$
$$=L'_{m+n+1}\left(p(z)\right)+L''\left(p(z)\right)-\sum_{j<m+n+1}\frac{1}{(z+m)+n-j}\gamma_j\left(p(z)\right)$$
$$=L'_{m+n+1}\left(p(z)\right)+L''\left(p(z)\right)-\sum_{0\leq\ell\leq m+n}\frac{1}{z-\ell}\gamma_{m+n-\ell}\left(p(z)\right).$$

- This shows that $\tilde{L}(p(z))$ is meromorphic near $z=0$ with a simple pole such that

$$\begin{aligned}\operatorname{Res}_{z=0}\tilde{L}\left(p(z)\right)&=\operatorname{Res}_{z=0}z^{-1}\gamma_{m+n}\left(p(z)\right)\\&=-\gamma_{m+n}\left(p(0)\right)\\&=-\gamma_{m+n}\left(p\right).\end{aligned}$$

Analytic Extension of $k_P(x,x)$

Setup

$U \subset \mathbb{R}^n$ is an open set.

Reminder

1. If $P \in \Psi^{<-n}(U)$, then $k_P(x,x) \in C^\infty(U)$.
2. If $P = p(x,D)$ with $p(x,\xi) \in S^{<-n}(U \times \mathbb{R}^n)$, then

$$k_P(x,x) = \check{p}_{\xi \to y}(x,0) = L\left(p(x,\cdot)\right).$$

Proposition

The map $\Psi^{<-n}(U) \ni P \to k_P(x,x) \in C^\infty(U)$ has a unique holomorphic extension $\Psi^{\mathbb{C}\backslash\mathbb{Z}}(U) \ni P \to t_P(x) \in C^\infty(U)$.

Analytic Extension of $k_P(x,x)$

Proof.

- Let $P \in \Psi^q(U)$, $q \in \mathbb{C}\setminus\mathbb{Z}$, and put
$$P = p(x,D) + R,$$
with $p(x,\xi) \in S^q(U \times \mathbb{R}^n)$ and $R \in \Psi^{-\infty}(U)$.
- If $\Re q < -n$, then
$$k_P(x,x) = \check{p}_{\xi\to y}(x,0) + k_R(x,x) = L\left(p(x,\cdot)\right) + k_R(x,x).$$
- In general, we set
$$t_P(x) = \tilde{L}\left(p(x,\cdot)\right) + k_R(x,x).$$
- It can be shown that the r.h.s. does not depend on the choice of the pair (p,R).
- It can be also shown that $t_P(x) \in C^\infty(U)$.
- This extends the map $P \to k_P(x,x)$ to $\Psi^{\mathbb{C}\setminus\mathbb{Z}}(U)$.

$\square$

Proof.

- Let $P(z)$, $z \in \Omega$, be a holomorphic family in $\Psi^{\mathbb{C}\mathbb{Z}}(U)$.
- We may write
$$P(z) = p(z)(x,D) + R(z),$$
where $p(z)(x,\xi)$ and $R(z)$ are holomorphic families in $S^{\mathbb{C}\mathbb{Z}}(U \times \mathbb{R}^n)$ and $\Psi^{-\infty}(U)$.
- We then have
$$t_{P(z)}(x) = \tilde{L}\left(p(z)(x,\cdot)\right) + k_{R(z)}(x,x).$$
- Here $k_{R(z)}(x,x)$ is a holomorphic family in $C^\infty(U)$.
- It can be checked that $\tilde{L}(p(z)(x,\cdot))$ is a holomorphic family in $C^\infty(U)$ as well.
- Thus, $t_{P(z)}(x)$, $z \in \Omega$, is a a holomorphic family in $C^\infty(U)$.
- It follows that the linear map $P \to t_P(x)$ is holomorphic.
- This is the unique holomorphic extension of $P \to k_P(x,x)$. □

Analytic Extension of $k_P(x,x)$

Definition

If $P \in \Psi^m(U)$, $m \in \mathbb{Z}$, $m \geq -n$, has symbol $p(x,\xi) \sim \sum p_{m-j}(x,\xi)$, then we set

$$c_P(x) = \int_{\mathbb{S}^{n-1}} p_{-n}(x,\xi) đ\xi.$$

Remarks

1. $c_P(x)$ is a smooth function on U.
2. We make the convention that $c_P(x) = 0$ for $m \in \mathbb{Z}$, $m < -n$.

