CHAPTER 5

Connes' Quantized Calculus

The quantized calculus of Connes [**Co**] aims to translate the main tools of the infinitesimal calculus into the operator theoretic language of quantum mechanics. It allows us to write down a dictionary between classical notions in the infinitesimal calculus and their quantum analogues.

5.1. Noncommutative Infinitesimal Calculus

In the sequel we let $\mathcal{H}$ be a separable Hilbert space. In practice $\mathcal{H}$ comes from a spectral triple $(\mathcal{A}, cH, D)$ and we shall set

$$F := \operatorname{Sign}(D) = D|D|^{-1}.$$

The first few lines of this dictionary between classical notions in the infinitesimal calculus and their quantum analogues are the following:

Classical	Quantum
Complex variable	Operator on $\mathcal{H}$
Real variable	Selfadjoint operator on $\mathcal{H}$
Infinitesimal variable	Compact operator on $\mathcal{H}$
Infinitesimal of order $\alpha > 0$	Compact operator T such that $\mu_n(T) = \mathrm{O}(n^{-\alpha})$
Differential $df = \sum \frac{\partial}{\partial x^j} dx^j$	$da := [F, a]$.
Integral	Dixmier Trace $⨍$

The first two lines comes directly from quantum mechanics. In that formalism the observables are selfadjoint operators and the values that can be observed from an observable are given by its spectrum. In addition, as we saw in Chapter 1, we have a holomorphic functional calculus for *any* bounded operator on $\mathcal{H}$, but there is a continuous functional calculus only for normal operators, including selfadjoint operators.

Intuitively, an infinitesimal can be thought as an object which is smaller than ϵ for any $\epsilon > 0$. For an operator T the condition $\|T\| < \epsilon$ for all $\epsilon > 0$ holds only when $T = 0$, but it can be relaxed into the following:

For any $\epsilon > 0$ there is a finite-dimensional subspace E such that $\|T|E^{\perp}\| < \epsilon$.

As shown by Proposition 4.1.6 this latter condition is equivalent to T being compact.

By Proposition 4.1.6 we also know that an operator T is compact if and only if its singular values $\mu_n(T)$ converge to 0 as $n \to \infty$. Thus the compactness of T can be measured by the decay of its singular values. Thus, an infinitesimal operator of order α, $\alpha > 0$, is just a compact operator T such that

$$\mu_n(T) = \mathrm{O}(n^{-\alpha}).$$

If $\alpha > 1$ and we set $p := \alpha^{-1}$, then we see that the set of infinitesimal operators of order α agrees with the operator ideal $\mathcal{L}^{(p,\infty)}$ introduced in the previous chapter.

For $\alpha = 1$ every infinitesimal of order 1 is contained in the ideal $\mathcal{L}^{(1,\infty)}$, but there are operators in $\mathcal{L}^{(1,\infty)}$ that are not infinitesimal operators of order 1.

The following show that intuitive rules of the infinitesimal calculus are satisfied.

LEMMA 5.1.1. *For $j = 1, 2$ let T_j be an infinitesimal operator of order α_j.*

(1) *$T_1 + T_2$ is an infinitesimal operator of order $\min(\alpha_1, \alpha_2)$.*
(2) *T_1T_2 is an infinitesimal operator of order $\alpha_1\alpha_2$.*

PROOF. Thanks to (4.7) we have

$$\mu_{m+n}(T_1 + T_2) \leq \mu_m(T_1) + \mu_n(T_2) \qquad \forall m, n \in \mathbb{N}_0. \tag{5.1}$$

Let $n \in \mathbb{N}_0$. Since $n \geq 2[\frac{n}{2}]$, by (4.3) we have $\mu_n(T_1 + T_2) = \mu_{[\frac{n}{2}]+[\frac{n}{2}]}(T_1 + T_2)$, and hence using (5.1) we get

$$\mu_n(T_1 + T_2) \leq \mu_{[\frac{n}{2}]}(T_1) + \mu_{[\frac{n}{2}]}(T_2). \tag{5.2}$$

Set $\alpha = \min(\alpha_1, \alpha_2)$. Then, as $[\frac{n}{2}] \geq \frac{n-1}{2}$, we have

$$\mu_{[\frac{n}{2}]}(T_j) = \mathrm{O}\left(\left[\frac{n}{2}\right]^{-\alpha_j}\right) = \mathrm{O}(n^{-\alpha_j}) = \mathrm{O}(n^{-\alpha}). \tag{5.3}$$

Combining this with (5.2) then shows that $T_1 + T_2$ is an infinitesimal operator of order α.

Next, by (4.9) we have

$$\mu_{n+m}(T_1T_2) \leq \mu_m(T_1)\mu_n(T_2) \qquad \forall m, n \in \mathbb{N}_0.$$

Therefore, for any $n \in \mathbb{N}_0$,

$$\mu_n(T_1T_2) \leq \mu_{[\frac{n}{2}]+[\frac{n}{2}]}(T_1T_2) \leq \mu_{[\frac{n}{2}]}(T_1)\mu_{[\frac{n}{2}]}(T_2).$$

Combining this with (5.3) shows that $\mu_n(T_1T_2) = \mathrm{O}(n^{-\alpha_1})\,\mathrm{O}(n^{-\alpha_2}) = \mathrm{O}(n^{-(\alpha_1+\alpha_2)})$, that is, T_1T_2 is an infinitesimal operator of order $\alpha_1\alpha_2$. The lemma is proved. $\square$

The differential $da := [F, a]$ is a derivation, and hence it satisfies Leibniz's Rule,

$$d(ab) = (da)b + adb, \qquad a, b \in \mathcal{A}.$$

In practice, the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is p-summable for some $p \geq 1$, that is,

$$\mu_n(D^{-1}) = \mathrm{O}(n^{-\frac{1}{p}}).$$

This means that D^{-1} is an infinitesimal operator of order $\frac{1}{p}$.

LEMMA 5.1.2 (cf. Chapter 11). *If $(\mathcal{A}, \mathcal{H}, D)$ is p-summable, then*

$$\mu_n([F, a]) = \mathrm{O}(n^{-\frac{1}{p}}) \qquad \forall a \in \mathcal{A}.$$

In other words, if the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is p-summable, then all the differentials $da = [F, a]$, $a \in \mathcal{A}$, are infinitesimal operators of order $\frac{1}{p}$.

If $x^1, \cdots, x^p$ form a system of local coordinates on a manifold M of dimension p, then $dx^1 \wedge \cdots \wedge dx^p$ can be thought as an "infinitesimal" of order 1 and the differentials $dx^1, \ldots, dx^p$ as "infinitesimals" of order $\frac{1}{p}$. We see that, similarly, if a^0, ..., a^p are in $\mathcal{A}$, then da^1, ..., da^p are infinitesimal operators of order $\frac{1}{p}$ and $a^0 da^1 \cdots da^p$ is an infinitesimal operator of order 1.

Next, the classical integral is a linear functional which the following properties:

(i) It is defined on infinitesimals of order 1.
(ii) It vanishes on infinitesimals of order > 1, i.e., we can integrate by neglecting higher-order infinitesimals (e.g., by approximating an integral by Riemann sums).
(iii) It is positive, i.e., the integral of a non-negative function is a non-negative number.
(iv) It vanishes on total differentials.

In the setting of quantized calculus we thus seek for a linear functional satisfying noncommutative analogues of the above conditions. The first three conditions can be easily translated into:

(i') Its domain contains the infinitesimal operators of order 1.
(ii') It vanishes on infinitesimal operators of order > 1.
(iii') It is positive, i.e., it takes on non-negative real values on the operators in its domain that are positive.

At least in the p-summable case, the condition (iv) corresponds to the vanishing of the quantum integral on operators of the form

$$d(a^0 da^1 \cdots da^p) = [F, a^0 da^1 \cdots da^p] \qquad a^j \in \mathcal{A}.$$

In general, we require the noncommutative integral to vanish on commutators,

$$[A, T],$$

where A ranges over $\mathcal{L}(\mathcal{H})$ and T ranges over the domain of the quantum integral. Thus we require

(iv') The quantum integral is a trace.

One candidate that comes to mind is the operator trace $T \to \operatorname{Trace}(T)$. This is a positive trace, but it satisfies none of the conditions (i')–(ii'). More precisely:

- The domain of the operator trace is the ideal $\mathcal{L}^1$ of trace-class operators, but an infinitesimal operator of order 1 need not be trace-class, e.g., if $\mu_n(T) = (n+1)^{-1}$, then $\sum \mu_n(T) = \infty$, and hence T is not trace-class.
- The operator trace does not vanish on all infinitesimal operators of order > 1, e.g., it does not vanish on (non-zero) finite-rank projections.

The solution for finding a positive linear trace satisfying (i')–(ii') is actually provided by the Dixmier trace. This trace was constructed by Dixmier [**Di**] as an example of a non-normal trace on $\mathcal{L}(\mathcal{H})$.

The rest of the chapter is devoted to presenting the construction of the Dixmier trace. The exposition follows closely [**CM**, Appendix A] (see also [**GVF**, Section 7.5]).

5.2. The Dixmier Ideal $\mathcal{L}^{(1,\infty)}$

As we shall see in the next section, the domain of the Dixmier trace is the Dixmier ideal $\mathcal{L}^{(1,\infty)}$ introduced in the previous chapter. It is defined as follows.

For $T \in \mathcal{K}$ set

$$\|T\|_{(1,\infty)} := \sup_{N\geq 2} \frac{\sigma_N(T)}{\log N}.$$

We then define

$$\mathcal{L}^{(1,\infty)} = \Big\{ T \in \mathcal{K};\ \|T\|_{(1,\infty)} < \infty \Big\}.$$

Equivalently,

$$\mathcal{L}^{(1,\infty)} = \Big\{ T \in \mathcal{K};\ \sigma_N(T) = \mathrm{O}(\log N) \Big\}.$$

Proposition 5.2.1 (See Chapter 4). *The following hold.*

(1) *$\mathcal{L}^{(1,\infty)}$ is a two-sided ideal and $\|.\|_{(1,\infty)}$ is a norm on $\mathcal{L}^{(1,\infty)}$ for which $\mathcal{L}^{(1,\infty)}$ is a Banach ideal. In particular,*

$$\|ATB\|_{(1,\infty)} \leq \|A\|\|T\|_{(1,\infty)}\|B\| \qquad \forall T \in \mathcal{L}^{(1,\infty)}\ \forall A, B \in \mathcal{L}(\mathcal{H}). \tag{5.4}$$

(2) *The Banach ideal $\mathcal{L}^{(1,\infty)}$ is not separable.*

(3) *Let $\mathcal{L}_0^{(1,\infty)}$ be the closure of finite-rank operators in $\mathcal{L}^{(1,\infty)}$. Then*

$$\mathcal{L}_0^{(1,\infty)} = \Big\{ T \in \mathcal{K};\ \sigma_N(T) = \mathrm{o}(\log N) \Big\}. \tag{5.5}$$

(4) *There are continuous inclusions,*

$$\mathcal{L}^1 \subset \mathcal{L}_0^{(1,\infty)} \qquad \textit{and} \qquad \mathcal{L}^{(1,\infty)} \subset \mathcal{K}.$$

(5) *There is a strict inclusion,*

$$\mathcal{L}^{(1,\infty)} \supsetneq \Big\{ T \in \mathcal{K};\ \mu_n(T) = \mathrm{O}(\frac{1}{n}) \Big\}. \tag{5.6}$$

Remark 5.2.2. It can be shown that a Banach ideal is separable if and only if the finite-rank operators are dense in it (see Chapter 4). Therefore (5.5) implies that $\mathcal{L}^{(1,\infty)}$ is not separable. Notice that $\mathcal{L}_0^{(1,\infty)}$ is a Banach ideal with respect to the norm $\|.\|_{(1,\infty)}$, since this is the closure in $\mathcal{L}^{(1,\infty)}$ of the ideal of finite-rank operators.

