CHAPTER 4

Singular Values and Schatten Classes

In this chapter we present basic results on characterstic values and ideals of compact operators. Part of the material of this chapter is succinctly presented in [**Co**, pp. 439–443] and [**GVF**, 310–317]. A thorough account on ideals of compact operators is given in the book of Gohberg-Krein [**GK**] (see also [**Si**]).

Throughout this chapter we let $\mathcal{H}$ be a separable Hilbert space.

4.1. Singular Values

DEFINITION 4.1.1 (Singular Values). *Let $T \in \mathcal{L}(\mathcal{H})$. For every $n \in \mathbb{N}_0$ the $(n+1)$-th singular value of T is*

$$\mu_n(T) := \inf\{\|T_{|E^\perp}\|;\ \dim E = n\}.$$

It immediately follows from this definition that, for any $T \in \mathcal{L}(\mathcal{H})$,

$$\mu_n(\lambda T) = |\lambda| \mu_n(T) \qquad \forall \lambda \in \mathbb{C}\ \forall n \in \mathbb{N}_0. \tag{4.1}$$

In the sequel we denote by $\mathcal{R}_n$ the space of operators $T \in \mathcal{L}(\mathcal{H})$ of rank $\leq n$.

PROPOSITION 4.1.2. *Let $T \in \mathcal{L}(\mathcal{H})$. Then*

$$\mu_n(T) = \operatorname{dist}(T, \mathcal{R}_n) \qquad \forall n \in \mathbb{N}_0, \tag{4.2}$$

$$\mu_m(T) \leq \mu_n(T) \qquad \forall m, n \in \mathbb{N}_0,\ m \geq n. \tag{4.3}$$

PROOF. Let $n \in \mathbb{N}_0$. By definition $\operatorname{dist}(T, \mathcal{R}_n) = \inf\{\|T - R\|;\ R \in \mathcal{R}_n\}$. Let E be an n-dimensional subspace of $\mathcal{H}$ and denote by Π_E the orthogonal projection onto E. Then $\operatorname{rk} T\Pi_E \leq \operatorname{rk} \Pi_E = n$, i.e., $T\Pi_E$ is contained in $\mathcal{R}_n$. Thus,

$$\operatorname{dist}(T, \mathcal{R}_n) \leq \|T - T\Pi_E\| = \|T(1 - \Pi_E)\| = \|T_{|E^\perp}\|,$$

from which we deduce that $\operatorname{dist}(T, \mathcal{R}_n) \leq \mu_n(T)$.

Conversely, let $R \in \mathcal{R}_n$. As $\operatorname{rk} R^* = \operatorname{rk} R \leq n$ there exists an n-dimensional subspace E of $\mathcal{H}$ containing $\operatorname{im} R^*$, so that $E^\perp$ is contained in $(\operatorname{im} R^*)^\perp = \ker R$. Thus,

$$\|T - R\| \geq \sup_{\substack{\xi \in \ker R \\ \|\xi\|=1}} \|(T - R)\xi\| = \sup_{\substack{\xi \in \ker R \\ \|\xi\|=1}} \|T\xi\| \geq \sup_{\substack{\xi \in E^\perp \\ \|\xi\|=1}} \|T\xi\| = \|T_{|E^\perp}\| \geq \mu_n(T).$$

This implies that $\operatorname{dist}(T, \mathcal{R}_n) \geq \mu_n(T)$, and hence $\mu_n(T) = \operatorname{dist}(T, \mathcal{R}_n)$.

Next, let $m \in \mathbb{N}_0$, $m \geq n$. Then $\mathcal{R}_m \supset \mathcal{R}_n$, and so $\operatorname{dist}(T, \mathcal{R}_m) \leq \operatorname{dist}(T, \mathcal{R}_n)$. Since $\mu_m(T) = \operatorname{dist}(T, \mathcal{R}_m)$ and $\mu_n(T) = \operatorname{dist}(T, \mathcal{R}_n)$, the inequality (4.3) follows. The proof is complete. ∎

PROPOSITION 4.1.3. *Let $T \in \mathcal{L}(\mathcal{H})$ and let $n \in \mathbb{N}_0$. Then*

$$\mu_n(T) = \mu_n(T^*) = \mu_n(|T|), \tag{4.4}$$

$$\mu_n(ATB) \leq \|A\| \mu_n(T) \|B\| \qquad \forall A, B \in \mathcal{L}(\mathcal{H}), \tag{4.5}$$

$$\mu_n(U^*TU) = \mu_n(T) \qquad \forall U \in \mathcal{L}(\mathcal{H}),\ U \text{ unitary}. \tag{4.6}$$

PROOF. Let A and B be in $\mathcal{L}(\mathcal{H})$, and let $R \in \mathcal{R}_n$. Then ARB is contained in $\mathcal{R}_n$ too, and hence

$$\mu_n(ATB) = \operatorname{dist}(ATB, \mathcal{R}_n) \leq \|ATB - ARB\| = \|A(T-R)B\| \leq \|A\| \|T-R\| \|B\|.$$

Thus $\mu_n(ATB) \leq \|A\| \operatorname{dist}(T, \mathcal{R}_n) \|B\| = \|A\| \mu_n(T) \|B\|$, proving (4.5).

If U is unitary and we take $A = U^*$ and $B = U$ in (4.5), then we get

$$\mu_n(U^*TU) \leq \|U^*\| \mu_n(T) \|U\| = \mu_n(T).$$

The above inequalities for U^*TU and U^* yield $\mu_n(T) \leq \mu_n(U^*TU)$. Thus $\mu_n(U^*TU)$ agrees with $\mu_n(T)$, proving (4.6).

It remains to prove (4.4). Let $T = U|T|$ be the polar decomposition of T. As $\|U\| = 1$ from (4.5) we get $\mu_n(T) = \mu_n(U|T|) \leq \|U\| \mu_n(|T|) = \mu_n(|T|)$. Since $|T| = U^*T$ we similarly have $\mu_n(|T|) \leq \mu_n(T)$, and hence $\mu_n(T) = \mu_n(|T|)$.

Finally, as $|T^*| = U^*|T|U$ and $|T| = U|T^*|U^*$, arguing as above shows that $\mu_n(|T|) = \mu_n(|T^*|)$. Since $\mu_n(|T^*|) = \mu_n(T^*)$ this implies that $\mu_n(|T|) = \mu_n(T^*)$, completing the proof. ☐

REMARK 4.1.4. Let $\mathcal{H}'$ be a separable Hilbert space, let $A \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ and let $B \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$. Then by arguing as above we can show that, for any $T \in \mathcal{L}(\mathcal{H})$,

$$\mu_n(ATB) \leq \|A\|_{\mathcal{L}(\mathcal{H}', \mathcal{H})} \mu_n(T) \|B\|_{\mathcal{L}(\mathcal{H}, \mathcal{H}')} \qquad \forall n \in \mathbb{N}_0.$$

In particular, if $U \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ is unitary, then we can show that, for any $T \in \mathcal{L}(\mathcal{H})$,

$$\mu_n(U^*TU) = \mu_n(T) \qquad \forall n \in \mathbb{N}_0.$$

Thus the singular values of T are invariant under the action of unitary isomorphisms.

PROPOSITION 4.1.5. *Let $S, T \in \mathcal{L}(\mathcal{H})$ and let $m, n \in \mathbb{N}_0$. Then*

$$\mu_{m+n}(S+T) \leq \mu_m(S) + \mu_n(T), \tag{4.7}$$

$$|\mu_m(S) - \mu_n(T)| \leq \|S - T\|, \tag{4.8}$$

$$\mu_{m+n}(ST) \leq \mu_m(S) \mu_n(T). \tag{4.9}$$

PROOF. Let $R \in \mathcal{R}_m$ and let $R' \in \mathcal{R}_n$. Then $R + R'$ is contained in $\mathcal{R}_{m+m}$, and hence

$$\mu_{m+n}(S+T) = \operatorname{dist}(S+T, \mathcal{R}_{m+n}) \leq \|S + T - (R + R')\| \leq \|S - R\| + \|T - R\|.$$

Thus $\mu_{m+n}(S+T) \leq \operatorname{dist}(S, \mathcal{R}_m) + \operatorname{dist}(T, \mathcal{R}_n) = \mu_m(S) + \mu_n(T)$, proving (4.7).