Analytic Extension of $k_P(x,x)$

Proposition

Let $P \in \Psi^m(U)$, $m \in \mathbb{Z}$. *Given any holomorphic gauging* $P(z)$ *for* P, *near* $z = 0$ *the map* $z \to t_{P(z)}(x)$ *has at worst a simple pole singularity such that*

$$\mathrm{Res}_{z=0}\, t_{P(z)}(x) = -c_P(x).$$

Proof.

- Put
$$P(z) = p(z)(x, D) + R(z),$$
where $p(z)(x,\xi)$ and $R(z)$ are holomorphic families in $S^{\bullet}(U \times \mathbb{R}^n)$ and $\Psi^{-\infty}(U)$.
- We have
$$t_{P(z)}(x) = \tilde{L}\left(p(z)(x,\cdot)\right) + k_{R(z)}(x,x).$$

□

Analytic Extension of $k_P(x,x)$

Proof.

- We have

$$t_{P(z)}(x) = \tilde{L}\left(p(z)(x,\cdot)\right) + k_{R(z)}(x,x).$$

- Here $k_{R(z)}(x,x)$ is a holomorphic family in $C^\infty(U)$.
- $\tilde{L}(p(z)(x,\cdot))$ has a simple pole at $z=0$.
- Thus $t_{P(z)}(x)$ has a simple pole at $z=0$ such that

$$\mathrm{Res}_{z=0}\, t_{P(z)}(x) = \mathrm{Res}_{z=0}\, \tilde{L}\left(p(z)(x,\cdot)\right) = -\int_{\mathbb{S}^{n-1}} p_{-n}(0)(x,\xi) đ\xi.$$

- Here $p_{-n}(0)(x,\xi)$ is the symbol of degree $-n$ of $P(0)=P$, and so

$$\int_{\mathbb{S}^{n-1}} p_{-n}(0)(x,\xi) đ\xi = c_P(x).$$

- This gives the result.

□

Analytic Extension of $k_P(x,x)$. Manifolds

Proposition

Let $\phi : U \rightarrow V$ *be a diffeomorphism, where* $V \subset \mathbb{R}^n$ *is open.*

1. *If* $P \in \Psi^{\mathbb{C}\setminus\mathbb{Z}}(V)$, *then*

$$t_{\phi^* P}(x) = |\phi'(x)| t_P(\phi(x)) \qquad \forall x \in V.$$

2. *If* $P \in \Psi^{\mathbb{Z}}(V)$, *then*

$$c_{\phi^* P}(x) = |\phi'(x)| c_P(\phi(x)) \qquad \forall x \in V.$$

Analytic Extension of $k_P(x,x)$. Manifolds

Proof.

- Let $P \in \Psi^q(V)$, $q \in \mathbb{C}$, and $P(z)$, $z \in \mathbb{C}$, a holomorphic gauging for P.
- Here $P(z)$ has order $z+q$.
- For $\Re(z+q) < -n$, we have

$$\begin{aligned} t_{\phi * P(z)} &= k_{\phi^* P(z)}(x,x) \\ &= |\phi'(x)| k_{P(z)}(\phi(x), \phi(x)) \\ &= |\phi'(x)| t_{P(z)}(\phi(x)). \end{aligned}$$

- Here $\phi^* P(z)$ is a holomorphic gauging for $\phi^* P(0) = \phi^* P$.
- Thus, by the analytic continuation principle,

$$t_{\phi * P(z)}(x) = |\phi'(x)| t_{P(z)}(\phi(x)) \qquad \forall z \in \mathbb{C},\ z+q \in \mathbb{C} \setminus \mathbb{Z}.$$

□

Analytic Extension of $k_P(x,x)$. Manifolds

Proof.