Remark 5.2.3. The continuous inclusions (5.6) hold for any non-trivial Banach ideal (see Chapter 4). In the case of $\mathcal{L}^{(1,\infty)}$, the inclusion of $\mathcal{L}^1$ in $\mathcal{L}_0^{(1,\infty)}$ follows from the fact that if $T \in \mathcal{L}^1$, then $\sigma_N(T) = \mathrm{O}(1) = \mathrm{o}(\log N)$. The continuity of the inclusions (5.6) can also be deduced from the fact that, for any $T \in \mathcal{K}$, we have

$$\|T\| = \mu_0(T) \leq \sigma_N(T) \leq \sum_{n\geq 0} \mu_n(T) = \|T\|_1 \qquad \forall N \in \mathbb{N},$$

which implies that

$$(\log 2)^{-1}\|T\| \leq \|T\|_{(1,\infty)} \leq (\log 2)^{-1}\|T\|_1 \qquad \forall T \in \mathcal{K}.$$

The Banach ideal $\mathcal{L}^{(1,\infty)}$ is in duality with the Macaev ideal $\mathcal{L}^{(\infty,1)}$. The latter can be defined as follows.

For $T \in \mathcal{K}$ we set

$$\|T\|_{(\infty,1)} := \sum_{n\geq 0}(n+1)^{-1}\mu_n(T).$$

We then define

$$\mathcal{L}^{(\infty,1)} = \Big\{T \in \mathcal{K};\ \|T\|_{(\infty,1)} < \infty\Big\}.$$

PROPOSITION 5.2.4 (See Chapter 4). *The following hold.*

(1) $\mathcal{L}^{(\infty,1)}$ *is a two-sided ideal and* $\|.\|_{(\infty,1)}$ *is a norm on* $\mathcal{L}^{(\infty,1)}$ *for which* $\mathcal{L}^{(\infty,1)}$ *is a Banach ideal.*

(2) *The Banach ideal* $\mathcal{L}^{(\infty,1)}$ *is separable and the finite-rank operators are dense in it.*

(3) *There are continuous inclusions,*

$$\mathcal{L}^1 \subset \mathcal{L}^{(\infty,1)} \subset \mathcal{K}.$$

(4) *There are isomorphisms,*

$$\mathcal{L}^{(1,\infty)} \simeq (\mathcal{L}^{(\infty,1)})' \qquad \textit{and} \qquad \mathcal{L}^{(\infty,1)} \simeq (\mathcal{L}_0^{(1,\infty)})'. \tag{5.7}$$

REMARK 5.2.5. As explained in Chapter 4, the duality isomorphisms (5.7) are inherited from the isometric isomorphism,

$$\mathcal{L}(\mathcal{H}) \ni S \longrightarrow (S,.) \in (\mathcal{L}^1)', \qquad (S,T) = \operatorname{Trace}(ST) \quad \forall T \in \mathcal{L}^1.$$

More precisely, if $S \in \mathcal{L}^{(\infty,1)}$ and $T \in \mathcal{L}^{(1,\infty)}$, then ST is trace-class and $(S,.)$ uniquely extends to a continuous linear form on $\mathcal{L}_0^{(1,\infty)}$. This yields the isomorphism from $\mathcal{L}^{(\infty,1)}$ onto $(\mathcal{L}_0^{(1,\infty)})'$. Similarly, if $S \in \mathcal{L}^{(1,\infty)}$, then $(S,.)$ uniquely extends to a continuous linear form on $\mathcal{L}^{(\infty,1)}$, which allows us to get an isomorphism from $\mathcal{L}^{(1,\infty)}$ onto $(\mathcal{L}^{(\infty,1)})'$. Notice also that these isomorphisms become isometries if we replace the norm $\|.\|_{(1,\infty)}$ by the equivalent norm,

$$\|T\|'_{(1,\infty)} := \sup_{N\geq 1}\frac{\sigma_N(T)}{\sum_{n<N}(n+1)^{-1}}.$$

In the sequel we let $\mathcal{H}'$ be a (separable) Hilbert space.

LEMMA 5.2.6. *Let* Φ *be a continuous* $*$*-homomorphism from* $\mathcal{L}(\mathcal{H})$ *to* $\mathcal{L}(\mathcal{H}')$ *such that*

$$\mu_n(\Phi(T)) = \mu_n(T) \qquad \forall T \in \mathcal{L}(\mathcal{H})\ \forall n \in \mathbb{N}_0. \tag{5.8}$$

Then Φ *induces an isometric linear map from* $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ *to* $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$.

PROOF. Using Proposition 4.1.6-(iii) and (5.8) we see that Φ maps $\mathcal{K}(\mathcal{H})$ to $\mathcal{K}(\mathcal{H}')$. Moreover, if $T \in \mathcal{K}$, then (5.8) implies that $\sigma_N(\Phi(T)) = \sigma_N(T)$ for all $N \in \mathbb{N}$, and hence $\|\Phi(T)\|_{(1,\infty)} = \|T\|_{(1,\infty)}$. Thus Φ gives rise to an isometric linear map from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ to $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$. The lemma is proved. $\square$

Let $S : \mathcal{H}' \to \mathcal{H}$ be a continuous linear isomorphism from $\mathcal{H}'$ onto $\mathcal{H}$. We denote by γ_S the conjugation by S, i.e., the map

$$\mathcal{L}(\mathcal{H}) \ni T \longrightarrow S^{-1}TS \in \mathcal{L}(\mathcal{H}').$$

This a continuous isomorphism from $\mathcal{L}(\mathcal{H})$ onto $\mathcal{L}(\mathcal{H}')$.

Proposition 5.2.7. *The conjugation by S gives rise to a continuous isomorphism from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ onto $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$, and hence*

$$\mathcal{L}^{(1,\infty)}(\mathcal{H}') = S^{-1}\mathcal{L}^{(1,\infty)}(\mathcal{H})S.$$

Furthermore, if S is unitary, then this isomorphism is isometric.

Proof. Let $S \in \mathcal{L}(\mathcal{H})$ be invertible. Since $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$ both γ_S and its inverse $\gamma_S^{-1} = \gamma_{S^{-1}}$ maps $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ to itself, and hence γ_S induces a linear map from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ to itself. Furthermore, this linear map is continuous, for by (5.4) we have

$$\|S^{-1}TS\|_{(1,\infty)} \leq \|S\|\|S^{-1}\|\|T\|_{(1,\infty)} \qquad \forall T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}).$$

Let $U \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ be unitary. Then by Remark 4.1.4 we have

$$\mu_n(U^*TU) = \mu_n(T) \qquad \forall T \in \mathcal{L}(\mathcal{H}),$$

that is, γ_U satisfies (5.8). Thus, it follows from Lemma 5.2.6 that γ_U induces a linear isometry from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ to $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$. Similarly, its inverse $\gamma_U^{-1} = \gamma_{U^*}$ induces a linear isometry from $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$ to $\mathcal{L}^{(1,\infty)}(\mathcal{H})$, so γ_U induces an isometric linear isomorphism from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ onto $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$.

Now, let $S \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ be a general isomorphism. Set $|S| = (S^*S)^{\frac{1}{2}}$ and $U = S|S|^{-1}$. Then $|S|$ is an invertible element of $\mathcal{L}(\mathcal{H}')$ and U is an unitary element of $\mathcal{L}(\mathcal{H}', \mathcal{H})$, for it is invertible and, as $|S|^{-1}S^*S|S|^{-1} = |S|^{-1}|S|^2|S|^{-1}$, for any $\xi \in \mathcal{H}'$, we have

$$\|U\xi\|_{\mathcal{H}}^2 = \langle S|S|^{-1}\xi, S|S|^{-1}\xi\rangle_{\mathcal{H}} = \langle \xi, |S|^{-1}S^*S|S|^{-1}\xi\rangle_{\mathcal{H}'} = \langle \xi, \xi\rangle_{\mathcal{H}} = \|\xi\|_{\mathcal{H}'}^2.$$

Thus, by the first two parts of the proof $\gamma_{|S|}$ induces a continuous linear isomorphism from $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$ onto itself and γ_U induces an isometric linear isomorphism from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ onto $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$. Since $S = U|S|$, and hence $\gamma_S = \gamma_{|S|} \circ \gamma_U$, it follows that γ_S a continuous linear isomorphism from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ onto $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$, proving the proposition. $\square$

In particular, if we let $\mathcal{H}'$ be the Hilbert space with same underlying vector space as $\mathcal{H}$ and equipped with an equivalent inner product and we let S be the identity map, then we obtain:

Corollary 5.2.8. *Neither $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ nor its topology depend on the choice of the inner product of $\mathcal{H}$.*

5.3. The Dixmier Trace

In this section, we shall construct the Dixmier trace as a trace on the Banach ideal $\mathcal{L}^{(1,\infty)}$. It will occur from the analysis of the logarithmic divergency of the partial traces,

$$\sigma_N(T) = \sum_{n<N} \mu_n(T), \qquad T \in \mathcal{K}, \quad N \in \mathbb{N}.$$

The first step is to extend the definition of $\sigma_N(T)$ to non-integer values of N. To this end recall that by Proposition 4.9.2 we have

$$\sigma_N(S+T) \leq \sigma_N(S) + \sigma_N(T) \qquad \forall S, T \in \mathcal{K}. \tag{5.9}$$

Lemma 5.3.1. *Let $N \in \mathbb{N}$. Then, for all $T \in \mathcal{K}$,*

$$\sigma_N(T) = \inf\{\|x\|_1 + N\|y\|;\ (x,y) \in \mathcal{L}^1 \times \mathcal{K} \text{ and } x + y = T\}. \tag{5.10}$$

PROOF. Let $T \in \mathcal{K}$. Let $(x, y) \in \mathcal{L}^1 \times \mathcal{K}$ be such that $x + y = T$. Using (5.9) we get

$$\sigma_N(T) = \sigma_N(x + y) \leq \sigma_N(x) + \sigma_N(y).$$

Notice that by (4.93) we have $\sigma_N(y) \leq N\|y\|$. Moreover,

$$\sigma_N(x) = \sum_{n<N} \mu_n(x) \leq \sum_{n\geq 0} \mu_n(x) = \|x\|_1.$$

Therefore, we have

$$\sigma_N(T) \leq \|x\|_1 + N\|y\|.$$

It then follows that

$$\sigma_N(T) \leq \inf\{\|x\|_1 + N\|y\|;\ (x, y) \in \mathcal{L}^1 \times \mathcal{K} \text{ and } x + y = T\}. \tag{5.11}$$

Let $T = U|T|$ be the polar decomposition of T, and let $(\xi_n)_{n\geq 0}$ be an orthonormal family such that $|T|\xi_n = \mu_n(T)\xi_n$ for all $n \in \mathbb{N}_0$. Denote by Π_N the orthogonal projection onto the span of $\xi_0, \ldots, \xi_{N-1}$. Define

$$x_N := (|T| - \mu_N(T))\Pi_N \qquad \text{and} \qquad y_N := \mu_N(T)\Pi_N + |T|(1 - \Pi_N). \tag{5.12}$$

Notice that $x_N + y_N = |T|$. Thus,

$$T = U|T| = Ux_N + Uy_N. \tag{5.13}$$

Notice also that, as Ux_N has finite-rank, this is a trace-class operator. In addition, as $\Pi_N = \sum_{n<N} \xi_n \otimes \xi_n^*$ and, by (4.15), $|T| = \sum_{n\geq 0} \mu_n(T)\xi_n \otimes \xi_n^*$, we have

$$x_N = \sum_{n<N} (\mu_n(T) - \mu_N(T))\xi_n \otimes \xi_n^*, \tag{5.14}$$

$$y_N = \sum_{n<N} \mu_N(T)\xi_n \otimes \xi_n^* + \sum_{n\geq N} \mu_n(T)\xi_n \otimes \xi_n^*. \tag{5.15}$$