If in (4.7) we substitute $S - T$ for S and we take $m = 0$, then we obtain

$$\mu_n(S) \leq \mu_0(S-T) + \mu_n(T) = \|S - T\| + \mu_n(T),$$

that is, $\mu_n(S) - \mu_n(T) \leq \|S - T\|$. Interchanging S and T yields

$$\mu_n(T) - \mu_n(S) \leq \|S - T\|,$$

so we see that $|\mu_m(S) - \mu_n(T)| \leq \|S - T\|$, i.e., (4.8) holds true.

It remains to prove (4.9). To this end let $R \in \mathcal{R}_m$ and let $R' \in \mathcal{R}_n$. Then

$$(S-R)(T-R') = ST - R'', \qquad R'' := SR' + R(T-R').$$

Observe that SR' is contained in $\mathcal{R}_n$ and $R(T-R')$ is contained in $\mathcal{R}_m$, so R'' is contained in $\mathcal{R}_{m+n}$. Thus,

$$\mu_{n+m}(ST) \leq \|ST - R''\| = \|(S-R)(T-R')\| \leq \|S-R\|\|T-R'\|.$$

From this we deduce that $\mu_{m+n}(ST) \leq \operatorname{dist}(S, \mathcal{R}_m)\operatorname{dist}(T, \mathcal{R}_n) = \mu_m(S)\mu_n(T)$, completing the proof. □

PROPOSITION 4.1.6. *Let $T \in \mathcal{L}(\mathcal{H})$. Then the following are equivalent:*

(i) *T is a compact operator.*
(ii) *T is the norm-limit of finite-rank operators.*
(iii) *$\mu_n(T) \to 0$ as $n \to 0$.*
(iv) *For any $\epsilon > 0$ there exists a finite-dimensional subspace E of $\mathcal{H}$ such that $\|T_{|E^\perp}\| < \epsilon$.*

PROOF. The implication (i) ⇒ (iv) follows from Proposition 3.4.2 (iv)–(v). Therefore, to show that the conditions (i)–(iv) are equivalent it is enough to prove the implications (ii) ⇒ (i), (iii) ⇒ (ii), (iv) ⇒ (iii).

• (ii) ⇒ (i): As any finite rank operator is compact and $\mathcal{K}$ is closed, any norm-limit of finite rank operators remains in $\mathcal{K}$. Thus (ii) implies (i).

• (iii) ⇒ (ii): Suppose that $\lim_{n\to\infty} \mu_n(T) = 0$. Let $k \in \mathbb{N}$. Then there exists $n_k \in \mathbb{N}_0$ such that $\operatorname{dist}(T, \mathcal{R}_{n_k}) = \mu_{n_k}(T) < k^{-1}$. Thus there exists $R_k \in \mathcal{R}_{n_k}$ such that $\|T - R_k\| \leq 2k^{-1}$. Then the sequence $(R_k)_{k\geq 1}$ converges to T in norm, so T is a norm-limit of finite rank operators. This proves that (iii) implies (ii).

• (iv) ⇒ (iii): Assume that (iv) holds. Let $\epsilon > 0$. Then there exists a finite-dimensional subspace $E \subset \mathcal{H}$ such that $\|T_{|E^\perp}\| < \epsilon$. Set $n_0 = \dim E$. Then we have $\mu_{n_0}(T) \leq \|T_{|E^\perp}\| < \epsilon$. Since by (4.3) the sequence $(\mu_n(T))_{n\geq 0}$ is decreasing, we see that $\mu_n(T) < \epsilon$ for all $n \geq n_0$. This proves that $\mu_n(T) \to 0$ as $n \to 0$. Thus (iv) implies (iii). The proof is complete. □

THEOREM 4.1.7 (Min-Max Principle). *Let $T \in \mathcal{K}$. Then, for all $n \in \mathbb{N}_0$,*

$$\mu_n(T) = (n+1)\text{-th eigenvalue of } |T| \text{ counted with multiplicity.} \tag{4.10}$$

PROOF. Since $\mu_n(T) = \mu_n(|T|)$ it enough to prove the result for $|T|$, which allows us to assume T positive. For any $k \in \mathbb{N}_0$ let λ_k be the $(k+1)$-th eigenvalue of T counted with multiplicity and let $(\xi_k)_{k\geq 0}$ be an orthonormal basis of $\mathcal{H}$ such that $T\xi_k = \lambda_k \xi_k$ for all $k \in \mathbb{N}_0$.

For $k \in \mathbb{N}$ denote by E_k be the k-dimensional subspace of $\mathcal{H}$ spanned by the vectors $\xi_0, \ldots, \xi_{k-1}$. Then by (3.18) we have

$$\|T_{|E_n^\perp}\| = \sup_{k\geq n} \lambda_k = \lambda_n.$$

Thus, by the very definition of $\mu_n(T)$ we have

$$\mu_n(T) \leq \|T_{|E_n^\perp}\| = \lambda_n. \tag{4.11}$$

Conversely, let E be an n-dimensional subspace of $\mathcal{H}$ and denote by Π the orthogonal projection onto E. Since $\Pi_{|E_{n+1}}$ maps the $(n+1)$-dimensional space E_{n+1} to the n-dimensional subspace E, it cannot be one-to-one. Therefore, there exists a unit vector ξ which is contained in E_{n+1} and in $\ker \Pi = E^{\perp}$. In particular $\xi = \sum_{k\leq n} \alpha_k \xi_k$ with $\sum_{k\leq n} |\alpha_k|^2 = 1$. Thus,

$$\|T_{|E^{\perp}}\|^2 \geq \|T\xi\|^2 = \Big\|\sum_{k\leq n} \alpha_k T\xi_k\Big\|^2 = \Big\|\sum_{k\leq n} \alpha_k \lambda_k \xi_k\Big\|^2 = \sum_{k\leq n} |\alpha_k|^2 \lambda_k^2.$$

Since $\lambda_n \leq \lambda_k$ for all $k \leq n$, we deduce that

$$\|T_{|E^{\perp}}\|^2 \geq \lambda_n \sum_{k\leq n} |\alpha_k|^2 = \lambda_n.$$

As the inequality $\|T_{|E^{\perp}}\| \geq \lambda_n$ is valid for any n-dimensional subspace of $\mathcal{H}$, it follows that $\mu_n(T) \geq \lambda_n$. Together with (4.11) this shows that $\lambda_n = \mu_n(T)$. The proposition is thus proved. □

Let us now look at some interesting consequences of the min-max principle. First, in the light of (4.10) the inequality (4.8) shows the continuity of the eigenvalues of *positive* compact operators.

Another interesting consequence is the following.

PROPOSITION 4.1.8. *Let $T \in \mathcal{K}$ have polar decomposition $T = U|T|$ and let $(\xi_n)_{n\geq 0}$ be an orthonormal family such that $|T|\xi_n = \mu_n(T)\xi_n$ for all $n \in \mathbb{N}_0$.*

(1) *We have*

$$T = \sum_{n\geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*,$$

where the series converges in norm.

(2) *Let f be a non-negative non-decreasing function on $[0, \infty)$ which is continuous and vanishes at 0. Then*

$$\mu_n(f(|T|) = f(\mu_n(T)) \qquad \forall n \in \mathbb{N}_0, \tag{4.12}$$

$$f(|T|) = \sum_{n\geq 0} f(\mu_n(T))\xi_n \otimes \xi_n^*, \tag{4.13}$$

where the series converges in norm.