- We have

$$t_{\phi*P(z)}(x) = |\phi'(x)| t_{P(z)}(\phi(x)) \qquad \forall z \in \mathbb{C},\ z+q \in \mathbb{C}\backslash\mathbb{Z}.$$

- If $q \in \mathbb{C}\backslash\mathbb{Z}$, then for $z=0$ this gives

$$t_{\phi*P}(x) = |\phi'(x)| t_{P(z)}(\phi(x)).$$

- If $q \in \mathbb{Z}$, then

$$\begin{aligned} c_{\phi^*P}(x) &= -\operatorname{Res}_{z=0} t_{\phi*P(z)}(x) \\ &= -\operatorname{Res}_{z=0} |\phi'(x)| t_{P(z)}(\phi(x)) \\ &= -|\phi'(x)| c_P(\phi(x)). \end{aligned}$$

□

Analytic Extension of $k_P(x,x)$. Manifolds

Setup

M^n = smooth manifold.

Reminder

If $P \in \Psi^{<-n}(M)$, then $k_P(x,x)$ is a C^∞-density, i.e., $k_P(x,x) \in C^\infty(M, |\Lambda|(M))$.

Proposition

If $P \in \Psi^q(M)$, $q \in \mathbb{C}\setminus\mathbb{Z}$.

1. *There is a unique C^∞-density $t_P(x) \in C^\infty(M, |\Lambda|(M))$ s.t.*

$$\kappa_*(t_P(x)_{|U}) = t_{\kappa_*(P_{|U})}(x) \quad \text{for every chart } \kappa : U \to V.$$

2. *If $\Re q < -n$, then $t_P(x) = k_P(x,x)$.*

Analytic Extension of $k_P(x,x)$. Manifolds

Proposition

If $P \in \Psi^{\mathbb{Z}}(M)$, then there is a unique C^∞-density $c_P(x) \in C^\infty(M, |\Lambda|(M))$ s.t.

$$\kappa_*(c_P(x)_{|U}) = c_{\kappa_*(P_{|U})}(x) \quad \textit{for every chart } \kappa : U \to V.$$

Remark

If $P \in \Psi^{-n}(M)$ and g is a Riemannian metric on M, then

$$c_P(x) = \left(\int_{S_x^* M} \sigma_{-n}(P)(x,\xi) đ\xi \right) \nu(g)(x)$$

Here $\sigma_{-n}(P)(x,\xi)$ is the principal symbol of P and $\nu(g)$ is the Riemannian density.

Analytic Extension of $k_P(x,x)$. Manifolds

Setup

$\mathcal{E}^r$ = smooth vector bundle over M of rank r.

Reminder

If $P \in \Psi^{<-n}(M,\mathcal{E})$, then $k_P(x,x)$ is a smooth $\mathrm{End}(\mathcal{E})$-valued density, i.e., a section

$$k_P(x,x) \in C^\infty\left(M, \mathrm{End}(\mathcal{E}) \otimes |\Lambda|(M)\right).$$

Analytic Extension of $k_P(x,x)$. Manifolds

Proposition

1. *If* $P \in \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})$*, then there is a unique* $\mathrm{End}(\mathcal{E})$*-valued density* $t_P(x)$ *in* $C^\infty(M, \mathrm{End}(\mathcal{E}) \otimes |\Lambda|(M))$ *s.t.*

$$\tau_*(t_P(x)_{|U}) = t_{\tau_*(P_{|U})}(x) \quad \text{for every triv. } \tau : \mathcal{E}_{|U} \rightarrow U \times \mathbb{C}^r.$$

2. *If* $P \in \Psi^{\mathbb{Z}}(M,\mathcal{E})$*, then there is a unique* $\mathrm{End}(\mathcal{E})$*-valued density* $c_P(x)$ *in* $C^\infty(M, \mathrm{End}(\mathcal{E}) \otimes |\Lambda|(M))$ *s.t.*

$$\tau_*(c_P(x)_{|U}) = c_{\tau_*(P_{|U})}(x) \quad \text{for every triv. } \tau : \mathcal{E}_{|U} \rightarrow U \times \mathbb{C}^r.$$

Remarks

1. $\tau_*(P_{|U})$ is an operator in $\Psi^\bullet(M,\mathbb{C}^r)$, and so $t_{\tau_*(P_{|U})}(x)$ and $c_{\tau_*(P_{|U})}(x)$ make sense as elements of

$$M_r(C^\infty(M, |\Lambda|(M)) = C^\infty(M, \mathrm{End}(\mathbb{C}^r) \otimes |\Lambda|(M))$$