It follows from (5.14) and the min-max principle that $\mu_n(x_N)$ is equal to $\mu_n(T) - \mu_N(T)$ for $n < N$ and is zero for $n \geq N$. Thus,

$$\|x_N\|_1 = \sum_{n\geq 0} \mu_n(x_N) = \sum_{n<N} (\mu_n(T) - \mu_N(T)) = \sigma_N(T) - N\mu_N(T).$$

Since by Proposition 3.1.8 $\|U\| \leq 1$, combining this with (4.20) gives

$$\|Ux_N\|_1 \leq \|U\|\|x_N\|_1 \leq \sigma_N(T) - N\mu_N(T). \tag{5.16}$$

As for y_N, it follows from (5.15) that y_N is a positive operator whose greatest eigenvalue is $\mu_N(T)$, and so using the min-max principle we get

$$\|y_N\| = \mu_0(y_N) = \mu_N(T),$$

and hence

$$\|Uy_N\| \leq \|U\|\|y_N\| \leq \mu_N(T). \tag{5.17}$$

Combining this with (5.16) gives

$$\|Ux_N\|_1 + N\|Uy_N\| \leq \sigma_N(T) - N\mu_N(T) + N\mu_N(T) = \sigma_N(T).$$

In view of (5.13) it follows that

$$\sigma_N(T) \geq \inf\{\|x\|_1 + N\|y\|;\ (x, y) \in \mathcal{L}^1 \times \mathcal{K} \text{ and } x + y = T\}.$$

Combining this with (5.11) then proves the lemma. $\square$

The previous lemma allow us to extend the definition of σ_N to non-integer values of N.

DEFINITION 5.3.2. *Let $T \in \mathcal{K}$. Then, for any $\lambda \geq 0$, we define*

$$\sigma_\lambda(T) = \inf\{\|x\|_1 + N\|y\|;\ (x,y) \in \mathcal{L}^1 \times \mathcal{K} \text{ and } x + y = T\}.$$

Lemma 5.3.1 shows that, when λ is an integer, the above definition of $\sigma_\lambda(T)$ agrees with that given in (4.90).

LEMMA 5.3.3. *Let $T \in \mathcal{K}$. Then*

(i) *The function $\lambda \to \sigma_\lambda(T)$ is concave.*

(ii) *For any $\lambda \geq 0$, we have*

$$\sigma_\lambda(T) = \sigma_N(T) + \alpha\mu_N(T), \tag{5.18}$$

$$= (1-\alpha)\sigma_N(T) + \alpha\sigma_{N+1}(T), \tag{5.19}$$

where we have set $N = [\lambda]$ and $\alpha = \lambda - [\lambda]$.

PROOF. Let $\lambda, \mu \in [0,\infty)$ and let $\alpha \in [0,1]$. For any $(x,y) \in \mathcal{L}^1 \times \mathcal{K}$ be such that $x + y = T$ we have

$$\begin{aligned}\|x\|_1 + (\alpha\lambda + (1-\alpha)\mu)\|y\| &= \alpha(\|x\|_1 + \lambda\|y\|) + (1-\alpha)(\|x\|_1 + \mu\|y\|)\\ &\geq \alpha\sigma_\lambda(T) + (1-\alpha)\sigma_\mu(T).\end{aligned}$$

Thus,

$$\sigma_{\alpha\lambda+(1-\alpha)\mu}(T) \geq \alpha\sigma_\lambda(T) + (1-\alpha)\sigma_\mu(T),$$

which shows that the function $\lambda \to \sigma_\lambda(T)$ is concave.

Let $\lambda \in [0,\infty)$ and set $N = [\lambda]$ and $\alpha = \lambda - N$. As $\sigma_{N+1}(T) = \sigma_N(T) + \mu_N(T)$, we have

$$\begin{aligned}(1-\alpha)\sigma_N(T) + \alpha\sigma_{N+1}(T) &= (1-\alpha)\sigma_N(T) + \alpha(\sigma_N(T) + \mu_N(T))\\ &= \sigma_N(T) + \alpha\mu_N(T).\end{aligned} \tag{5.20}$$

Notice also that $\lambda = (1-\alpha)N + \alpha(N+1)$. As the function $\lambda \to \sigma_\lambda(T)$ is concave, it follows that

$$\sigma_\lambda(T) \geq (1-\alpha)\sigma_N(T) + \alpha\sigma_{N+1}(T). \tag{5.21}$$

Let $T = U|T|$ be the polar decomposition of T and let x_N and y_N be as in (5.12), so that $T = Ux_N + Uy_N$ and, as in (5.16) and (5.17), we have

$$\|Ux_N\|_1 \leq \sigma_N(T) - N\mu_N(T) \quad \text{and} \quad \|Uy_N\| \leq \mu_N(T).$$

Then

$$\sigma_\lambda(T) \leq \|Ux_N\|_1 + \lambda\|y_N\| \leq \sigma_N(T) + (\lambda - N)\mu_N(T).$$

Combining this with (5.20) and (5.21) proves (5.18) and (5.19). The proof is complete. $\square$

REMARK 5.3.4. The equality (5.18) can be rewritten as

$$\sigma_\lambda(T) = \int_0^\lambda \mu_{[u]}(T)du.$$

Thus, when T is positive, $\sigma_\lambda(T)$ can be seen as the cut-off by λ of the trace,

$$\operatorname{Trace} T = \int_0^\infty \mu_{[u]}(T)du.$$

Remark 5.3.5. It follows from (5.19) that the function $\lambda \to \sigma_\lambda(T)$ is affine between $\sigma_N(T)$ and $\sigma_{N+1}(T)$, so this function agrees with the affine interpolation of the $\sigma_N(T)$'s.

Remark 5.3.6. Since the σ_N's are norms on $\mathcal{K}$, it follows from (5.19) that the σ_λ's too are norms. Thus, for any $\lambda \geq 0$, we have

$$\sigma_\lambda(cT) = |c|\sigma_\lambda(T) \qquad \forall T \in \mathcal{K}\ \forall c \in \mathbb{C}, \tag{5.22}$$

$$\sigma_\lambda(S+T) \leq \sigma_\lambda(S) + \sigma_\lambda(T) \qquad \forall S, T \in \mathcal{K}. \tag{5.23}$$

Lemma 5.3.7. *Let T_1 and T_2 be positive compact operators. Then*

$$\sigma_{\lambda_1+\lambda_2}(T_1+T_2) \geq \sigma_{\lambda_1}(T_1) + \sigma_{\lambda_2}(T_2) \qquad \forall \lambda_j \geq 0. \tag{5.24}$$

Proof. For $j=1,2$ let $N_j \in \mathbb{N}$ and let us show that

$$\sigma_{N_1+N_2}(T_1+T_2) \geq \sigma_{N_1}(T_1) + \sigma_{N_2}(T_2). \tag{5.25}$$

For $j=1,2$ let E_j be a subspace of $\mathcal{H}$ of dimension N_j, and let E be a subspace of dimension N_1+N_1 containing E_1 and E_2. Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$ such that $\xi_0, \ldots, \xi_{N_1-1}$ span E_1 and $\xi_0, \ldots, \xi_{N-1}$ span E. Then

$$\mathrm{Trace}(T_1\Pi_{E_1}) = \sum_{n\geq 0}\langle \xi_n, T_1\Pi_{E_1}\xi_n\rangle = \sum_{n<N_1}\langle \xi_n, T_1\xi_n\rangle.$$

Similarly, we have

$$\mathrm{Trace}(T_1\Pi_E) = \sum_{n<N}\langle \xi_n, T_1\xi_n\rangle.$$

As T_1 is positive $\langle \xi_n, T_1\xi_n\rangle \geq 0$ for all $n \in \mathbb{N}_0$, and hence

$$\mathrm{Trace}(T_1\Pi_{E_1}) \leq \mathrm{Trace}(T_1\Pi_E).$$

Similarly $\mathrm{Trace}(T_2\Pi_{E_2}) \leq \mathrm{Trace}(T_2\Pi_E)$, and hence using (4.95) we get

$$\mathrm{Trace}(T_1\Pi_{E_1}) + \mathrm{Trace}(T_2\Pi_{E_2}) \leq \mathrm{Trace}\left((T_1+T_2)\Pi_E\right) \leq \sigma_{N_1+N_2}(T_1+T_2).$$

Thanks to (4.95) taking the supremum of $\mathrm{Trace}(T_1\Pi_{E_1}) + \mathrm{Trace}(T_2\Pi_{E_2})$ over all subspaces E_1 of dimension N_1 and all subspaces E_2 of dimension N_2 gives (5.25).

Now, for $j=1,2$ let λ_j be a non-negative real number and set $N_j = [\lambda_j]$ and $\alpha_j = \lambda_j - N_j$. In addition, define $\lambda = \lambda_1 + \lambda_2$ and set $N = [\lambda]$ and $\alpha = \lambda - N$. Notice that either $N = N_1 + N_2$ or $N = N_1 + N_2 + 1$.

Assume that $N = N_1 + N_2$. Then $\alpha = \alpha_1 + \alpha_2$, and hence by (5.19) we have

$$\begin{aligned}\sigma_\lambda(T_1+T_2) &= (1-\alpha_1-\alpha_2)\sigma_N(T_1+T_2) + (\alpha_1+\alpha_2)\sigma_{N+1}(T_1+T_2)\\ &= (1-\alpha_1-\alpha_2)\sigma_{N_1+N_2}(T_1+T_2) + (\alpha_1+\alpha_2)\sigma_{N_1+N_2+1}(T_1+T_2).\end{aligned} \tag{5.26}$$

By (5.25) we have

$$\sigma_{N_1+N_2}(T_1+T_2) \geq \sigma_{N_1}(T_1) + \sigma_{N_2}(T_2), \tag{5.27}$$

$$\sigma_{N_1+N_2+1}(T_1+T_2) \geq \sigma_{N_1+1}(T_1) + \sigma_{N_2}(T_2), \tag{5.28}$$

$$\sigma_{N_1+N_2+1}(T_1+T_2) \geq \sigma_{N_1}(T_1) + \sigma_{N_2+1}(T_2). \tag{5.29}$$

Combining this with (5.19) and (5.26) we get

$$\begin{aligned}
\sigma_\lambda(T_1+T_2) \geq{}& (1-\alpha_1-\alpha_2)(\sigma_{N_1}(T_1)+\sigma_{N_2}(T_2)) \\
&+\alpha_1(\sigma_{N_1+1}(T_1)+\sigma_{N_2}(T_2))+\alpha_2(\sigma_{N_1}(T_1)+\sigma_{N_2+1}(T_2)), \\
\geq{}& (1-\alpha_1)\sigma_{N_1}(T_1)+\alpha_1\sigma_{N_1+1}(T_1)+(1-\alpha_2)\sigma_{N_2}(T_2)+\alpha_2\sigma_{N_2+1}(T_2) \\
\geq{}& \sigma_{\lambda_1}(T_1)+\sigma_{\lambda_2}(T_2). \qquad (5.30)
\end{aligned}$$

Suppose now that $N=N_1+N_2+1$. Then $\alpha=\alpha_1+\alpha_2-1$, and hence (5.19) gives

$$\begin{aligned}
&\sigma_\lambda(T_1+T_2)=(2-\alpha_1-\alpha_2)\sigma_{N+1}(T_1+T_2)+(\alpha_1+\alpha_2-1)\sigma_{N+2}(T_1+T_2) \\
&=[(1-\alpha_1)+(1-\alpha_2)]\sigma_{N_1+N_2+1}(T_1+T_2)+(\alpha_1+\alpha_2-1)\sigma_{N_1+1+N_2+1}(T_1+T_2).
\end{aligned}$$

By (5.25) we have

$$\sigma_{N_1+1+N_2+1}(T_1+T_2)\geq\sigma_{N_1+1}(T_1)+\sigma_{N_2+1}(T_2),$$

Combining this with (5.19) and (5.27)–(5.29) we get

$$\begin{aligned}
\sigma_\lambda(T_1+T_2) \geq{}&(1-\alpha_1)(\sigma_{N_1}(T_1)+\sigma_{N_2+1}(T_2))+(1-\alpha_2)(\sigma_{N_1+1}(T_1)+\sigma_{N_2}(T_2)) \\
&+(\alpha_1+\alpha_2-1)(\sigma_{N_1+1}(T_1)+\sigma_{N_2+1}(T_2)), \\
\geq{}&(1-\alpha_1)\sigma_{N_1}(T_1)+\alpha_1\sigma_{N_1+1}(T_1)+(1-\alpha_2)\sigma_{N_2}(T_2)+\alpha_2\sigma_{N_2+1}(T_2), \\
\geq{}&\sigma_{\lambda_1}(T_1)+\sigma_{\lambda_2}(T_2).
\end{aligned}$$

The proof is complete. □

In the sequel we denote by $\mathcal{L}_+^{(1,\infty)}$ the cone of positive operators of $\mathcal{L}^{(1,\infty)}$.