PROOF. Let E be the closed subspace spanned by all the vectors ξ_n. Observe that E contains all the eigenvectors of $|T|$ associated to a non-zero eigenvalue. Therefore, E contains $(\ker |T|)^{\perp}$, and hence $E^{\perp}$ is contained in $\ker |T|$. Thus if we let $(\eta_k)_{k\in I}$ be an orthonormal basis of $E^{\perp}$, then $\{\eta_k\}_{k\in I} \cup \{\xi_n\}_{n\in\mathbb{N}_0}$ is an orthonormal basis of $\mathcal{H}$ which respect to which $|T|$ is diagonal, namely,

$$|T|\eta_k = 0 \quad \forall k \in I \qquad \text{and} \qquad |T|\xi_n = \mu_n(T)\xi_n \quad \forall n \in \mathbb{N}_0. \tag{4.14}$$

Then by 3.19 we have

$$|T| = \sum_{k\in I} 0.\eta_k \otimes \eta_k^* + \sum_{n\geq 0} \mu_n(T)\xi_n \otimes \xi_n^* = \sum_{n\geq 0} \mu_n(T)\xi_n \otimes \xi_n^*, \tag{4.15}$$

where the series converge in norm. Since $T = U|T|$ this gives

$$T = U \sum_{n\geq 0} \mu_n(T)\xi_n \otimes \xi_n^* = \sum_{n\geq 0} \mu_n(T) U\Big(\xi_n \otimes \xi_n^*\Big) = \sum_{n\geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*,$$

where the series converge in norm.

Let f be a non-negative non-decreasing function on $[0, \infty)$ which is continuous and vanishes at 0. Then using (4.14) and Proposition 3.4.8 we see that $f(|T|)$ is compact and we have

$$f(|T|) = \sum_{k \in I} f(0) \eta_k \otimes \eta_k^* + \sum_{n \geq 0} f(\mu_n(T)) \xi_n \otimes \xi_n^* = \sum_{n \geq 0} f(\mu_n(T)) \xi_n \otimes \xi_n^*,$$

where the series converges in norm. Since f is nondecreasing it follows from this that the $(n+1)$-th eigenvalue of $f(|T|)$ is $f(\mu_n(T))$. Therefore, using the min-max principle we deduce that

$$\mu_n(f(|T|) = f(\mu_n(T)) \qquad \forall n \in \mathbb{N}_0,$$

which completes the proof. □

EXAMPLE 4.1.9. Let $p > 0$ and let $T \in \mathcal{K}$. Applying the above lemma to $f(t) = t^p$ (with the convention that $0^p = 0$) shows that

$$\mu_n(|T|^p) = \mu_n(T)^p \qquad \forall n \in \mathbb{N}_0.$$

4.2. Trace-class operators

For all $T \in \mathcal{L}(\mathcal{H})$ we set

$$\|T\|_1 := \sum_{n \geq 0} \mu_n(T).$$

We then define

$$\mathcal{L}^1 := \{T \in \mathcal{L}(\mathcal{H});\ \|T\|_1 < \infty\}. \tag{4.16}$$

The elements of $\mathcal{L}^1$ are called *trace-class operators*.

Observe that if $\sum \mu_n(T) < \infty$, then $\mu_n(T) \to 0$ as $n \to \infty$, and so using Proposition 4.1.6 we see that T is a compact operator. Thus any trace-class operator is compact. Moreover,

$$\|T\| = \mu_0(T) \leq \|T\|_1 \qquad \forall T \in \mathcal{L}(\mathcal{H}). \tag{4.17}$$

Notice also that it follows from (4.1), (4.4) and (4.5) that

$$\|T\|_1 = \|T^*\|_1 = \||T|\|_1 \qquad \forall T \in \mathcal{L}(\mathcal{H}), \tag{4.18}$$

$$\|\lambda T\|_1 = |\lambda| \|T\|_1 \qquad \forall T \in \mathcal{L}(\mathcal{H})\ \forall \lambda \in \mathbb{C}, \tag{4.19}$$

$$\|ATB\|_1 \leq \|A\| \|T\|_1 \|B\| \qquad \forall A, T, B \in \mathcal{L}(\mathcal{H}). \tag{4.20}$$

As an immediate consequence of (4.18) we see that, for all $T \in \mathcal{L}(\mathcal{H})$,

$$T \in \mathcal{L}^1 \Longrightarrow T^* \in \mathcal{L}^1 \Longrightarrow |T| \in \mathcal{L}^1.$$

In the sequel we denote by $\mathcal{L}(\mathcal{H})_+$ the cone of operators in $\mathcal{L}(\mathcal{H})$ that are positive.

LEMMA 4.2.1. *Let $T \in \mathcal{L}(\mathcal{H})_+$. Then, for any orthonormal basis $(\xi_n)_{n \geq 0}$, we have*

$$\|T\|_1 = \sum_{n \geq 0} \langle \xi_n, T\xi_n \rangle. \tag{4.21}$$

PROOF. Let us first assume that T is compact. By Theorem 3.4.7 and Theorem 4.1.7 there exists an orthonormal basis $(\eta_n)_{n\geq 0}$ such that $T\eta_n = \mu_n(T)\eta_n$ for all $n \in \mathbb{N}_0$. Then

$$\|T\|_1 = \sum_{n\geq 0} \mu_n(T) = \sum_{n\geq 0} \langle \eta_n, T\eta_n\rangle = \sum_{n\geq 0} \langle T^{\frac{1}{2}}\eta_n, T^{\frac{1}{2}}\eta_n\rangle = \sum_{n\geq 0} \|T^{\frac{1}{2}}\eta_n\|^2.$$

Since $(\xi_n)_{\geq 0}$ is an orthonormal basis, we also have

$$\sum_{n\geq 0} \|T^{\frac{1}{2}}\eta_n\|^2 = \sum_{n\geq 0}\Big(\sum_{k\geq 0} |\langle \xi_k, T^{\frac{1}{2}}\eta_n\rangle|^2\Big) = \sum_{k\geq 0}\Big(\sum_{n\geq 0} |\langle T^{\frac{1}{2}}\xi_k, \eta_n\rangle|^2\Big)$$
$$= \sum_{k\geq 0} \|T^{\frac{1}{2}}\xi_n\|^2 = \sum_{k\geq 0} \langle \xi_k, T\xi_k\rangle,$$

proving the lemma when T is compact.

Suppose now that T is not compact. Then T is not trace-class, and hence $\|T\|_1 = \infty$. For $N \in \mathbb{N}$ denote by E_N the span of $\xi_0, \ldots, \xi_{N-1}$. Moreover, since T is positive we have $T = (T^{\frac{1}{2}})(T^{\frac{1}{2}})$, and so as $\mathcal{K}$ is a two-sided ideal $T^{\frac{1}{2}}$ cannot be compact. It then follows from Proposition 3.4.2-(v) that the sequence $(\|(T^{\frac{1}{2}})_{|E_N^{\perp}}\|)_{N\geq 1}$ does not converge to 0. Since it is a non-increasing sequence of non-negative numbers this means there is $c > 0$ such that $\|(T^{\frac{1}{2}})_{|E_N^{\perp}}\| > c$ for all $N \in \mathbb{N}$.