2. If $P \in \Psi^{<-n}(M,\mathcal{E})$, then $t_P(x) = k_P(x,x)$.

Analytic Extension of $k_P(x,x)$. Manifolds

Theorem (Guillemin, Wodzicki, Kontsevich-Vishik)

1. *The map* $\Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E}) \ni P \rightarrow t_P(x) \in C^\infty(M, \mathrm{End}(\mathcal{E}) \otimes |\Lambda|(M))$ *is the unique holomorphic extension to* $\Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})$ *of the map* $\Psi^{<-n}(M,\mathcal{E}) \ni P \rightarrow k_P(x,x) \in C^\infty(M, \mathrm{End}(\mathcal{E}) \otimes |\Lambda|(M))$.
2. *Let* $P \in \Psi^{\mathbb{Z}}(M)$. *Given any holomorphic gauging* $P(z)$ *for* P, *the map* $z \rightarrow t_{P(z)}(x)$ *has near* $z=0$ *at worst a simple pole singularity such that*
$$\mathrm{Res}_{z=0}\, t_{P(z)}(x) = -c_P(x).$$

Assumption

M^n is a closed manifold

Reminder

1. If $\rho(x) \in C^\infty(M, \mathrm{End}(\mathcal{E}) \otimes |\Lambda|(M))$, then $\mathrm{tr}_{\mathcal{E}}[\rho(x)] \in C^\infty(M, |\Lambda|(M))$.
2. If $P \in \Psi^{<-n}(M, \mathcal{E})$, then P is trace-class, and
$$\mathrm{Tr}[P] = \int_M \mathrm{tr}_{\mathcal{E}}\left[k_P(x,x)\right].$$
3. In particular, if $P \in \Psi^{<-n}(M, M)$, then
$$\mathrm{Tr}[P] = \int_M k_P(x,x).$$

Noncommutative Residue and Canonical Trace

Definition (Kontsevich-Vishik)

The canonical trace $\operatorname{TR} : \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M, \mathcal{E}) \to \mathbb{C}$ is defined by

$$\operatorname{TR}(P) = \int_M \operatorname{tr}_{\mathcal{E}} \left[t_P(x) \right], \qquad P \in \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M, \mathcal{E}).$$

Remarks

- If $P \in \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M)$, then
$$\operatorname{TR}(P) = \int_M t_P(x).$$
- If $P \in \Psi^{<-n}(M, \mathcal{E})$, then $t_P(x) = k_P(x, x)$, and so we have
$$\operatorname{TR}(P) = \int_M \operatorname{tr}_{\mathcal{E}}[k_P(x, x)] = \operatorname{Tr}(P).$$

Noncommutative Residue and Canonical Trace

Definition (Guillemin, Wodzicki)

The noncommutative residue $\operatorname{Res} : \Psi^{\mathbb{Z}}(M, \mathcal{E}) \to \mathbb{C}$ is given by

$$\operatorname{Res}(P) = \int_M \operatorname{tr}_{\mathcal{E}}[c_P(x)], \qquad P \in \Psi^{\mathbb{Z}}(M, \mathcal{E}).$$

Remarks

1. If $P \in \Psi^{\mathbb{Z}}(M)$, then

$$\operatorname{Res}(P) = \int_M c_P(x).$$

2. If $P \in \Psi^{-n}(M, \mathcal{E})$, then

$$\operatorname{Res}(P) = \int_{S^*M} \operatorname{tr}_{\mathcal{E}_x}[\sigma_{-n}(P)(x,\xi)]\, dx đ\xi.$$

Noncommutative Residue and Canonical Trace

Remark

The noncommutative residue is annihilated by the following classes of operators:

- ΨDOs of order $< -n$, including smoothing operators.
- Differential operators (since such operators don't have homogeneous symbols of negative degree).