Let $T\in\mathcal{L}_+^{(1,\infty)}$. For $\lambda\geq e$ we define

$$\tau_\lambda(T):=\frac{1}{\log\lambda}\int_e^\lambda\frac{\sigma_u(T)}{\log u}\frac{du}{u}. \qquad (5.31)$$

In other words, the function $\lambda\to\tau_\lambda(T)$ is the Cesāro mean of $\frac{\sigma_\lambda(T)}{\log\lambda}$ with respect to the Haar measure $\frac{du}{u}$ of the multiplicative group $\mathbb{R}_+^*$.

Lemma 5.3.8. *We have*

$$\sigma_\lambda(T)\leq 2\|T\|_{(1,\infty)}\log\lambda \qquad \forall\lambda\geq 2. \qquad (5.32)$$

$$0\leq\tau_\lambda(T)\leq 2\|T\|_{(1,\infty)} \qquad \forall\lambda\geq e. \qquad (5.33)$$

Proof. Let $\lambda\in[2,\infty)$ and set $N=[\lambda]$. Then

$$\frac{\sigma_\lambda(T)}{\log\lambda}\leq\frac{\sigma_{N+1}}{\log N}\leq\frac{\log(N+1)}{\log N}.\frac{\sigma_{N+1}(T)}{\log(N+1)}\leq 2\|T\|_{(1,\infty)}$$

where we have used the fact that $\sup_{u\geq 2}\frac{\log(u+1)}{\log u}=\frac{\log 2}{\log 3}\leq 2$. It follows from this, that for all $\lambda\geq e$, we have

$$\tau_\lambda(T)=\frac{1}{\log\lambda}\int_e^\lambda\frac{\sigma_u(T)}{\log u}\frac{du}{u}\leq\frac{1}{\log\lambda}\int_e^\lambda 2\|T\|_{(1,\infty)}\frac{du}{u}\leq 2\|T\|_{(1,\infty)}.$$

The lemma is proved. □

In particular, this lemma shows that the function $\lambda \to \tau_\lambda(T)$ is contained in $C_b[e,\infty)$, the C^*-algebra of bounded continuous functions on the interval $[e,\infty)$.

The interest of considering the above Cesāro mean stems from the fact that, while $\frac{\sigma_\lambda(T)}{\log\lambda}$ is not additive with respect to T, its Cesāro mean is asymptotically additive as $\lambda\to\infty$. Namely, we have:

Lemma 5.3.9. *Let T_1 and T_2 be in $\mathcal{L}_+^{(1,\infty)}$. Then, for all $\lambda \geq e$,*

$$|\tau_\lambda(T_1+T_2)-\tau_\lambda(T_1)-\tau_\lambda(T_2)| \leq 2\|T_1+T_2\|_{(1,\infty)}\frac{(\log(\log\lambda)+2)}{\log\lambda}.$$

Proof. First, it follows from (5.23) that

$$\tau_\lambda(T_1+T_2)\leq \tau_\lambda(T_1)+\tau_\lambda(T_2) \qquad \forall\lambda\geq e. \tag{5.34}$$

It remains to find an upper bound for $\tau_\lambda(T_1)+\tau_\lambda(T_2)-\tau_\lambda(T_1+T_2)$. By Lemma 5.3.7 we have

$$\sigma_u(T_1)+\sigma_u(T_2)\leq \sigma_{2u}(T_1+T_2) \qquad \forall u\geq e.$$

Thus,

$$\tau_\lambda(T_1)+\tau_\lambda(T_2)\leq \frac{1}{\log\lambda}\int_e^\lambda \frac{\sigma_{2u}(T_1+T_2)}{\log u}\frac{du}{u} = \frac{1}{\log\lambda}\int_{2e}^{2\lambda}\frac{\sigma_u(T_1+T_2)}{\log(\frac{u}{2})}\frac{du}{u}.$$

Since $\tau_\lambda(T_1+T_2)=\frac{1}{\log\lambda}\int_e^\lambda \frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}$ we deduce that

$$\begin{aligned}(\log\lambda)\{\tau_\lambda(T_1)+\tau_\lambda(T_2)-\tau_\lambda(T_1+T_2)\}\\ \leq \int_{2e}^{2\lambda}\frac{\sigma_u(T_1+T_2)}{\log(\frac{u}{2})}\frac{du}{u}-\int_e^\lambda\frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}=\delta+\delta',\end{aligned} \tag{5.35}$$

where we have let

$$\delta=\int_{2e}^{2\lambda}\sigma_u(T_1+T_2)\left(\frac{1}{\log(\frac{u}{2})}-\frac{1}{\log u}\right)\frac{du}{u},$$
$$\delta'=\int_{2e}^{2\lambda}\frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}-\int_e^\lambda\frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}.$$

Notice that $\frac{1}{\log(\frac{u}{2})}-\frac{1}{\log u}=\frac{\log 2}{\log u\log\frac{u}{2}}$. Since $\sigma_u(T_1+T_2)\leq 2\|T_1+T_2\|_{(1,\infty)}\log u$, we then see that

$$\delta\leq 2\|T_1+T_2\|_{(1,\infty)}\log 2\int_{2e}^{2\lambda}\frac{1}{\log\frac{u}{2}}\frac{du}{u}=2\|T_1+T_2\|_{(1,\infty)}(\log 2)\log(\log\lambda). \tag{5.36}$$

In addition, we have

$$\begin{aligned}\delta'=\int_{2e}^{e}\frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}+\int_e^{2\lambda}\frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}+\int_\lambda^e\frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}\\ \leq\int_\lambda^{2\lambda}\frac{\sigma_u(T_1+T_2)}{\log u}\frac{du}{u}.\end{aligned}$$

Since $\sigma_u(T_1+T_2)\leq 2\|T_1+T_2\|_{(1,\infty)}\log u$, it follows that

$$\delta'\leq 2\|T_1+T_2\|_{(1,\infty)}\int_\lambda^{2\lambda}\frac{du}{u}=\|T_1+T_2\|_{(1,\infty)}\int_1^2\frac{du}{u}=\|T_1+T_2\|_{(1,\infty)}\log 2.$$

Combining this with (5.34), (5.35) and (5.36) proves the lemma. $\square$

Next, recall that $C_0[e,\infty)$ is a closed two-sided ideal of $C_b[e,\infty)$. Therefore, the quotient

$$\mathcal{Q} = C_b[e,\infty)/C_0[e,\infty).$$

is a (commutative) C^*-algebra with respect to the quotient norm,

$$\|[f]\|_{\mathcal{Q}} = \inf_{g\in C_0[e,\infty)} \|f+g\|_\infty \qquad \forall f \in C_b[e,\infty),$$

where $[f]$ denotes the class of f in $\mathcal{Q}$. Notice also that

$$\|[f]\|_{\mathcal{Q}} \leq \|f\| \qquad \forall f \in C_b[e,\infty). \tag{5.37}$$

Notice also that $\mathcal{Q}$ is a commutative C^*-algebra.

We define a map $\tau : \mathcal{L}_+^{(1,\infty)} \to \mathcal{Q}$ by

$$\tau(T) = \text{class of } \lambda \to \tau_\lambda(T) \text{ in } \mathcal{Q} \qquad \forall T \in \mathcal{L}_+^{(1,\infty)}.$$

Lemma 5.3.10. *The following hold.*

$$\tau(cT) = c\tau(T) \qquad \forall T \in \mathcal{L}_+^{(1,\infty)}\ \forall c \geq 0. \tag{5.38}$$

$$\tau(T_1+T_2) = \tau(T_1)+\tau(T_2) \qquad \forall T_j \in \mathcal{L}_+^{(1,\infty)}. \tag{5.39}$$

$$\|\tau(T)\| \leq 2\|T\|_{(1,\infty)} \qquad \forall T \in \mathcal{L}_+^{(1,\infty)}, \tag{5.40}$$

$$\tau(U^*TU) = \tau(T) \qquad \forall T \in \mathcal{L}_+^{(1,\infty)}\ \forall U \in \mathcal{L}(\mathcal{H}),\ U \text{ unitary.} \tag{5.41}$$

Proof. Let T_1 and T_2 be in $\mathcal{L}_+^{(1,\infty)}$. It follows from Lemma 5.3.9 that

$$\tau_\lambda(T_1+T_2) = \tau_\lambda(T_1)+\tau_\lambda(T_2) \mod C_0[e,\infty),$$

and hence $\tau(T_1+T_2) = \tau(T_1)+\tau(T_2)$.

Let $T \in \mathcal{L}_+^{(1,\infty)}$. Then by (5.33) and (5.37) we have

$$\|\tau(T)\|_{\mathcal{Q}} \leq \sup_{\lambda\geq e} \tau_\lambda(T) \leq 2\|T\|_{(1,\infty)}.$$

In addition, for any $c \in [0,\infty)$, it follows from (5.22) and (5.31) that that $\tau_\lambda(cT) = c\tau_\lambda(T)$ for all $\lambda \geq e$, and hence $\tau(cT) = c\tau(T)$.

Let $U \in \mathcal{L}(\mathcal{H})$ be unitary. Then $U^*TU \in \mathcal{L}_+^{(1,\infty)}$ and by (4.6) $\mu_n(U^*TU) = \mu_n(T)$ for all $n \in \mathbb{N}_0$, and hence $\sigma_N(U^*TU) = \sigma_N(T)$ for all $N \in \mathbb{N}$. Combining this with (5.19) and (5.31) then shows that $\sigma_\lambda(U^*TU) = \sigma_\lambda(T)$ and $\tau_\lambda(U^*TU) = \tau_\lambda(T)$ for all $\lambda \geq e$, and hence $\tau(U^*TU) = \tau(T)$. The proof is complete. □

In the sequel we say that $x \in \mathcal{Q}$ is *positive* if $x = y^*y$ for some $y \in \mathcal{Q}$. Thus the condition (i) just says that a state on $\mathcal{Q}$ must take non-negative real values on positive elements of $\mathcal{Q}$.

It is not difficult to check that $x \in \mathbb{Q}$ if $x = [f]$ for some non-negative function $f \in C_b[0,\infty)$. In particular, for any $T \in \mathcal{L}_+^{(1,\infty)}$ the class $\tau(T)$ is a positive element of $\mathcal{Q}$.