Let $N \in \mathbb{N}$ and let $\xi \in E_N^{\perp}$ be such that $\|\xi\| = 1$ and $\|T^{\frac{1}{2}}\xi\| > c$. Notice that

$$\|T^{\frac{1}{2}}\xi\| = \Big\|\sum_{n\geq N} \langle \xi_n, \xi\rangle T^{\frac{1}{2}}\xi_n\Big\| \leq \sum_{n\geq N} |\langle \xi_n, \xi\rangle| \|T^{\frac{1}{2}}\xi_n\| \leq \Big(\sum_{n\geq N} |\langle \xi_n, \xi\rangle|^2\Big)^{\frac{1}{2}} \Big(\sum_{n\geq N} \|T^{\frac{1}{2}}\xi_n\|^2\Big)^{\frac{1}{2}}.$$

Since $\sum_{n\geq N} |\langle \xi_n, \xi\rangle|^2 = \|\xi\|^2 = 1$ and $\|T^{\frac{1}{2}}\xi_n\|^2 = \langle T^{\frac{1}{2}}\xi_n, T^{\frac{1}{2}}\xi_n\rangle = \langle \xi_n, T\xi_n\rangle$, it follows that

$$c^2 < \|T\xi\|^2 \leq \sum_{n\geq N} \langle \xi_n, T\xi_n\rangle \qquad \forall N \in \mathbb{N}.$$

Therefore, the series $\sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle$ diverges, i.e., $\sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle = \infty = \|T\|_1$. The proof is complete. □

DEFINITION 4.2.2. *Let $T \in \mathcal{L}(\mathcal{H})_+$. Then the trace of T is defined to be*

$$\operatorname{Trace} T := \sum_{n\geq 0} \langle \xi_n, T\xi_n\rangle, \tag{4.22}$$

where $(\xi_n)_{n\geq 0}$ is any orthonormal basis.

Using (4.4) we see that, for all $T \in \mathcal{L}(\mathcal{H})$,

$$\|T\|_1 = \||T|\|_1 = \operatorname{Trace} |T|.$$

In particular,

$$T \in \mathcal{L}^1 \Longleftrightarrow \operatorname{Trace} |T| < \infty.$$

We shall now extend the definition of $\operatorname{Trace} T$ to all operators $T \in \mathcal{L}^1$.

LEMMA 4.2.3. *Let $T \in \mathcal{L}(\mathcal{H})$. Then, for any orthonormal basis $(\xi_n)_{n\geq 0}$,*

$$\sum_{n\geq 0} |\langle \xi_n, T\xi_n\rangle| \leq \|T\|_1.$$

Proof. Let $T = U|T|$ be the polar decomposition of T. Then

$$|\langle \xi_n, T\xi_n \rangle| = |\langle \xi_n, U|T|\xi_n \rangle| = |\langle |T|^{\frac{1}{2}} U^* \xi_n, |T|^{\frac{1}{2}} \xi_n \rangle \leq \| |T|^{\frac{1}{2}} U^* \xi_n \| \| |T|^{\frac{1}{2}} \xi_n \|.$$

Thus,

$$\sum_{n\geq 0} |\langle \xi_n, T\xi_n \rangle| \leq \sum_{n\geq 0} \| |T|^{\frac{1}{2}} U^* \xi_n \| \| |T|^{\frac{1}{2}} \xi_n \| \leq \Big(\sum_{n\geq 0} \| |T|^{\frac{1}{2}} U^* \xi_n \|^2 \Big)^{\frac{1}{2}} \Big(\sum_{n\geq 0} \| |T|^{\frac{1}{2}} \xi_n \|^2 \Big)^{\frac{1}{2}}. \tag{4.23}$$

Using Lemma 4.2.1 and (4.2) we get

$$\sum_{n\geq 0} \| |T|^{\frac{1}{2}} \xi_n \|^2 = \sum_{n\geq 0} \langle |T|^{\frac{1}{2}} \xi_n, |T|^{\frac{1}{2}} \xi_n \rangle = \sum_{n\geq 0} \langle \xi_n, |T| \xi_n \rangle = \operatorname{Trace} |T| = \|T\|_1.$$

Similarly, we have

$$\sum_{n\geq 0} \| |T|^{\frac{1}{2}} U^* \xi_n \|^2 = \sum_{n\geq 0} \langle |T|^{\frac{1}{2}} U^* \xi_n, |T|^{\frac{1}{2}} U^* \xi_n \rangle = \sum_{n\geq 0} \langle \xi_n, U|T|U^* \xi_n \rangle.$$

Proposition 3.1.8) tells us that $U^*|T|U = |T^*|$, and so using Lemma 4.2.1 and (4.2) we get

$$\sum_{n\geq 0} \| |T|^{\frac{1}{2}} U^* \xi_n \|^2 = \sum_{n\geq 0} \langle \xi_n, |T^*| \xi_n \rangle = \operatorname{Trace} |T^*| = \|T^*\|_1 = \|T\|_1. \tag{4.24}$$

Combining (4.23) with (4.2) and (4.24) gives

$$\sum_{n\geq 0} |\langle \xi_n, T\xi_n \rangle| \leq \|T\|_1^{\frac{1}{2}} \|T\|_1^{\frac{1}{2}} = \|T\|_1,$$

proving the lemma. □

Lemma 4.2.4. *We have*

$$\|S + T\|_1 \leq \|S\|_1 + \|T\|_1 \qquad \forall S, T \in \mathcal{L}(\mathcal{H}). \tag{4.25}$$

Proof. Let $S, T \in \mathcal{L}(\mathcal{H})$ and let $S+T = U|S+T|$ be the polar decompositions of $S + T$. Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$. Upon writing

$$|S + T| = U^*(S + T) = U^*S + U^*T$$

and using Lemma 4.2.1 and Lemma 4.2.3 we get

$$\|S + T\|_1 = \sum_{n\geq 0} \langle \xi_n, |S + T| \xi_n \rangle \leq \sum_{n\geq 0} |\langle \xi_n, U^*S\xi_n \rangle| + \sum_{n\geq 0} |\langle \xi_n, U^*T\xi_n \rangle| \leq \|U^*S\|_1 + \|U^*T\|_1.$$

Combining this with (4.20) gives

$$\|S + T\|_1 \leq \|U^*\| \|S\|_1 + \|U^*\| \|T\|_1 \leq \|S\|_1 + \|T\|_1,$$

proving the lemma. □

Proposition 4.2.5. *The following hold.*

(1) *$\mathcal{L}^1$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$.*
(2) *$\|.\|_1$ is a norm on $\mathcal{L}^1$ with respect to which $\mathcal{L}^1$ is a Banach space.*

PROOF. It follows from (4.17), (4.19), (4.20) and (4.25) that $\mathcal{L}^1$ is a two-sided ideal and $\|.\|_1$ is a seminorm on $\mathcal{L}^1$.

It remains to show that $\mathcal{L}^1$ is complete with respect to the norm $\|.\|_1$. Thus, let $(T_n)_{n\geq 0}$ be a sequence with values in $\mathcal{L}^1$ which is Cauchy with respect to $\|.\|_1$. Since by (4.17) $\|T_n - T_p\| \leq \|T_p - T_n\|_1$ we see that $(T_n)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{L}(\mathcal{H})$, and hence converges to some operator T in $\mathcal{L}(\mathcal{H})$.

Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\|T_n - T_p\| < \epsilon$ for all $n, p \geq N$. By (4.8), for all $k \in \mathbb{N}$,

$$|\mu_k(T_n - T) - \mu_k(T_n - T_p)| \leq \|T_p - T\|.$$

Since $\|T_p - T\| \to 0$ as $p \to \infty$, we see that $\mu_k(T_n - T) = \lim_{p\to\infty} \mu_k(T_n - T_p)$. Therefore,

$$\sum_{k=0}^{M} \mu_k(T_n - T) = \lim_{p\to\infty} \sum_{k=0}^{M} \mu_k(T_n - T) \qquad \forall M \in \mathbb{N}.$$

Since for $n, p \geq N$ we have

$$\sum_{k=0}^{M} \mu_k(T_n - T) \leq \sum_{k=0}^{\infty} \mu_k(T_n - T) = \|T_n - T_p\|_1 < \epsilon,$$

we deduce that, if $n \geq N$, then $\sum_{k=0}^{M} \mu_k(T_n - T) \leq \epsilon$ for all $M \in \mathbb{N}$. Thus, for all $n \geq N$, we have

$$\|T_n - T\|_1 = \sum_{k=0}^{\infty} \mu_k(T_n - T) \leq \epsilon. \tag{4.26}$$

Therefore $T_n - T$ is in $\mathcal{L}^1$ and, as $\mathcal{L}^1$ is a subspace containing T_n, we see that T is trace-class. Then (4.26) shows that T_n converges to T with respect to the norm $\|.\|_1$. Thus $\mathcal{L}^1$ is complete with respect to the norm $\|.\|_1$, completing the proof. ☐

REMARK 4.2.6. It follows from (4.17) that the inclusions of $\mathcal{L}^1$ into $\mathcal{K}$ and $\mathcal{L}(\mathcal{H})$ are continuous.