Noncommutative Residue and Canonical Trace

Theorem (Guillemin, Wodzicki, Kontsevich-Vishik)

1. *The canonical trace* $\mathrm{TR} : \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M, \mathcal{E}) \to \mathbb{C}$ *is the unique holomorphic extension to* $\Psi^{\mathbb{C}\setminus\mathbb{Z}}(M, \mathcal{E})$ *of the ordinary trace* $\mathrm{Tr} : \Psi^{<-n}(M, \mathcal{E}) \to \mathbb{C}$.
2. *Let* $P \in \Psi^{\mathbb{Z}}(M, \mathcal{E})$. *Given any holomorphic gauging* $P(z)$, $z \in \mathbb{C}$, *for* P, *near* $z = 0$ *the function* $z \to \mathrm{TR}[P(z)]$ *has a simple pole such that*

$$\mathrm{Res}_{z=0}\, \mathrm{TR}\,[P(z)] = -\,\mathrm{Res}(P).$$

Noncommutative Residue and Canonical Trace

Proposition

Let $P_1 \in \Psi^{q_1}(M, \mathcal{E})$ *and* $P_2 \in \Psi^{q_2}(M, \mathcal{E})$.

1. *If* $q_1 + q_2 \in \mathbb{C} \backslash \mathbb{Z}$, *then*
$$\mathrm{TR}\left[P_1 P_2\right] = \mathrm{TR}\left[P_2 P_1\right].$$
2. *If* $q_1 + q_2 \in \mathbb{Z}$, *then*
$$\mathrm{Res}\left[P_1 P_2\right] = \mathrm{Res}\left[P_2 P_1\right].$$

Corollary

The noncommutative residue is a trace on the algebra $\Psi^{\mathbb{Z}}(M, \mathcal{E})$.

Proof of the Proposition.

We may assume $\Re q_1 \geq \Re q_1$.

- For $j = 1, 2$ let $P_j(z)$ be a holomorphic gauging for P_j.
- For $\Re(z + q_1) \leq 0$ the operator $P_1(z)$ is bounded
- For $\Re(z + q_2) < -n$, the operator $P_2(z)$ is trace-class.
- Thus, for $\Re z < \min(-\Re q_1, -(n + \Re q_2))$, we have

$$\begin{aligned} \mathrm{TR}\left[P_1(z)P_2(z)\right] &= \mathrm{Tr}\left[P_1(z)P_2(z)\right] \\ &= \mathrm{Tr}\left[P_2(z)P_1(z)\right] = \mathrm{TR}\left[P_2(z)P_1(z)\right]. \end{aligned}$$

- By analytic continuation we get

$$\mathrm{TR}\left[P_1(z)P_2(z)\right] = \mathrm{TR}\left[P_2(z)P_1(z)\right], \quad z + q_1 + q_2 \in \mathbb{C}\setminus\mathbb{Z}.$$

- If $q_1 + q_2 \in \mathbb{C}\setminus\mathbb{Z}$, then for $z = 0$ we get

$$\mathrm{TR}\left[P_1 P_2\right] = \mathrm{TR}\left[P_1(0)P_2(0)\right] = \mathrm{TR}\left[P_2(0)P_1(0)\right] = \mathrm{TR}\left[P_2 P_1\right].$$

□

Proof of the Proposition.

Assume $q_1 + q_2 \in \mathbb{Z}$.

- Here $P_1(z/2)P_2(z/2)$ is a holomorphic gauging for $P_1 P_2$, since it has order $(z/2 + q_1) + (z/2 + q_2) = z + q_1 + q_2$ and $P_1(0)P_2(0) = P_1 P_2$.
- Likewise, $P_2(z/2)P_1(z/2)$ is a holomorphic gauging for $P_2 P_1$.
- Thus,

$$\begin{aligned}\operatorname{Res}[P_1 P_2] &= -\operatorname{Res}_{z=0} \operatorname{TR}[P_1(z/2)P_2(z/2)] \\ &= -\operatorname{Res}_{z=0} \operatorname{TR}[P_2(z/2)P_1(z/2)] \\ &= \operatorname{Res}[P_2 P_1].\end{aligned}$$

□

Noncommutative Residue and Canonical Trace

Theorem (Wodzicki)

If M is connected and $n \geq 2$, then the noncommutative residue is the unique trace on $\Psi^{\mathbb{Z}}(M, \mathcal{E})$ up to constant multiple.

Remark

Alternative proofs by Guillemin, Lesch, Lescure-Paycha, RP.