We shall also write $x_1 \leq x_2$ to mean that $x_2 - x_1$ is positive. Notice that if $x \in \mathcal{Q}$ is selfadjoint, then $-\|x\|_{\mathcal{Q}}.1 \leq x \leq \|x\|_{\mathcal{Q}}.1$, for the functions $\|x\|_{\mathcal{Q}} \pm t$ are non-negative on $\operatorname{Sp} x \subset [0, \|x\|_{\mathcal{Q}}]$.

Definition 5.3.11. *A* state *on $\mathcal{Q}$ is a linear map $\omega : \mathcal{Q} \to \mathbb{C}$ such that*

(i) *ω is* positive*, i.e., $\omega(x) \geq 0$ if x is positive;*
(ii) *ω is* normalized*, i.e., $\omega(1) = 1$.*

We denote by $\Omega(\mathcal{Q})$ the set of states on $\mathcal{Q}$

EXAMPLE 5.3.12. Any character $\chi : \mathcal{Q} \to \mathbb{C}$ is a state on $\mathcal{Q}$. Thus, there are plenty of states on $\mathcal{Q}$.

LEMMA 5.3.13. *Let ω be a state on $\mathcal{Q}$. Then*

(i) $\omega(x^*) = \overline{\omega(x)}$ *for all* $x \in \mathcal{Q}$.

(ii) ω *is a continuous linear form on* $\mathcal{Q}$. *In fact,*

$$|\omega(x)| \leq \|x\|_{\mathcal{Q}} \qquad \forall x \in \mathcal{Q}. \tag{5.42}$$

PROOF. Let $x \in \mathcal{Q}$ be selfadjoint. As above-mentioned $\|x\|_{\mathcal{Q}}.1 \pm x$ are positive, and hence using the positivity of ω and the fact that $\omega(1) = 1$ we get

$$0 \leq \omega(\|x\|_{\mathcal{Q}} \pm x) = \|x\|_{\mathcal{Q}}\omega(1) \pm \omega(x) = \|x\|_{\mathcal{Q}} \pm \omega(x).$$

Thus,

$$|\omega(x)| \leq \|x\|_{\mathcal{Q}} \qquad \forall x \in \mathcal{Q},\ x \text{ selfadjoint}. \tag{5.43}$$

Assume now that x is any element of $\mathcal{Q}$ and let us write $x = x_1 + ix_2$ with $x_1 = \frac{x+x^*}{2}$ and $x_2 = \frac{x-x^*}{2i}$. Then

$$\omega(x) = \omega(x_1) + i\omega(x_2).$$

As x_1 and x_2 are selfadjoint, both $\omega(x_1)$ and $\omega(x_2)$ are real numbers, and hence

$$\omega(x^*) = \omega(x_1 - ix_2) = \omega(x_1) - i\omega(x_2) = \overline{\omega(x_1) + i\omega(x_2)} = \overline{\omega(x)}.$$

In particular,

$$\Re(\omega(x)) = \omega(x_1) = \omega\left(\frac{x+x^*}{2}\right). \tag{5.44}$$

It remains to show that $|\omega(x)| \leq \|x\|_{\mathcal{Q}}$. Since this inequality is obvious when $\omega(x) = 0$, we may assume $\omega(x) \neq 0$. Set $\alpha = \frac{\overline{\omega(x)}}{|\omega(x)|}$. Then $\omega(\alpha x) = \alpha\omega(x) = |\omega(x)|$, and hence using (5.44) we get

$$|\omega(x)| = \Re(\omega(\alpha x)) = \omega\left(\frac{(\alpha x) + (\alpha x)^*}{2}\right).$$

Since $\frac{1}{2}\left((\alpha x) + (\alpha x)^*\right)$ is selfadjoint, using (5.43) we obtain

$$|\omega(x)| \leq \left\|\frac{(\alpha x) + (\alpha x)^*}{2}\right\|_{\mathcal{Q}} \leq \frac{1}{2}(\|\alpha x\|_{\mathcal{Q}} + \|\overline{\alpha}x^*\|_{\mathcal{Q}}) = \|x\|_{\mathcal{Q}},$$

completing the proof.

□

The states on $\mathcal{Q}$ should be thought of as generalizations of the limit at ∞ of a function on $[e, \infty)$, very much in the same way Banach limits are generalizations of the limit of sequence. More precisely, we have

PROPOSITION 5.3.14.

(1) $\Omega(\mathcal{Q})$ *separates the points of* $\mathcal{Q}$.

(2) *Let* $f \in C_b[e, \infty)$. *Then*

$$\lim_{\lambda \to \infty} f(\lambda) = L \Longleftrightarrow \omega([f]) = L \quad \forall \omega \in \Omega(\mathcal{Q}).$$

Proof. Let $x \in \mathcal{Q} \setminus 0$. Denote by $\operatorname{Sp} Q$ the spectrum of $\mathcal{Q}$, i.e., the set of characters on $\mathcal{Q}$ endowed with the weak topology of $\mathcal{Q}^*$. Since $\mathcal{Q}$ is a commutative C^*-algebra, the injectivity of the Gel'fand transform $G_{\mathcal{Q}} : \mathcal{Q} \to C(\operatorname{Sp} \mathcal{Q})$, implies that there exists $\chi \in \operatorname{Sp} Q$ such that $\chi(x) \neq 0$. Since χ is a state, this shows that $\Omega(\mathcal{Q})$ separates the points of $\mathcal{Q}$.

Let $f \in C_b[e, \infty)$. Then

$$\lim_{\lambda \to \infty} f(\lambda) = L \Longleftrightarrow f = L \bmod C_0[e, \infty) \Longleftrightarrow [f] = L \text{ in } \mathcal{Q}.$$

Since $\Omega(\mathcal{Q})$ separates the point of $\mathcal{Q}$ and $\omega(L) = L\omega(1) = L$, it follows that

$$\lim_{\lambda \to \infty} f(\lambda) = L \Longleftrightarrow \omega([f]) = \omega(L)\ \forall \omega \in \Omega(\mathcal{Q}) \Longleftrightarrow \omega([f]) = L\ \forall \omega \in \Omega(\mathcal{Q}),$$

completing the proof. $\square$

Let ω be a state on $\mathcal{Q}$. We define a functional $\operatorname{Tr}_\omega : \mathcal{L}_+^{(1,\infty)} \to [0, \infty)$ by letting

$$\operatorname{Tr}_\omega T := \omega(\tau(T)) \qquad \forall T \in \mathcal{L}_+^{(1,\infty)}.$$

Notice that as ω is linear and τ

We shall now extend Tr_ω into a linear form on $\mathcal{L}^{(1,\infty)}$. To this end we recall the following result, a proof of which can be found in Chapter 4.

Lemma 5.3.15. *Let $\mathcal{I}$ be a two-sided ideal of $\mathcal{L}(\mathcal{H})$. Then*

(i) *For any $T \in \mathcal{L}(\mathcal{H})$, we have*

$$T \in \mathcal{I} \Longrightarrow |T| \in \mathcal{I} \longrightarrow T^* \in \mathcal{I}. \tag{5.45}$$

(ii) *Any $T \in \mathcal{I}$ can be written in the form,*

$$T = (T_1 - T_2) + i(T_3 - T_4) \qquad \text{with } T_j \in \mathcal{I} \cap \mathcal{L}(\mathcal{H})_+. \tag{5.46}$$

Proposition 5.3.16. *The functional Tr_ω uniquely extends to a linear form*

$$\operatorname{Tr}_\omega : \mathcal{L}^{(1,\infty)} \longrightarrow \mathbb{C}.$$

Proof. Thanks to (5.39) and the linearity of ω we have

$$\operatorname{Tr}_\omega(T_1 + T_2) = \operatorname{Tr}_\omega T_1 + \operatorname{Tr}_\omega T_2 \qquad \forall T_j \in \mathcal{L}_+^{(1,\infty)}. \tag{5.47}$$

Let $T \in \mathcal{L}^{(1,\infty)}$. Thanks to (5.46) we can write $T = T_1 - T_2 + i(T_3 - T_4)$ with $T_j \in \mathcal{L}_+^{(1,\infty)}$. Let $T = T_1' - T_2' + i(T_3' - T_4')$ be another such decomposition with $T_j' \in \mathcal{L}_+^{(1,\infty)}$. Observe that

$$\frac{1}{2}(T + T^*) = T_1 - T_2 = T_1' - T_2' \quad \text{and} \quad \frac{1}{2i}(T + T^*) = T_3 - T_4 = T_3' - T_4',$$

and hence $T_1 + T_2' = T_1' + T_2$ and $T_3 + T_4' = T_3' + T_4$. Therefore, using (5.47) we get

$$\begin{aligned} \operatorname{Tr}_\omega(T_1) + \operatorname{Tr}_\omega(T_2') &= \operatorname{Tr}_\omega(T_1') + \operatorname{Tr}_\omega(T_2'), \\ \operatorname{Tr}_\omega(T_3) + \operatorname{Tr}_\omega(T_4') &= \operatorname{Tr}_\omega(T_3') + \operatorname{Tr}_\omega(T_4'). \end{aligned}$$

Thus,

$$\begin{aligned} &\operatorname{Tr}_\omega(T_1) - \operatorname{Tr}_\omega(T_2) + i(\operatorname{Tr}_\omega(T_3) - \operatorname{Tr}_\omega(T_4)) \\ &\qquad = \operatorname{Tr}_\omega(T_1') - \operatorname{Tr}_\omega(T_2') + i(\operatorname{Tr}_\omega(T_3') - \operatorname{Tr}_\omega(T_4')). \end{aligned}$$

This shows that the value of the right-hand side above is the same for any decomposition $T = T_1 - T_2 + i(T_3 - T_4)$ with $T_j \in \mathcal{L}_+^{(1,\infty)}$, and hence depends only on T. We then define

$$\operatorname{Tr}_\omega(T) := \operatorname{Tr}_\omega(T_1) - \operatorname{Tr}_\omega(T_2) + i(\operatorname{Tr}_\omega(T_3) - \operatorname{Tr}_\omega(T_4)), \tag{5.48}$$

where the T_j's are any operators in $\mathcal{L}_+^{(1,\infty)}$ such that $T = T_1 - T_2 + i(T_3 - T_4)$.

Let $S \in \mathcal{L}^{(1,\infty)}$ and let us write $S = S_1 - S_2 + i(S_3 - S_4)$ with $S_j \in \mathcal{L}_+^{(1,\infty)}$. Then $S + T = (S_1 + T_1) - (S_2 + T_2) + i\,((S_3 + T_3) - (S_4 + T_4))$. Observe that each operator $S_j + T_j$ is in $\mathcal{L}^{(1,\infty)}$ and is positive by Corollary 3.1.3. Therefore, using (5.48) and (5.47) we get

$$\begin{aligned}(5.49)\quad &\operatorname{Tr}_\omega(S+T) = \operatorname{Tr}_\omega(S_1+T_1) - \operatorname{Tr}_\omega(S_2+T_2) + i(\operatorname{Tr}_\omega(S_3+T_3) - \operatorname{Tr}_\omega(S_4+T_4))\\ &= \operatorname{Tr}_\omega(S_1) - \operatorname{Tr}_\omega(S_2) + i(\operatorname{Tr}_\omega(S_3) - \operatorname{Tr}_\omega(S_4)) + \operatorname{Tr}_\omega(T_1) - \operatorname{Tr}_\omega(T_2) + i(\operatorname{Tr}_\omega(T_3) - \operatorname{Tr}_\omega(T_4))\\ &= \operatorname{Tr}_\omega S + \operatorname{Tr}_\omega T,\end{aligned}$$

showing that Tr_ω is additive on $\mathcal{L}^{(1,\infty)}$.