LEMMA 4.2.7. *Any $T \in \mathcal{L}^1$ can be written in the form*

$$T = T_1 - T_2 + iT_3 - iT_4, \qquad T_j \in \mathcal{L}^1 \cap \mathcal{L}(\mathcal{H})_+.$$

PROOF. Let $T \in \mathcal{L}^1$. Then $T = S_+ + iS_-$ with $S_\pm = \frac{1}{2}(T \pm T^*)$. The operators $S_\pm$ are selfadjoint and are contained in $\mathcal{L}^1$ by (4.2). Therefore, the proof reduces to proving that any selfadjoint trace-class operator can be written as the difference of two positive trace-class operators.

Let $T \in \mathcal{L}^1$ be selfadjoint. Then $T = T_+ - T_-$ with $T_\pm = \frac{1}{2}(|T| \pm T)$. Observe that by (4.2) the operators $T_\pm$ are trace-class. In addition, observe that $T_\pm = f_\pm(T)$ where $f_\pm(t) = \frac{1}{2}(|t| \pm t)$. As the functions $f_\pm$ are nonnegative on $\operatorname{Sp} T \subset \mathbb{R}$, the operators $T_\pm$ are positive by Corollary 3.1.4. We deduce from this that T can be written as the difference of two positive trace-class operators. This completes the proof. ☐

We are now ready to prove the following.

PROPOSITION 4.2.8. *Let $T \in \mathcal{L}^1$. For any orthonormal basis $(\xi_n)_{n\geq 0}$ the series*

$$\sum_{n\geq 0} \langle \xi_n, T\xi_n \rangle$$

is absolutely convergent and the value of its sum does not depend on the choice of the orthonormal basis $(\xi_n)_{n\geq 0}$.

PROOF. Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$. It follows from Lemma 4.2.3 that the series $\sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle$ is absolutely convergent. Moreover, Lemma 4.2.7 allows us to write $T = T_1 - T_2 + iT_3 - iT_4$ with $T_j \in \mathcal{L}^1 \cap \mathcal{L}(\mathcal{H})_+$. Then using Lemma 4.2.1 we get

$$\begin{aligned}\sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle = \sum_{n\geq 0}\langle \xi_n, T_1\xi_n\rangle - \sum_{n\geq 0}\langle \xi_n, T_2\xi_n\rangle + i\sum_{n\geq 0}\langle \xi_n, T_3\xi_n\rangle - i\sum_{n\geq 0}\langle \xi_n, T_4\xi_n\rangle \\ = \operatorname{Trace} T_1 - \operatorname{Trace} T_2 + i \operatorname{Trace} T_3 - i \operatorname{Trace} T_4.\end{aligned}$$

Therefore the value of the sum $\sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle$ does not depend on the orthonormal basis $(\xi_n)_{n\geq 0}$, proving the proposition. ☐

DEFINITION 4.2.9. *The trace of an operator* $T \in \mathcal{L}^1$ *is defined to be*

$$\operatorname{Trace}(T) := \sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle,$$

where $(\xi_n)_{n\geq 0}$ *is any orthonormal basis of* $\mathcal{H}$.

LEMMA 4.2.10. *Any* $A \in \mathcal{L}(\mathcal{H})$ *is linear combination of* 4 *unitaries.*

PROOF. Upon writing $A = \|A\|(T_+ + iT_-)$ with $T_\pm = \frac{1}{2\|A\|}(T \pm T^*)$ we see that the proof reduces to showing that any selfadjoint operator $A \in \mathcal{L}(\mathcal{H})$ with $\|A\| \leq 1$ is a linear combination of two unitaries.

Let $A \in \mathcal{L}(\mathcal{H})$ be selfadjoint and such that $\|A\| \leq 1$. Then the spectrum of A is contained in $[-1, 1]$, and so we can write

$$A = \frac{1}{2}(U + U^*), \qquad U := A + i\sqrt{1 - A^2}.$$

Observe that $U = f(A)$ with $f(t) = 1 + i\sqrt{1-t^2}$ and $f(t)\overline{f(t)} = 1$, so by continuous functional calculus $U^*U = UU^* = f(A)\bar{f}(A) = 1$, i.e., U is unitary. Thus A is the linear combination of two unitaries. This completes the proof. ☐

PROPOSITION 4.2.11. *The map* $T \to \operatorname{Trace}(T)$ *is a linear form on* $\mathcal{L}^1$ *such that, for any* $T \in \mathcal{L}^1$, *we have*

$$|\operatorname{Trace}(T)| \leq \|T\|_1, \tag{4.27}$$

$$\operatorname{Trace}(T^*) = \overline{\operatorname{Trace}(T)}, \tag{4.28}$$

$$\operatorname{Trace}(AT) = \operatorname{Trace}(TA) \qquad \forall A \in \mathcal{L}(\mathcal{H}). \tag{4.29}$$

PROOF. It is immediate from its definition that the map $T \to \operatorname{Trace}(T)$ is linear. Moreover, if $T \in \mathcal{L}^1$, then by Lemma 4.2.3, for any orthonormal basis $(\xi_n)_{n\geq 0}$,

$$|\operatorname{Trace}(T)| = |\sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle| \leq \sum_{n\geq 0}|\langle \xi_n, T\xi_n\rangle| \leq \|T\|_1.$$

Moreover, by (4.2) the adjoint T^* is trace-class. Then, for any orthonormal basis $(\xi_n)_{n\geq 0}$, we have

$$\operatorname{Trace}(T^*) = \sum_{n\geq 0}\langle \xi_n, T^*\xi_n\rangle = \sum_{n\geq 0}\overline{\langle \xi_n, T\xi_n\rangle} = \overline{\operatorname{Trace}(T)}.$$

Let $U \in \mathcal{L}(\mathcal{H})$ be unitary. Then

$$\operatorname{Trace}(U^*TU) = \sum_{n\geq 0} \langle \xi_n, U^*TU\xi_n\rangle = \sum_{n\geq 0} \langle U\xi_n, TU\xi_n\rangle.$$

Since U is unitary $(U\xi_n)_{n\geq 0}$ is an orthonormal basis of $\mathcal{H}$. Therefore, in view of the definition of $\operatorname{Trace}(T)$ we see that $\sum_{n\geq 0}\langle U\xi_n, TU\xi_n\rangle = \operatorname{Trace}(T)$. Thus $\operatorname{Trace}(U^*TU) = \operatorname{Trace}(T)$. Upon replacing T by UT we deduce that

$$\operatorname{Trace}(TU) = \operatorname{Trace}(UT) \qquad \forall U \in \mathcal{L}(\mathcal{H}) \text{ unitary}.$$

As by Lemma 4.2.10 any $A \in \mathcal{L}(\mathcal{H})$ is the linear combination of 4 unitaries, it follows that

$$\operatorname{Trace}(AT) = \operatorname{Trace}(TA) \qquad \forall A \in \mathcal{L}(\mathcal{H}).$$

This shows that $T \to \operatorname{Trace}(T)$ is a trace on $\mathcal{L}^1$. The proof is complete. $\square$

Example 4.2.12. Let ξ and η be nonzero vectors in $\mathcal{H}$. Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$ such that $\xi_0 = \|\xi\|^{-1}\xi$. Then, for any $n \in \mathbb{N}_0$,

$$\langle \xi_n, (\xi\otimes\eta^*)\xi_n\rangle = \langle \xi_n, \xi\rangle\langle \eta, \xi_n\rangle = \delta_{n,0}\|\xi\|^{-1}\langle \eta, \xi_0\rangle = \delta_{n,0}\langle \eta, \xi\rangle.$$

Therefore, we get

$$\operatorname{Trace}(\xi\otimes\eta^*) = \sum_{n\geq 0}\langle \xi_n, (\xi\otimes\eta^*)\xi_n\rangle = \langle \eta, \xi\rangle. \tag{4.30}$$

Example 4.2.13. Let $T \in \mathcal{L}^1$ be normal. Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$ in which T diagonalizes, i.e., $T\lambda_n = \lambda_n\xi_n$ forall $n \in \mathbb{N}_0$. Then

$$\operatorname{Trace} T = \sum_{n\geq 0}\langle \xi_n, T\xi_n\rangle = \sum_{n\geq 0}\langle \xi_n, \lambda_n\xi_n\rangle = \sum_{n\geq 0}\lambda_n, \tag{4.31}$$

that is, $\operatorname{Trace} T$ is the sum of the eigenvalues of T counted with multiplicity.