Zeta Functions of Elliptic Operators

Setup

- M is equipped with a smooth positive density and $\mathcal{E}$ is endowed with a (smooth) Hermitian metric.
- $P \in \Psi^m(M, \mathcal{E})$, $m > 0$, is elliptic with $\sigma_m(P)(x, \xi) > 0$ and is selfadjoint and ≥ 0.

Reminder

Under the above assumptions, the spectrum of P can be arranged as a non-decreasing sequence,

$$0 \leq \lambda_0(P) \leq \lambda_1(P) \leq \cdots .$$

where each eigenvalue is repeated according to multiplicity.

Zeta Functions of Elliptic Operators

Reminder

- P^z, $z \in \mathbb{C}$, is a holomorphic family of ΨDOs of order mz.
- We have
$$P^{z_1+z_2} = P^{z_1} P^{z_2}, \qquad P^z\big|_{z=0} = 1 - \Pi_0(P),$$
where $\Pi_0(P)$ is the orthogonal projection onto $\ker P$.
- $\sigma_{mz}(P^z)(x,\xi) = \sigma_m(P)(x,\xi)^z$.
- $\Pi_0(P)$ is a smoothing operator.

Consequence

$P^{z_0+z/m}$, $z \in \mathbb{C}$, is a holomorphic gauging for P^{z_0} for every $z_0 \in \mathbb{C}$.

Zeta Functions of Elliptic Operators

Remark

- P^{-z} has order $-mz$, and hence is trace-class for $\Re z > nm^{-1}$.
- For $\Re z > mn^{-1}$ we have

$$\mathrm{Tr}\left[P^{-z}\right] = \sum_{\lambda_j(P)>0} \lambda_j(P)^{-z},$$

Definition

The zeta function of P is

$$\zeta_P(z) := \sum_{\lambda_j(P)>0} \lambda_j(P)^{-z}, \qquad \Re z > nm^{-1}.$$

Remark

- $\zeta_P(z) = \mathrm{Tr}[P^{-z}] = \mathrm{TR}[P^{-z}]$ for $\Re z > nm^{-1}$.
- As P^{-z}, $\Re z > nm^{-1}$, is a holomorphic family in $\Psi^{<-n}(M)$, we see that $\zeta_P(z)$ is holomorphic for $\Re z > nm^{-1}$.

Zeta Functions of Elliptic Operators

Notation

$$\Sigma := \left\{ \frac{n-j}{m};\ j \geq 0,\ j \neq n \right\} \subseteq \mathbb{R} \setminus 0.$$

Theorem

1. *The zeta function* $\zeta_P(z)$ *has a meromorphic continuation to* $\mathbb{C}$ *with at worst simple poles on* Σ*.*
2. *If* $\sigma \in \Sigma$*, then*

$$m \operatorname{Res}_{z=\sigma} \zeta_P(z) = \operatorname{Res}\left[P^{-\sigma}\right].$$

3. *The function* $\zeta_P(z)$ *is always regular at* $z = 0$*.*

Proof.

- For $\Re z > nm^{-1}$ we have
$$\zeta_P(z) = \mathrm{Tr}\left[P^{-z}\right] = \mathrm{TR}\left[P^{-z}\right].$$
- P^{-z}, $z \in \mathbb{C}$, is a holomorphic family of order $-mz$.
- Set $\Sigma_0 = \Sigma \cup \{0\} = \{(n-j)m^{-1}; j \geq 0\}$. We have
$$\begin{aligned} -m\sigma \in -n + \mathbb{N}_0 &\Longleftrightarrow -m\sigma = -n + j \text{ for some } j \geq 0 \\ &\Longleftrightarrow \sigma = (n-j)m^{-1} \text{ for some } j \geq 0 \\ &\Longleftrightarrow \sigma \in \Sigma_0. \end{aligned}$$
- Thus, P^{-z}, $z \in \mathbb{C} \setminus \Sigma_0$, is a holomorphic family in $\Psi^{\mathbb{C}\setminus\mathbb{Z}}(M, \mathcal{E}) \cup \Psi^{<-n}(M, \mathcal{E})$.
- This ensures $\mathrm{TR}[P^{-z}]$ is holomorphic on $\mathbb{C} \setminus \Sigma_0$.
- Therefore, this an analytic extension of $\zeta_P(z)$ to $\mathbb{C} \setminus \Sigma_0$. □

Proof.