Next, combining (5.38) with the linearity of ω we get

$$\operatorname{Tr}_\omega(\lambda T) = \lambda \operatorname{Tr}_\omega T \qquad \forall T \in \mathcal{L}_+^{(1,\infty)}\ \forall \lambda \geq 0. \tag{5.50}$$

Let $T \in \mathcal{L}^{(1,\infty)}$ and let us write $T = T_1 - T_2 + i(T_3 - T_4)$ with $T_j \in \mathcal{L}_+^{(1,\infty)}$. Let λ be a non-negative real number. Then $\lambda T = \lambda T_1 - \lambda T_2 + i(\lambda T_3 - \lambda T_4)$. Since each operator λT_j is positive, using (5.48) and (5.50) we get

$$\begin{aligned}(5.51)\quad \operatorname{Tr}_\omega(\lambda T) &= \operatorname{Tr}_\omega(\lambda T_1) - \operatorname{Tr}_\omega(\lambda T_2) + i(\operatorname{Tr}_\omega(\lambda T_3) - \operatorname{Tr}_\omega(\lambda T_4))\\ &= \lambda\left(\operatorname{Tr}_\omega(T_1) - \operatorname{Tr}_\omega(T_2) + i(\operatorname{Tr}_\omega(T_3) - \operatorname{Tr}_\omega(T_4))\right) = \lambda \operatorname{Tr}_\omega(T).\end{aligned}$$

Notice also that $-T = (T_2 - T_1) + i(T_4 - T_3)$ and $iT = (T_4 - T_3) + i(T_1 - T_2)$, and hence from (5.48) we get

$$\begin{aligned}\operatorname{Tr}_\omega(-T) &= \operatorname{Tr}_\omega(T_2) - \operatorname{Tr}_\omega(T_1) + i(\operatorname{Tr}_\omega(T_4) - \operatorname{Tr}_\omega(T_3)) = -\operatorname{Tr}_\omega(T),\\ \operatorname{Tr}_\omega(iT) &= \operatorname{Tr}_\omega(T_4) - \operatorname{Tr}_\omega(T_3) + i(\operatorname{Tr}_\omega(T_1) - \operatorname{Tr}_\omega(T_2)) = i\operatorname{Tr}_\omega(T).\end{aligned}$$

Let $\lambda \in \mathbb{C}$ and let us write $\lambda = \lambda_1 - \lambda_2 + i(\lambda_3 - \lambda_4)$ with $\lambda_j \geq 0$. Then combining the additivity of Tr_ω with (5.51)–(5.3) gives

$$\begin{aligned}\operatorname{Tr}_\omega(\lambda T) &= \operatorname{Tr}_\omega\left(\lambda_1 T + (-\lambda_2 T) + (i\lambda_3 T) + (-i\lambda_4 T)\right)\\ &= \operatorname{Tr}_\omega(\lambda_1 T) + \operatorname{Tr}_\omega(-\lambda_2 T) + \operatorname{Tr}_\omega(i\lambda_3 T) + \operatorname{Tr}_\omega(-\lambda_4 T)\\ &= (\lambda_1 - \lambda_2 + i\lambda_3 - i\lambda_4)\operatorname{Tr}_\omega T = \lambda \operatorname{Tr}_\omega T.\end{aligned}$$

Together with (5.49) this shows that Tr_ω is a linear map.

Finally, since (5.46) shows that $\mathcal{L}_+^{(1,\infty)}$ spans $\mathcal{L}^{(1,\infty)}$, any linear map that agrees with Tr_ω on $\mathcal{L}_+^{(1,\infty)}$ must agree with Tr_ω on all $\mathcal{L}^{(1,\infty)}$. Thus Tr_ω, as defined in (5.48), is the unique linear extension to $\mathcal{L}^{(1,\infty)}$ of $\operatorname{Tr}_{\omega|\mathcal{L}_+^{(1,\infty)}} = \omega \circ \tau$. The proof is complete. □

Proposition 5.3.17. *The following hold.*

(i) Tr_ω *is a continuous linear form on* $\mathcal{L}^{(1,\infty)}$. *In fact,*

$$|\operatorname{Tr}_\omega T| \leq 2\|T\|_{(1,\infty)} \qquad \forall T \in \mathcal{L}^{(1,\infty)}.$$

(ii) Tr_ω *is positive, i.e.,* $\mathrm{Tr}_\omega T \geq 0$ *for all* $T \in \mathcal{L}_+^{(1,\infty)}$.

(iii) $\mathrm{Tr}_\omega T^* = \overline{\mathrm{Tr}_\omega T}$ *for all* $T \in \mathcal{L}^{(1,\infty)}$.

(iii) Tr_ω *is a trace, that is,*

$$\mathrm{Tr}_\omega(TA) = \mathrm{Tr}_\omega(AT) \qquad \forall T \in \mathcal{L}^{(1,\infty)}\ \forall A \in \mathcal{L}(\mathcal{H}).$$

Proof. Let $T \in \mathcal{L}_+^{(1,\infty)}$. Then $\tau(T)$ is a positive element of $\mathcal{Q}$. As ω is a state, and hence is a positive linear form, it follows that $\mathrm{Tr}_\omega(T) = \omega(\tau(T)) \geq 0$. Thus Tr_ω is positive. In addition, combining (5.40) and (5.42) gives

$$|\mathrm{Tr}_\omega(T)| = \omega(\tau(T)) \leq \|\tau(T)\|_{\mathcal{Q}} \leq 2\|T\|_{(1,\infty)}. \tag{5.52}$$

Let $T \in \mathcal{L}^{(1,\infty)}$ be selfadjoint. Using (5.45) we see that $|T| \pm T$ is an element of $\mathcal{L}_+^{(1,\infty)}$. Thus,

$$0 \leq \mathrm{Tr}_\omega(|T| \pm T) = \mathrm{Tr}_\omega |T| \pm \mathrm{Tr}_\omega T.$$

Therefore, using (5.52) we get

$$|\mathrm{Tr}_\omega T| \leq \mathrm{Tr}_\omega |T| \leq 2\||T|\|_{(1,\infty)} = 2\|T\|_{(1,\infty)}.$$

Granted this, we then can argue as in the proof of Lemma 5.3.13 to show that, for all $T \in \mathcal{L}^{(1,\infty)}$, we have

$$\mathrm{Tr}_\omega T^* = \overline{\mathrm{Tr}_\omega T} \qquad \text{and} \qquad |\mathrm{Tr}_\omega T| \leq \|T\|_{(1,\infty)}.$$

Next, let $U \in \mathcal{L}(\mathcal{H})$ be unitary. Using (5.41) we see that, for any $T \in \mathcal{L}_+^{(1,\infty)}$,

$$\mathrm{Tr}_\omega(U^*TU) = \omega(\tau(U^*TU)) = \omega(\tau(T)) = \mathrm{Tr}_\omega(T).$$

Thus $T \rightarrow \mathrm{Tr}_\omega(U^*T)$ is a linear form on $\mathcal{L}^{(1,\infty)}$ that agrees with Tr_ω on $\mathcal{L}_+^{(1,\infty)}$, and so by Proposition 5.3.16 it agrees with Tr_ω on all $\mathcal{L}^{(1,\infty)}$. Thus,

$$\mathrm{Tr}_\omega(U^*TU) = \mathrm{Tr}_\omega(T) \qquad \forall T \in \mathcal{L}^{(1,\infty)}.$$

Upon changing T by UT this shows that

$$\mathrm{Tr}_\omega(TU) = \mathrm{Tr}_\omega(UT) \qquad \forall T \in \mathcal{L}^{(1,\infty)}.$$

Since by Lemma 4.2.10 the unitary operators span $\mathcal{L}(\mathcal{H})$, it follows that

$$\mathrm{Tr}_\omega(TA) = \mathrm{Tr}_\omega(AT) \qquad \forall T \in \mathcal{L}^{(1,\infty)}\ \forall A \in \mathcal{L}(\mathcal{H}),$$

that is, Tr_ω is a trace. The proof is complete. □

Definition 5.3.18. *The functional* Tr_ω *from Proposition 5.3.16 is called the Dixmier trace associated to* ω.

In the sequel we let $\mathcal{H}'$ be a separable Hilbert space.

Lemma 5.3.19. *Let* Φ *be a continuous* $*$*-homomorphism from* $\mathcal{L}(\mathcal{H})$ *to* $\mathcal{L}(\mathcal{H}')$ *satisfying (5.8). Then*

$$\mathrm{Tr}_{\omega,\mathcal{H}'}(\Phi(T)) = \mathrm{Tr}_{\omega,\mathcal{H}}(T) \qquad \forall T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}).$$

Proof. We know by Lemma 5.2.6 that ϕ induces an isometric linear map from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ to $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$. Moreover, as by assumption Φ is a homomorphism of C^*-algebras, it maps $\mathcal{L}(\mathcal{H})_+$ to $\mathcal{L}(\mathcal{H}')_+$. Thus Φ maps $\mathcal{L}^{(1,\infty)}(\mathcal{H})_+$ to $\mathcal{L}^{(1,\infty)}(\mathcal{H}')_+$.

Let $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}')_+$. As shown in proof of Lemma 5.2.6, the property (5.8) implies that $\sigma_u(\Phi(T)) = \sigma_u(T)$ for all $u \geq e$. Incidentally, $\tau_\lambda(\Phi(T)) = \tau_\lambda(T)$ for all $\lambda \geq e$, and hence $\tau(\Phi(T)) = \tau(T)$, which immediately implies that $\mathrm{Tr}_{\omega,\mathcal{H}'}(\Phi(T)) = \mathrm{Tr}_{\omega,\mathcal{H}}(T)$.

It follows from all this that $\operatorname{Tr}_{\omega,\mathcal{H}'} \circ \Phi$ is a well-defined linear map on $\mathcal{L}^{(1,\infty)}$ which agrees with $\operatorname{Tr}_{\omega,\mathcal{H}}$ on $\mathcal{L}^{(1,\infty)}(\mathcal{H})_+$, and so it follows from Proposition 5.3.16 that $\operatorname{Tr}_{\omega,\mathcal{H}'} \circ \Phi$ and $\operatorname{Tr}_{\omega,\mathcal{H}}$ agree on all $\mathcal{L}^{(1,\infty)}(\mathcal{H})$, proving the lemma. □

Proposition 5.3.20. *Let $S : \mathcal{H}' \to \mathcal{H}$ be a continuous linear isomorphism from $\mathcal{H}'$ onto $\mathcal{H}$. Then*

$$\operatorname{Tr}_{\omega,\mathcal{H}'}(S^{-1}TS) = \operatorname{Tr}_{\omega,\mathcal{H}} T \qquad \forall T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}).$$

Proof. Recall that by Proposition 5.2.7 the conjugation by S gives rise to a continuous isomorphism from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ onto $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$. In addition, set $|S| = (S^*S)^{\frac{1}{2}}$ and $U = S|S|^{-1}$. As shown in the proof of Proposition 5.2.7 $|S|$ is an invertible element of $\mathcal{L}(\mathcal{H}')$ and U is a unitary element of $\mathcal{L}(\mathcal{H}',\mathcal{H})$.