4.3. Duality between $\mathcal{L}^1$ and $\mathcal{L}(\mathcal{H})$

The Banach space $\mathcal{L}^1$ is in duality with the Banach spaces $\mathcal{L}(\mathcal{H})$ and $\mathcal{K}$. This can be seen as follows.

If S and T are in $\mathcal{L}(\mathcal{H})$ and one of these operators is trace-class, we set

$$(S,T) := \operatorname{Trace}(ST).$$

Thanks to (4.20) and (4.27) we see that, if $S \in \mathcal{L}^1$ and $T \in \mathcal{L}(H)$, then

$$|(T,S)| = |(S,T)| = |\operatorname{Trace}(ST)| \leq \|ST\|_1 \leq \|S\|_1\|T\|. \tag{4.32}$$

From this we deduce the following:

- For any $S \in \mathcal{L}(\mathcal{H})$ the map $(S,.) : \mathcal{L}^1 \ni T \to \operatorname{Trace}(ST)$ is a continuous linear form on $\mathcal{L}^1$.
- For any $S \in \mathcal{L}^1$ the map $(S,.) : \mathcal{L}(\mathcal{H}) \in T \to \operatorname{Trace}(ST)$ gives rise to a continuous linear form on $\mathcal{L}(\mathcal{H})$.

Lemma 4.3.1. *For any ξ and η in $\mathcal{H}$, we have*

$$\|\xi\otimes\eta^*\| = \|\xi\otimes\eta^*\|_1 = \|\xi\|\|\eta\|.$$

PROOF. First, in view of the definition (3.12) of $\xi \otimes \eta^*$ we have

$$\|\xi \otimes \eta^*\| = \sup_{\|\zeta\|=1} \|(\xi \otimes \eta^*)\zeta\| = \sup_{\|\zeta\|=1} \|\langle \eta, \zeta\rangle \xi\| = (\sup_{\|\zeta\|=1} |\langle \eta, \zeta\rangle|)\|\xi\| = \|\eta\|\|\xi\|.$$

Moreover, for any ζ_1 and ζ_2 in $\mathcal{H}$ we have

$$\langle \zeta_1, (\xi \otimes \eta^*)\zeta_2\rangle = \langle \eta, \zeta_2\rangle \langle \zeta_1, \xi\rangle = \langle \langle \xi, \zeta_1\rangle \eta, \zeta_2\rangle = \langle (\eta \otimes \xi^*)\zeta_1, \zeta_2\rangle,$$

which shows that

$$(\xi \otimes \eta^*)^* = \eta \otimes \xi^*. \tag{4.33}$$

We then have

$$(\xi \otimes \eta^*)^*(\xi \otimes \eta^*) = (\eta \otimes \xi^*)(\xi \otimes \eta^*) = \|\xi\|^2(\eta \otimes \eta^*).$$

Notice that $\|\eta\|^{-2}(\eta \otimes \eta^*)$ is the orthogonal projection onto $\mathbb{C}\eta$, and hence, as any orthogonal projection, $\|\eta\|^{-2}(\eta \otimes \eta^*)$ is a positive operator which agrees with its square root, i.e., $\eta \otimes \eta^*$ is positive and $(\eta \otimes \eta^*)^{\frac{1}{2}} = \|\eta\|^{-1}(\eta \otimes \eta^*)$. Thus,

$$|\xi \otimes \eta^*| = ((\xi \otimes \eta^*)^*(\xi \otimes \eta^*))^{\frac{1}{2}} = (\|\xi\|^2(\eta \otimes \eta^*))^{\frac{1}{2}} = \|\xi\|\|\eta\|^{-1}(\eta \otimes \eta^*).$$

Combining this with (4.30) we obtain

$$\|\xi \otimes \eta^*\|_1 = \operatorname{Trace} |\xi \otimes \eta^*| = \operatorname{Trace} \left(\|\xi\|\|\eta\|^{-1}(\eta \otimes \eta)^*\right) = \|\xi\|\|\eta\|,$$

completing the proof. □

LEMMA 4.3.2. *The following hold.*

(i) *For all* $S \in \mathcal{L}^1$,

$$\|S\|_1 = \sup_{\|T\|=1} |\operatorname{Trace}(ST)|. \tag{4.34}$$

(ii) *For all* $S \in \mathcal{L}(\mathcal{H})$,

$$\|S\| = \sup_{\|T\|_1=1} |\operatorname{Trace}(ST)|. \tag{4.35}$$

PROOF. Let $S \in \mathcal{L}^1$. Then by (4.32), for any $T \in \mathcal{L}(\mathcal{H})$ of norm 1, we have

$$|\operatorname{Trace}(ST)| \leq \|S\|_1\|T\| = \|S\|_1. \tag{4.36}$$

Let $S = U|S|$ be the polar decomposition of S. Then by Proposition 3.1.7 $\|U^*\| = \|U\| = 1$ and $U^*S = |S|$, and hence

$$\operatorname{Trace}(SU^*) = \operatorname{Trace}(U^*S) = \operatorname{Trace} |S| = \|S\|_1.$$

Together with (4.36) this gives (4.34).

Now, let $S \in \mathcal{L}(\mathcal{H})$. We may assume $S \neq 0$, since otherwise the equality (4.35) is trivially satisfied. By (4.32), for any T in the unit sphere of $\mathcal{L}^1$,

$$|\operatorname{Trace}(ST)| \leq \|S\|\|T\|_1 = \|S\|. \tag{4.37}$$

Moreover, as

$$\sup_{\|\xi\|=1} \langle \xi, |S|\xi\rangle = \sup_{\|\xi\|=1} \||S|^{\frac{1}{2}}\xi\|^2 = \||S|^{\frac{1}{2}}\|^2 = \|(S|^{\frac{1}{2}})^*S|^{\frac{1}{2}}\| = \||S|\| = \|S\|,$$

we see that, for any $\epsilon \in (0, \|S\|)$, there exists a unit vector $\xi \in \mathcal{H}$ such that

$$\langle \xi, |S|\xi\rangle \geq \|S\| - \epsilon. \tag{4.38}$$

Let $S = U|S|$ be the polar decomposition of S. Since $\|U\| = 1$, we have $\|U\xi\| \leq \|\xi\| = 1$. Moreover, by Lemma 3.1.6 and Proposition 3.1.7 $\ker U = \ker S = \ker |S|$, and so (4.38) implies that $U\xi$ is not in $\ker U$. Therefore, using Lemma 4.3.1 we see that

$$\|\xi \otimes (U\xi)^*\|_1 = \|\xi\|\|U\xi\| = \|U\xi\| \in (0,1].$$

In addition, using (4.30) and the fact that $|S| = U^*S$ we get

$$\operatorname{Trace}[S(\xi \otimes (U\xi)^*)] = \operatorname{Trace}((S\xi) \otimes (U\xi)^*) = \langle U\xi, S\xi\rangle = \langle \xi, U^*S\xi\rangle = \langle \xi, |S|\xi|\rangle.$$

Combining this with (4.38) then gives

$$\operatorname{Trace}[S(\xi \otimes (U\xi)^*)] \geq \|S\| - \epsilon. \tag{4.39}$$

Set $T = c^{-1}(\xi \otimes (U\xi)^*)$, with $c = \|\xi \otimes (U\xi)^*\|_1$. Then $\|T\|_1 = 1$ and, as $c^{-1} \geq 1$, using (4.39) we see that, for any $\epsilon \in (0, \|S\|)$, we have

$$\operatorname{Trace} ST = c^{-1} \operatorname{Trace}[S(\xi \otimes (U\xi)^*)] \geq c^{-1}(\|S\| - \epsilon) \geq \|S\| - \epsilon.$$

Combining this with (4.37) yields (4.35). The proposition is thus proved. □

In the sequel we denote by $\mathcal{R}_\infty$ the subspace of $\mathcal{L}(\mathcal{H})$ consisting of finite rank operators. This is the subspace of $\mathcal{H}$ spanned by the rank 1, i.e., by all operators of the form (3.12).