- Let $\sigma \in \Sigma_0$. In that case $P^{-\sigma}$ has order $-m\sigma \in -n + \mathbb{N}_0$.
- Here $P^{-\sigma+z/m}$, $z \in \mathbb{C}$, is a holomorphic gauging for $P^{-\sigma}$.
- Thus, the function $F(z) := \mathrm{TR}[P^{-\sigma+z/m}]$ has a simple pole near $z = 0$ s.t.

$$\mathrm{Res}_{z=0} F(z) = -\operatorname{Res}\left[P^{-\sigma}\right].$$

- Here $F(z) = \mathrm{TR}[P^{-\sigma+z/m}] = \zeta_P(\sigma - z/m)$, i.e., $\zeta_P(z) = F(m(\sigma - z))$.
- Thus, $\zeta_P(z)$ has a simple pole near $z = \sigma$ s.t.

$$\mathrm{Res}_{z=\sigma} \zeta_P(z) = -m^{-1} \mathrm{Res}_{z=0} F(z) = m^{-1} \operatorname{Res}\left[P^{-\sigma}\right].$$

□

Proof.

- For $\sigma = 0$ we get
$$m \operatorname{Res}_{z=0} \zeta_P(z) = \operatorname{Res}\left[P^{-0}\right] = \operatorname{Res}\left[1 - \Pi_0(P)\right].$$
- As 1 is a differential operator and $\Pi_0(P)$ is a smoothing operator, we have
$$\operatorname{Res}[1] = \operatorname{Res}\left[\Pi_0(P)\right] = 0.$$
- Thus,
$$m \operatorname{Res}_{z=0} \zeta_P(z) = \operatorname{Res}\left[1 - \Pi_0(P)\right] = 0.$$
- This shows that $\zeta_P(z)$ does not have a pole at $z = 0$, and hence it is regular there. □

Zeta Functions of Elliptic Operators

Setup

$A \in \Psi^a(M, \mathcal{E})$, $a \in \mathbb{R}$, and

$$\Sigma_a := \left\{ \frac{n+a-j}{m};\ j \geq 0 \right\}$$

Facts

- AP^{-z} is a holomorphic family of ΨDOs of order $a - mz$.
- In particular, AP^{-z} is trace-class for $\Re z > n + a$.
- We also have

$$a - m\sigma \in -n + \mathbb{N}_0 \Longleftrightarrow \sigma \in \Sigma_a.$$

- It follows that AP^{-z}, $z \in \mathbb{C} \setminus \Sigma_a$ is a holomorphic family in $\Psi^{<-n}(M, \mathcal{E}) \cup \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M, \mathcal{E})$.

Zeta Functions of Elliptic Operators

Definition

$$\zeta_P(A; z) := \operatorname{Tr}\left[AP^{-z}\right], \qquad z \in \mathbb{C} \setminus \Sigma_a.$$

Proposition

1. *The zeta function $\zeta_P(A; z)$ has a meromorphic continuation to $\mathbb{C}$ with at worst simple poles on Σ_a.*
2. *If $\sigma \in \Sigma_a$, then*

$$m \operatorname{Res}_{z=\sigma} \zeta_P(A; z) = \operatorname{Res}\left[AP^{-\sigma}\right].$$

Theorem (Ikehara's Tauberian Theorem; see Shubin)

Assume $N(t)$, $t \geq 0$, is a non-decreasing function s.t.

(i) $N(t) = 0$ *near* $t = 0$.

(ii) *The integral* $\zeta(z) := \int t^{-z} dN(t)$ *converges for* $\Re z > q > 0$.