Notice that, as $\operatorname{Tr}_{\omega,\mathcal{H}'}$ is a trace on $\mathcal{L}^{(1,\infty)}(\mathcal{H}')$, for any $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}')$, we have

$$\operatorname{Tr}_\omega(|S|^{-1}T|S|) = \operatorname{Tr}_\omega(T|S|S^{-1}) = \operatorname{Tr}_\omega(T). \tag{5.53}$$

Furthermore, as U is unitary, Remark 4.1.4 tells us that the conjugation by U satisfies (5.8), and hence by Lemma 5.3.19 we have

$$\operatorname{Tr}_{\omega,\mathcal{H}'}(U^*TU) = \operatorname{Tr}_{\omega,\mathcal{H}}(T) \qquad \forall T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}). \tag{5.54}$$

Since $S = U|S|$, combining (5.53) and (5.54) we see that, for all $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$,

$$\operatorname{Tr}_{\omega,\mathcal{H}'}(S^{-1}TS) = \operatorname{Tr}_{\omega,\mathcal{H}'}\left(|S|^{-1}(U^*TU)|S|\right) = \operatorname{Tr}_{\omega,\mathcal{H}'} U^*T|U) = \operatorname{Tr}_{\omega,\mathcal{H}}(T),$$

proving the proposition. □

In particular, if we let $\mathcal{H}'$ be the hilbert space with same underlying vector space as $\mathcal{H}$ and equipped with an equivalent inner product, and we let S be the identity map, then we obtain

Corollary 5.3.21. *The Dixmier trace Tr_ω does not depend on the choice of the inner product of $\mathcal{H}$.*

Definition 5.3.22. *An operator $T \in \mathcal{L}^{(1,\infty)}$ is said to be measurable if the value of $\operatorname{Tr}_\omega T$ is independent of the choice of the state ω.*

We denote by $\mathcal{M}$ the subspace of $\mathcal{L}^{(1,\infty)}$ consisting of all measurable operators.

Definition 5.3.23. *The Dixmier trace of $T \in \mathcal{M}$, denoted $⨍ T$, is defined by*

$$⨍ T := \operatorname{Tr}_\omega T, \qquad \omega \text{ any state on } \mathcal{Q}.$$

We shall refer to the functional $⨍ : T \to ⨍ T$ as *the Dixmier trace* on $\mathcal{M}$.

Proposition 5.3.24. *The following hold.*

1. *$\mathcal{M}$ is a closed subspace of $\mathcal{L}^{(1,\infty)}$ on which the Dixmier trace $⨍$ is a continuous linear form.*
2. *Let $\mathcal{H}'$ be a (separable) Hilbert space and let Φ be a continuous $*$-homomorphism from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H}')$ satisfying (5.8). Then $\Phi(\mathcal{M}(\mathcal{H})) \subset \mathcal{M}(\mathcal{H}')$ and*

$$⨍_{\mathcal{H}'} \Phi(T) = ⨍_{\mathcal{H}} T \qquad \forall T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}).$$

(3) *Let $S : \mathcal{H} \to \mathcal{H}'$ be a continuous linear isomorphism from $\mathcal{H}$ onto a Hilbert space $\mathcal{H}'$. Then $\mathcal{M}(\mathcal{H}') = S^{-1}\mathcal{M}(\mathcal{H})S$ and*

$$\fint_{\mathcal{H}'} S^{-1}TS = \fint_{\mathcal{H}} T \qquad \forall T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}).$$

(4) *$\mathcal{M}$ and $\fint$ don't depend on the choice of the inner product on $\mathcal{H}$.*

Proof. By definition

$$\mathcal{M} = \bigcap_{\omega,\omega' \in \Omega(\mathcal{Q})} \{T \in \mathcal{L}^{(1,\infty)};\ \mathrm{Tr}_\omega(T) = \mathrm{Tr}_{\omega'} T\}.$$

Since each Dixmier trace Tr_ω is a continuous linear form on $\mathcal{L}^{(1,\infty)}$ it follows that $\mathcal{M}$ is a closed subspace of $\mathcal{M}$. In addition, since $\fint$ agrees on $\mathcal{M}$ with any Dixmier trace, it follows from Proposition 5.3.17 that $\fint$ is a continuous linear form on $\mathcal{M}$. This proves the first part of the proposition. The other parts immediately follow from Lemma 5.3.19 and Proposition 5.3.20. $\square$

Proposition 5.3.25. *The following hold.*

(1) *For any $T \in \mathcal{L}_+^{(1,\infty)}$,*

$$\left(T \in \mathcal{M} \text{ and } \fint T = L\right) \Longleftrightarrow \lim_{\lambda\to\infty} \tau_\lambda(T) = L.$$

(2) *For any $T \in \mathcal{K}$ positive,*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n<N} \mu_n(T) = L \Longrightarrow \left(T \in \mathcal{M} \text{ and } \fint T = L\right).$$

Proof. Let $T \in \mathcal{L}_+^{(1,\infty)}$. Then it follows from Proposition 5.3.14 that

$$\lim_{\lambda\to\infty} \tau_\lambda(T) = L \Leftrightarrow \Big(\omega(\tau(T)) = L\ \forall \omega \in \Omega(\mathcal{Q})\Big) \Leftrightarrow \left(T \in \mathcal{M} \text{ and } \fint T = L\right).$$

Next, let T be a positive compact operator such that

$$\lim_{N\to\infty} \frac{1}{\log N} \sigma_N(T) = L.$$

Then $\sigma_N(T) = \mathrm{O}(\log N)$, i.e., T is contained in $\mathcal{L}^{(1,\infty)}$.

Let $\lambda \in [e,\infty)$ and set $N = [\lambda]$. Then we have

$$\frac{\sigma_\lambda(T)}{\log\lambda} \leq \frac{\sigma_{N+1}(T)}{\log N} = \frac{\log(N+1)}{\log N}.\frac{\sigma_{N+1}(T)}{\log(N+1)},$$
$$\frac{\sigma_\lambda(T)}{\log\lambda} \geq \frac{\sigma_N(T)}{\log(N+1)} = \frac{\log N}{\log(N+1)}.\frac{\sigma_N(T)}{\log N}.$$

Since $\frac{\log(N+1)}{\log N}$ and $\frac{\log N}{\log(N+1)}$ both converge to 1 as $N \to \infty$, it follows that

$$\lim_{\lambda\to\infty} \frac{\sigma_u(T)}{\log u} \longrightarrow L \qquad \text{as } \lambda \to \infty.$$

If $L \neq 0$, then as $\lambda \to \infty$ we have $\frac{\sigma_\lambda(T)}{\log\lambda}\frac{1}{\lambda} \sim \frac{L}{\lambda}$, and hence

$$\int_e^\lambda \frac{\sigma_u(T)}{\log u}\frac{du}{u} \sim L \int_e^\lambda \frac{du}{u} \sim L\log\lambda.$$

If $L = 0$, then $\frac{\sigma_\lambda(T)}{\log\lambda}\frac{1}{\lambda} = \mathrm{o}(\frac{1}{\lambda})$, and hence

$$\int_e^\lambda \frac{\sigma_u(T)}{\log u}\frac{du}{u} = \mathrm{o}\left(\int_e^\lambda \frac{du}{u}\right) = \mathrm{o}(\log\lambda).$$

In both cases, we deduce that

$$\tau_\lambda(T) = \frac{1}{\log\lambda}\int_e^\lambda \frac{\sigma_u(T)}{\log u}\frac{du}{u} \longrightarrow L \qquad \text{as } \lambda \to \infty.$$

It then follows from the first part of the proposition that T is measurable and $⨍ T = L$. The proof is complete. □

PROPOSITION 5.3.26. *The following types of operators are measurable and have a vanishing Dixmier trace:*

(i) *Any operator in* $\mathcal{L}_0^{(1,\infty)}$.
(ii) *Any trace-class operator.*
(iii) *Any infinitesimal operator of order* > 1.

PROOF. Since $\mathcal{L}_0^{(1,\infty)}$ contains $\mathcal{L}^1$ and the infinitesimal operators of order > 1, (ii) and (iii) follows from (i). Therefore, we only have to prove that if $T \in \mathcal{L}_0^{(1,\infty)}$ then T is measurable and its Dixmier trace is zero.

Since $\mathcal{L}_0^{(1,\infty)}$ is a two-sided ideal (cf. Remark 5.2.2), Lemma 5.3.15 allows us to write T as $T = T_1 - T_2 + i(T_3 - T_4)$ with $T_j \in \mathcal{L}_0^{(1,\infty)} \cap \mathcal{L}(\mathcal{H})_+$. Since $\sigma_N(T_j) = \mathrm{o}(\log N)$, it follows from Proposition 5.3.25 that T_j is measurable and have a vanishing Dixmier trace. By linearity the same is true for T. The proof is complete. □

Next, we shall make use of the following Tauberian theorem to obtain a measurability criterion.

LEMMA 5.3.27 (See [**GVF**, Lemmas 7.19–7.20]). *Let* $(\lambda_n)_{n\geq 0}$ *be a nonincreasing sequence of positive real numbers such that*

(i) $\sum_{n\geq 0}\lambda_n^s < \infty$ *for all* $s > 1$.
(ii) $\lim_{s\to 1^+}(s-1)\sum_{n\geq 0}\lambda_n^s = 1$.

Then

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n<N}\lambda_n = 1.$$

PROPOSITION 5.3.28. *Let* $T \in \mathcal{K}$ *be positive and assume there exists* $p > 0$ *such that the following holds*

(i) T^s *is trace-class for all* $s > p$.
(ii) $\lim_{s\to p^+}(s-p)\,\mathrm{Trace}\,T^s = L \neq 0$.

Then T^p *is measurable and*

$$⨍ T^p = \frac{L}{p}.$$

PROOF. For $n \in \mathbb{N}_0$ set $\lambda_n = pL^{-1}\mu_n(T^p)$. If $s > 0$, then by Proposition 4.1.8 we have $\mu_n(T^p) = \mu_n(T^{ps})$ for all $n \in N_0$, and hence

$$\sum_{n\geq 0}\lambda_n^s = \sum_{n\geq 0}(pL^{-1})^s\mu_n(T^p)^s = (pL^{-1})^s\sum_{n\geq 0}\mu_n(T^{ps}) = (pL^{-1})^s\,\mathrm{Trace}\,T^{ps}.$$

Therefore, we see that

- The series $\sum_{n\geq 0}\lambda_n^s$ is convergent for all $s>1$.
- As $s\rightarrow 1^+$ we have

$$\sum_{n\geq 0}\lambda_n^s=(pL^{-1})^s\operatorname{Trace}T^{ps}=(pL^{-1})(sp-p)^{-1}L+\mathrm{o}(1)=(s-1)+\mathrm{o}(1).$$

We then can apply Lemma 5.3.27 to deduce that $\lim_{N\rightarrow\infty}\frac{1}{\log N}\sum_{n<N}\lambda_n=1$. Since $\mu_n(T^p)=Lp^{-1}\lambda_n$ this gives

$$\lim_{N\rightarrow\infty}\frac{1}{\log N}\sum_{n<N}\mu_n(T^p)=\frac{L}{p}. \tag{5.55}$$

It then follows from Proposition 5.3.25 that T^p is measurable and $⨍ T^p=\frac{L}{p}$, proving the proposition. $\square$

Example 5.3.29. Let $\Delta=-(\partial_{x_1}^2+\ldots+\partial_{x_n}^2)$ be the (positive) Laplacian on the n-dimensional torus $\mathbb{T}^n=\mathbb{R}^n/(2\pi\mathbb{Z})^n$. For $k\in\mathbb{Z}^n$ set $e_k=(2\pi)^{-\frac{n}{2}}e^{ik.x}$. Then $(e_k)_{k\in\mathbb{Z}^n}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$. Furthermore,

$$\Delta e_k=|k|^2e_k\qquad\forall k\in\mathbb{Z}^n.$$

Thus $(|k|^2)_{k\in\mathbb{Z}^n}$ is the family of eigenvalues of Δ counted with multiplicity. We order it in a non-decreasing sequence $(\lambda_j)_{j\geq 0}$, i.e., λ_j is the $(j+1)$'th eigenvalue of Δ counted with multiplicity. (In fact, as 0 is an eigenvalue with multiplicity 1, if $j\geq 1$ then λ_j is the j'th nonzero eigenvalue counted with multiplicity.)