We know that the closure of $\mathcal{R}_\infty$ in $\mathcal{L}(\mathcal{H})$ is $\mathcal{K}$ (cf. Proposition 4.1.6). In addition, the following holds.

Lemma 4.3.3. *The finite-rank operators are dense in $\mathcal{L}^1$.*

Proof. Let $T \in \mathcal{L}^1$ have polar decomposition $T = U|T|$, and let $(\xi_n)_{n\geq 0}$ be an orthonormal family of $\mathcal{H}$ such that $T\xi_n = \mu_n(T)\xi_n$ for all $n \in \mathbb{N}_0$. Then by Proposition 4.1.8 we have

$$T = \sum_{n\geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*,$$

where the series converges in norm. By Lemma 4.3.1 we have

$$\|(U\xi_n) \otimes \xi_n^*\|_1 = \|U\xi_n\|\|\xi_n\| \leq \|U\|\|\xi_n\|^2 = 1,$$

and so we see that

$$\sum_{n\geq 0} \|\mu_n(T)(U\xi_n) \otimes \xi_n^*\|_1 \leq \sum_{n\geq 0} \mu_n(T) = \|T\|_1 < \infty.$$

Thus the series $\sum_{n\geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*$ converges in $\mathcal{L}^1$. Since it converges to T in $\mathcal{L}(\mathcal{H})$ and the inclusion of $\mathcal{L}^1$ into $\mathcal{L}(\mathcal{H})$ is continuous (cf. Remark 4.2.6), T is its sum in $\mathcal{L}^1$ too, that is, T is the limit in $\mathcal{L}^1$ of the finite-rank operators $\sum_{n<N} \mu_n(T)(U\xi_n) \otimes \xi_n^*$. This shows that finite-rank operators are dense in $\mathcal{L}^1$, proving the lemma. □

Proposition 4.3.4. *The map $\mathcal{L}(\mathcal{H}) \ni S \to (S,.) \in (\mathcal{L}^1)'$ is an isometric isomorphism.*

Proof. It follows from (4.35) that the map $\mathcal{L}(\mathcal{H}) \ni S \to (S,.) \in (\mathcal{L}^1)'$ is isometric. Therefore, in view of Lemma 1.1.8, in order to prove this map is an isometric isomorphism we only have to check it is onto.

Let $\varphi \in (\mathcal{L}^1)'$. Let S be the endomorphism of $\mathcal{H}$ defined by

$$\langle \eta, S\xi\rangle = \langle \varphi, \xi \otimes \eta^*\rangle \qquad \forall \xi, \eta \in \mathcal{H}.$$

Thanks to Lemma 4.3.1 we have

$$|\langle \eta, S\xi\rangle| = |\langle \varphi, \xi \otimes \eta^*\rangle| \leq \|\varphi\|_{(\mathcal{L}^1)'}\|\langle \varphi, \xi \otimes \eta^*\rangle\|_1 \leq \|\varphi\|_{(\mathcal{L}^1)'}\|\|\xi\|\|\eta\|.$$

Thus,

$$\sup_{\|\xi\|=1} \|S\xi\| = \sup_{\|\eta\|=1} \sup_{\|\xi\|=1} |\langle \eta, S\xi\rangle| \leq \|\varphi\|_{(\mathcal{L}^1)'},$$

showing that S is a continuous endomorphism, i.e., S is contained in $\mathcal{L}(\mathcal{H})$.

Now, thanks to 4.30, for any ξ and η in $\mathcal{H}$, we have

$$(4.40)\quad (S, \xi\otimes\eta^*) = \text{Trace}(S(\xi\otimes\eta^*)) = \text{Trace}((S\xi)\otimes\eta^*) = \langle \eta, S\xi\rangle = \langle \varphi, \xi\otimes\eta^*\rangle,$$

that is, $(S,.)$ and φ agree on operators of rank 1, and by linearity they agree on their span, namely, $\mathcal{R}_\infty$. Both $(S,.)$ and φ are continuous linear forms and by Lemma 4.3.3 the subspace $\mathcal{R}_\infty$ is dense in $\mathcal{L}^1$, so $(S,.)$ and φ agree on all $\mathcal{L}^1$. This proves that the map $\mathcal{L}(\mathcal{H}) \ni S \to (S,.) \in (\mathcal{L}^1)'$ is onto, completing the proof. □

It follows from the above the proposition that $\mathcal{L}(\mathcal{H})$ can be canonically identified with the dual of $\mathcal{L}^1$. For this reason $\mathcal{L}^1$ can be referred to as the *predual* of $\mathcal{L}(\mathcal{H})$.

The converse is not true. Namely, $\mathcal{L}^1$ is not the dual of $\mathcal{L}(\mathcal{H})$, but instead is that of the space $\mathcal{K}$ of compact operators. This is the content of the following.

Proposition 4.3.5. *The map $\mathcal{L}^1 \ni S \to (S,.) \in \mathcal{K}'$ is an isometric isomorphism.*

Proof. As in the proof of Proposition 4.3.4 we only have to prove that the map $\mathcal{L}^1 \ni S \to (S,.) \in \mathcal{K}'$ is onto. To this end let $\varphi \in \mathcal{K}'$. Since the inclusion of $\mathcal{L}^1$ into $\mathcal{K}$ is contionuous (cf. Remark 4.2.6), we see that φ induces a continuous linear form on $\mathcal{L}^1$, and hence by Proposition 4.3.4 there exists $S \in \mathcal{L}(\mathcal{H})$ such that

$$(4.41)\qquad \langle \varphi, T\rangle = \text{Trace}(ST) \qquad \forall T \in \mathcal{L}^1.$$

In particular, by (4.40) we have

$$(4.42)\qquad \langle \eta, S\xi\rangle = \langle \varphi, \xi\otimes\eta^*\rangle \qquad \forall \xi, \eta \in \mathcal{H}.$$

Let $S = U|S|$ be the polar decomposition of S and let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$. As by Proposition 3.1.7 $|S| = U^*S$, using (4.42) we see that, for any $n \in \mathbb{N}_0$, we have

$$(4.43)\qquad \langle \xi_n, |S|\xi_n\rangle = \langle \xi_n, U^*S\xi_n\rangle = \langle U\xi_n, S\xi_n\rangle = \langle \varphi, \xi_n\otimes(U\xi_n)^*\rangle.$$