(iii) $\zeta(z)$ *admits a meromorphic extension to a half-plane* $\Re z > q - \epsilon$, $\epsilon > 0$ *with only a simple pole at* $z = q$ *s.t.*

$$\mathrm{Res}_{z=q}\, \zeta(z) = A > 0.$$

Then, we have

$$N(t) \sim \frac{1}{q} A t^q \quad \text{as } t \to \infty.$$

Weyl's Law for Elliptic Operators

Definition

The counting function of P is

$$N_P(t) := \#\left\{j;\ 0 < \lambda_j(P) \leq t\right\}, \quad t > 0.$$

Remark

- We have
$$dN_P(t) = \sum_{\lambda_j(P)>0} \delta_{\lambda_j(P)}(t),$$
where $\delta_{\lambda_j(P)}(t)$ is the Dirac measure at $\lambda_j(P)$.
- Thus, for $\Re z > nm^{-1}$ we have
$$\zeta_P(z) = \sum_{\lambda_j(P)>0} \lambda_j(P)^{-z} = \int t^{-z} dN_P(t).$$

Weyl's Law for Elliptic Operators

Theorem (Weyl's Law; 1st Version)

As $t \to \infty$, we have

$$N_P(t) \sim \frac{1}{n}\left(\int_{S^*M} \mathrm{Tr}_{\mathcal{E}}\left[\sigma_m(P)(x,\xi)^{-\frac{n}{m}}\right] dx đ\xi \right) t^{\frac{n}{m}}.$$

Proof.

- For $\Re z > nm^{-1}$, we have $\int t^{-z} dN_P(t) = \zeta_P(z)$.
- The function $\zeta_P(z)$ is holomorphic for $\Re z > nm^{-1}$.
- It has a meromorphic continuation to the half-plane $\Re z > (n-1)m^{-1}$ with only a simple pole at $z = nm^{-1}$ s.t.

$$\mathrm{Res}_{z=nm^{-1}}\, \zeta_P(z) = \frac{1}{m}\, \mathrm{Res}\left[P^{-\frac{n}{m}}\right].$$

□

Weyl's Law for Elliptic Operators

Proof.

- Here $P^{-n/m}$ has order $-n$, and its principal symbol is
$$\sigma_{-n}(P^{-\frac{n}{m}})(x,\xi) = \sigma_m(P)(x,\xi)^{-\frac{n}{m}} > 0.$$
- Thus,
$$\begin{aligned}\operatorname{Res}\left[P^{-\frac{n}{m}}\right] &= \int_{S^*M} \operatorname{Tr}_{\mathcal{E}_x}\left\{\sigma_{-n}\left(P^{-\frac{n}{m}}\right)(x,\xi)\right\} dx đ\xi \\ &= \int_{S^*M} \operatorname{Tr}_{\mathcal{E}_x}\left[\sigma_m(P)(x,\xi)^{-\frac{n}{m}}\right] dx đ\xi > 0.\end{aligned}$$
- Therefore, we may apply Ikehara's Tauberian theorem to get
$$\begin{aligned}N_P(t) &= \frac{m}{n}\cdot\frac{1}{m}\operatorname{Res}\left[P^{-\frac{n}{m}}\right] t^{\frac{n}{m}} + \mathrm{o}\left(t^{\frac{n}{m}}\right) \\ &= \frac{1}{n}\left(\int_{S^*M} \operatorname{Tr}_{\mathcal{E}}\left[\sigma_m(P)(x,\xi)^{-\frac{n}{m}}\right] dx đ\xi\right) t^{\frac{n}{m}} + \mathrm{o}\left(t^{\frac{n}{m}}\right).\end{aligned}$$

This gives the result. □

Weyl's Law for Elliptic Operators

Lemma

We always have

$$\limsup_{j\to\infty} j\lambda_j(P)^{-\frac{m}{n}} = \limsup_{t\to\infty} t^{-\frac{n}{m}} N_P(t),$$

$$\liminf_{j\to\infty} j\lambda_j(P)^{-\frac{m}{n}} = \liminf_{t\to\infty} t^{-\frac{n}{m}} N_P(t).$$

Therefore, we obtain:

Theorem (Weyl's Law; 2nd Version)

As $j \to \infty$ *we have*

$$\lambda_j(P) \sim j^{\frac{m}{n}} \left(\frac{1}{n} \int_{S^*M} \mathrm{Tr}_{\mathcal{E}} \left[\sigma_m(P)(x,\xi)^{-\frac{n}{m}} \right] dx đ\xi \right)^{-\frac{m}{n}}.$$