Lemma 5.3.30 (Weyl Asymptotic). *As $j\rightarrow\infty$ we have*

$$\lambda_j\simeq\left(\frac{j}{c}\right)^{\frac{2}{n}},\qquad c:=\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}. \tag{5.56}$$

Proof. For $\lambda>0$ define

$$N_1(\lambda)=\#\{j\in\mathbb{N}_0;\ \lambda_j<\lambda\}\quad\text{and}\quad N_2(\lambda)=\#\{j\in\mathbb{N}_0;\ \lambda_j<\lambda\}.$$

Observe that

$$N_1(\lambda)=\#\{k\in\mathbb{Z}^n;\ |k|^2<\lambda\}=2^n\#\{k\in\mathbb{N}_0^n;\ |k|<\sqrt{\lambda}\}, \tag{5.57}$$

$$N_2(\lambda)=\#\{k\in\mathbb{Z}^n;\ |k|^2\leq\lambda\}=2^n\#\{k\in\mathbb{N}_0^n;\ |k|\leq\sqrt{\lambda}\}. \tag{5.58}$$

In addition, for any $j\in\mathbb{N}$, we have

$$N_1(\lambda_j)\leq j\leq N_2(\lambda_j). \tag{5.59}$$

For $r>0$ we denote by $B(0,r)$ the (open) ball of radius r about the origin in $\mathbb{R}^n$ and we set $B^+(0,r)=B(0,r)\bigcap[0,\infty)^n$. For $k=(k_1,\ldots,k_n)$ in $\mathbb{N}_0^n$ we set $I_k:=[k_1,k_1+1)\times\ldots\times[k_n,k_n+1)$. In addition, for any Borel set A we shall denote by $|A|$ its Lebesgue measure. For instance $|I_k|=1$ and $|B(0,r)|=2^{-n}cr^n$, with $c=|B(0,1)|=\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

If $x\in I_k$, then $|k|\leq|x|<|k|+\sqrt{n}$. Therefore, for any $\lambda>0$, we have

$$B^+(0,\sqrt{\lambda})\subset\bigcup_{|k|<\sqrt{\lambda}}I_k\qquad\text{and}\qquad\bigcup_{|k|\leq\sqrt{\lambda}}I_k\subset B^+(0,\sqrt{\lambda}+\sqrt{n}).$$

Thus,

$$|B^+(0,\sqrt{\lambda})| \le \Big|\bigcup_{|k|<\sqrt{\lambda}} I_k\Big| = \#\{k \in \mathbb{N}_0^n;\ |k| < \sqrt{\lambda}\},$$

$$|B^+(0,\sqrt{\lambda}+\sqrt{n})| \ge \Big|\bigcup_{|k|\le\sqrt{\lambda}} I_k\Big| = \#\{k \in \mathbb{N}_0^n;\ |k| \le \sqrt{\lambda}\}.$$

Since $|B^+(0,r)| = 2^{-n}|B(0,r)| = 2^{-n}cr^n$, using (5.57)–(5.58) we deduce that

$$c\lambda^{\frac{n}{2}} \le N_1(\lambda) \qquad \text{and} \qquad N_2(\lambda) \le c\lambda^{\frac{n}{2}}\left(1+\frac{\sqrt{n}}{\sqrt{\lambda}}\right)^n.$$

Combining this with (5.59) shows that, for any $j \in \mathbb{N}$, we have

$$c\lambda_j^{\frac{n}{2}} \le N_1(\lambda_j) \le j \le N_2(\lambda_j) \le c\lambda_j^{\frac{n}{2}}\left(1+\frac{\sqrt{n}}{\sqrt{\lambda_j}}\right)^n.$$

Thus,

$$\left(1+\frac{\sqrt{n}}{\sqrt{\lambda_j}}\right)^{-\frac{1}{2}} \le \lambda_j(cj^{-1})^{\frac{2}{n}} \le 1. \tag{5.60}$$

As the sequence $(\lambda)_{j\ge 0}$ is non-decreasing and unbounded it converges to ∞ and $(1+\frac{\sqrt{n}}{\sqrt{\lambda_j}})^{-\frac{1}{2}}$ converges to 1 as $j \to \infty$. Combining this with (5.60) shows that

$$\lambda_j(cj^{-1})^{\frac{2}{n}} \longrightarrow 1 \qquad \text{as } j \to \infty,$$

proving the claim. □

Next, the operator $\Delta^{-\frac{n}{2}}$ as the bounded operator on $L^2(\mathbb{T}^n)$ such that

$$\Delta^{-s}e_0 = 0 \qquad \text{and} \qquad \Delta^{-s}e_k = |k|^{-n}e_k \quad \forall k \in \mathbb{Z}^n \setminus 0.$$

Therefore, using Proposition 3.4.2 (iv) we see that $\Delta^{-\frac{n}{2}}$ is a positive compact operator. Moreover, as the definition of $\Delta^{-\frac{n}{2}}$ implies that $((\lambda_j)^{-\frac{n}{2}})_{j\ge 1}$ is the non-increasing sequence of its eigenvalues with multiplicity, the min-max principle insures us that, for all $j \in \mathbb{N}_0$,

$$\mu_j(\Delta^{-\frac{n}{2}}) = (j+1)\text{'th eigenvalue of } \Delta^{-\frac{n}{2}} \text{ counted with multiplicity} = (\lambda_{j+1})^{-\frac{n}{2}}.$$

Combining this with (5.56) shows that $\mu_j(\Delta^{-\frac{n}{2}}) \simeq \frac{c}{j}$ as $j \to \infty$. Thus,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \mu_j(\Delta^{-\frac{n}{2}}) = c.$$

We then may use Proposition 5.3.25 to deduce that $\Delta^{-\frac{n}{2}}$ is measurable and

$$⨍ \Delta^{-\frac{n}{2}} = c = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}.$$

Example 5.3.31. Let us present an example of an operator in $\mathcal{L}^{(1,\infty)}$ which is not measurable. To this end, let σ be the function on $(1,\infty)$ defined by

$$\sigma(t) = \log t\,\{4 + \cos(\log(\log t)) - \sin(\log(\log t))\}.$$

Notice that σ is a smooth function and

$$\sigma'(t) = \frac{2}{t}\left\{2 - \sin\left(\log(\log t)\right)\right\}$$
$$\sigma''(t) = \frac{-2}{t^2}\left\{2 - \sin\left(\log(\log t)\right) - (\log t)^{-1}\cos\left(\log(\log t)\right)\right\}.$$

Using (5.3.31) we see that $\sigma(t)$ is increasing on $(1,\infty)$ and

$$|\sigma'(t)| \leq \frac{6}{t} \qquad \forall t \geq 1. \tag{5.61}$$

Thus,

$$|\sigma(n+1) - \sigma(n)| \leq \frac{6}{n} \qquad \forall n \in \mathbb{N}. \tag{5.62}$$

In addition, using (5.3.31) we can check that $\sigma(t)$ is concave on (e,∞) and hence, for any integer $n \geq 3$, we have

$$\frac{1}{2}(\sigma(n+2) + \sigma(n)) \leq \sigma(n+1)$$

that is,

$$\sigma(n+2) - \sigma(n+1) \leq \sigma(n+1) - \sigma(n). \tag{5.63}$$

Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$ and let $T \in \mathcal{L}(\mathcal{H})$ be defined by

$$T\xi_n = \begin{cases} (\sigma(4) - \sigma(3))\xi_n & \text{for } n = 0,1,2, \\ (\sigma(n+1) - \sigma(n))\xi_n & \text{for } n \geq 3. \end{cases} \tag{5.64}$$

The operator T is positive and using Proposition 3.4.2-(iv) and (5.62) we see that T is a compact operator. Furthermore, by (5.63) the sequence $(\sigma(n+1)-\sigma(n))_{n\geq 3}$ is decreasing, and so using (5.64) and the min-max principle we deduce that

(5.65)
$$\mu_0(T) = \mu_1(T) = \mu_2(T) = \sigma(4) - \sigma(3) \quad \text{and} \quad \mu_n(T) = \sigma(n+1) - \sigma(n) \text{ for } n \geq 3.$$

Combining this with (5.62) we see that that $\mu_n(T) = \mathrm{O}(\frac{1}{n})$, and hence T is an element of $\mathcal{L}^{(1,\infty)}$.

Let N be an integer ≥ 4. Then (5.65) immediately implies that

$$\sigma_N(T) = \sum_{n<N} \mu_n(T) = \sigma(N) + a, \qquad a := 3\sigma(4) - 4\sigma(3).$$

Let $u \in [4,\infty)$ and set $N = [u]$ and $\alpha = u - N$. Then combining (5.19) and (5.3.31) we get

$$\sigma_u(T) = \alpha\sigma_{N+1}(T) + (1-\alpha)\sigma_N(T) = \alpha\sigma(N+1) + (1-\alpha)\sigma(N) + a,$$

and hence

$$|\sigma_u(T) - \sigma(u) - a| \leq \alpha|\sigma(N+1) - \sigma(u)| + (1-\alpha)|\sigma(N) - \sigma(u)|.$$

Thanks to (5.61) we have

$$|\sigma(N+1) - \sigma(u)| \leq \frac{6}{u},$$
$$|\sigma(N) - \sigma(u)| \leq \frac{6}{N} \leq \frac{N+1}{N}\frac{6}{N+1} \leq \frac{5}{4}\frac{6}{u} = \frac{15}{2u},$$

where we have used the fact that $\frac{t+1}{t} \leq \frac{5}{4}$ for all $t \geq 4$. Thus,

$$|\sigma_u(T) - \sigma(u) - a| \leq \frac{15}{2u} \qquad \forall u \geq 4,$$

and hence

$$\sigma_u(T) = \sigma(u) + \mathrm{O}(1) \qquad \text{as } u \to \infty.$$

It follows from this that, as $\lambda \to \infty$, we have

$$\tau_\lambda(T) = \frac{1}{\log\lambda}\int_e^\lambda \frac{\sigma_u(T)}{\log u}\frac{du}{u} = \frac{1}{\log\lambda}\int_e^\lambda \frac{\sigma(u)}{\log u}\frac{du}{u} + \mathrm{O}\left(\frac{1}{\log\lambda}\int_e^\lambda \frac{du}{u\log u}\right).$$

Since $\int_e^\lambda \frac{du}{u\log u} = \int_1^{\log\lambda}\frac{dv}{v} = \log\log\lambda = \mathrm{o}(\log\lambda)$, we see that

$$\tau_\lambda(T) = \frac{1}{\log\lambda}\int_e^\lambda \frac{\sigma(u)}{\log u}\frac{du}{u} + \mathrm{o}(1). \tag{5.66}$$

Next, we have

$$\int_e^\lambda \frac{\sigma(u)}{\log u}\frac{du}{u} = \int_1^{\log\lambda}\frac{\sigma(e^u)}{u}du = \int_1^{\log\lambda}(4+\cos(\log u) - \sin(\log u))\,du.$$

Observing that $\frac{d}{du}\left[u\,(4+\cos(\log u))\right] = (4+\cos(\log u) - \sin(\log u))$, we get

$$\int_e^\lambda \frac{\sigma(u)}{\log u}\frac{du}{u} = \log\lambda\,\{4+\cos\left(\log(\log\lambda)\right)\} - 4.$$

Combining this with (5.66) we then obtain

$$\tau_\lambda(T) = 4 + \cos\left(\log(\log\lambda)\right) + \mathrm{o}(1) \qquad \text{as } \lambda \to \infty.$$

Therefore $\tau_\lambda(T)$ does not have a limit as $\lambda \to \infty$. It then follows from Proposition 5.3.25 that T is an element of $\mathcal{L}^{(1,\infty)}$ which is not measurable.

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