Notice that thanks to (4.33) we have

$$\xi_n\otimes(U\xi_n)^* = ((U\xi_n)\otimes\xi_n^*)^* = (U(\xi_n\otimes\xi_n^*))^* = (\xi_n\otimes\xi_n^*)U^*,$$

and hence $\langle \xi_n, |S|\xi_n\rangle = \langle \varphi, (\xi_n\otimes\xi_n^*)U^*\rangle$. Therefore, for all $N \in \mathbb{N}$,

$$\sum_{n<N} \langle \xi_n, |S|\xi_n\rangle = \left|\langle \varphi, (\sum_{n<N} \xi_n\otimes\xi_n^*)U^*\rangle\right| \leq \|\varphi\|_{\mathcal{K}'} \left\|(\sum_{n<N} \xi_n\otimes\xi_n^*)U^*\right\|$$
$$\leq \|\varphi\|_{\mathcal{K}'} \left\|\sum_{n<N} \xi_n^*\otimes\xi_n\right\| \|U^*\|.$$

By Proposition 3.1.7 $\|U^*\| = \|U\| = 1$ and, as $\sum_{n<N} \xi_n\otimes\xi_n^*$ is the orthogonal projection onto the span of $\xi_0, \ldots, \xi_N$, this operator has norm 1 too. Thus,

$$\sum_{n<N} \langle \xi_n, |S|\xi_n\rangle \leq \|\varphi\|_{\mathcal{K}'} \qquad \forall N \in \mathbb{N}.$$

Using (4.18) and Lemma 4.2.1 we then get

$$\|S\|_1 = \||S|\|_1 = \sum_{n\geq 0} \langle \xi_n, |S|\xi_n\rangle \leq \|\varphi\|_{\mathcal{K}'} < \infty,$$

and hence S is trace-class.

Since S is in $\mathcal{L}^1$, the map $(S,.) : \mathcal{K} \ni T \rightarrow \text{Trace}(ST)$ is a continuous linear map on $\mathcal{K}$. Moreover, it follows from (4.41) that it agrees with φ on $\mathcal{L}^1$, and hence on $\mathcal{R}_\infty$. As by Proposition 4.1.6 $\mathcal{R}_\infty$ is dense in $\mathcal{K}$, we see that $(S,.)$ and φ agree on all $\mathcal{K}$, showing that the map $\mathcal{L}^1 \ni S \rightarrow (S,.) \in \mathcal{K}'$ is onto. The proof is complete. ☐

4.4. Hilbert-Schmidt Operators

For any $T \in \mathcal{L}(\mathcal{H})$ we define

$$\|T\|_2 = \left(\sum_{n\geq 0} \mu_n(T)^2\right)^{\frac{1}{2}}.$$

We then define

$$\mathcal{L}^2 := \{T \in \mathcal{L}(\mathcal{H});\ \|T\|_2 < \infty\}.$$

The elements of $\mathcal{L}^2$ are called Hilbert-Schmidt operators.

As in (4.18)–(4.20) we have

$$\|T\|_2 = \|T^*\|_2 = \||T|\|_2 \qquad \forall T \in \mathcal{L}(\mathcal{H}), \tag{4.44}$$

$$\|\lambda T\|_2 = |\lambda|\|T\|_2 \qquad \forall T \in \mathcal{L}(\mathcal{H})\ \forall \lambda \in \mathbb{C}, \tag{4.45}$$

$$\|ATB\|_2 \leq \|A\|\|T\|_2\|B\| \qquad \forall A, T, B \in \mathcal{L}(\mathcal{H}). \tag{4.46}$$

As an immediate consequence of (4.44) we see that, for any $T \in \mathcal{L}(\mathcal{H})$,

$$T \in \mathcal{L}^2 \Longleftrightarrow T^* \in \mathcal{L}^2 \Longleftrightarrow |T| \in \mathcal{L}^2.$$

Lemma 4.4.1. *The following hold.*

(i) *For any* $T \in \mathcal{L}(\mathcal{H})$,

$$\|T\| \leq \|T\|_2 \leq \|T\|^{\frac{1}{2}} \|T\|_1^{\frac{1}{2}}. \tag{4.47}$$

(ii) *We have the inclusions,*

$$\mathcal{L}^1 \subset \mathcal{L}^2 \subset \mathcal{K}. \tag{4.48}$$

Proof. Let $T \in \mathcal{L}(\mathcal{H})$. Since $\|T\| = \mu_0(T)$ we have $\|T\| \leq \|T\|_2$. Moreover, using (4.3) we see that, for any $n \in \mathbb{N}_0$,

$$\mu_n(T)^2 = \mu_n(T).\mu_n(T) \leq \mu_0(T)\mu_n(T) = \|T\|\mu_n(T).$$

Thus,

$$\sum_{n\geq 0} \mu_n(T)^2 \leq \|T\| \sum_{n\geq 0} \mu_n(T) = \|T\|\|T\|_1,$$

from which we get $\|T\|_2 \leq \|T\|^{\frac{1}{2}} \|T\|_1^{\frac{1}{2}}$. In particular, if $T \in \mathcal{L}^1$, then $\|T\|_2 < \infty$, and hence T is in $\mathcal{L}^2$. Thus $\mathcal{L}^1$ is contained in $\mathcal{L}^2$.

Let $T \in \mathcal{L}^2$. Then $\sum_{n\geq 0} \mu_n(T)^2 < \infty$, and hence $\lim_{n\rightarrow\infty} \mu_n(T) = 0$. Therefore T is compact by Proposition 4.1.6. This proves that $\mathcal{L}^2$ is contained in $\mathcal{K}$. ☐

Example 4.4.2. Let ξ and η be vectors in $\mathcal{H}$. By Lemma 4.3.1 both $\|\xi \otimes \eta^*\|$ and $\|\xi \otimes \eta^*\|_1$ are equal to $\|\xi\|\|\eta\|$, and so using (4.47) we get

$$\|\xi \otimes \eta^*\|_2 = \|\xi\|\|\eta\|.$$

LEMMA 4.4.3. *Let $T \in \mathcal{L}(\mathcal{H})$. Then, for any orthonormal basis $(\xi_n)_{n\geq 0}$, we have*

$$\|T\|_2^2 = \operatorname{Trace}|T|^2 = \sum_{n\geq 0} \|T\xi_n\|^2, \tag{4.49}$$

and hence

$$T \in \mathcal{L}^2 \Longleftrightarrow \operatorname{Trace}|T|^2 < \infty \Longleftrightarrow |T|^2 \in \mathcal{L}^1. \tag{4.50}$$

PROOF. Assume first that T is compact. Then thanks to (4.12) $\mu_n(|T|^2) = \mu_n(T)^2$ for all $n \in \mathbb{N}_0$, and so we have

$$\|T\|_2 = \left(\sum_{n\geq 0} \mu_n(|T|^2)\right)^{\frac{1}{2}} = (\operatorname{Trace}|T|^2)^{\frac{1}{2}}.$$

Suppose now that T is not compact. Then by (4.48) T is not in $\mathcal{L}^2$, and hence $\|T\|_2 = \infty$. By Proposition 3.4.4 the fact that T is not compact, implies that $|T|$ is not compact either. Observe further that, as $\lim_{t\to 0^+} t^{\frac{1}{2}} = 0$, it follows from Proposition 3.4.8 that, for any positive compact operator S, the operator $S^{\frac{1}{2}}$ is compact too. Therefore, if $|T|^2$ were compact, then $|T| = (|T|^2)^{\frac{1}{2}}$ would be compact too. Since $|T$ is not compact, we deduce that $|T|^2$ cannot be compact. Incidentally, $|T|^2$ is not trace-class, and hence

$$\operatorname{Trace}|T|^2 = \||T|^2\|_1 = \infty = \|T\|_2.$$

In general, for any $T \in \mathcal{L}(\mathcal{H})$, using Lemma 4.2.3 we get

$$\|T\|_2^2 = \operatorname{Trace}|T|^2 = \sum_{n\geq 0} \langle \xi_n, |T|^2 \xi_n\rangle. \tag{4.51}$$

Since $|T|^2 = T^*T$, for any $n \in \mathbb{N}_0$, we have

$$\langle \xi_n, |T|\xi_n\rangle = \langle \xi_n, T^*T\xi_n\rangle = \langle T\xi_n, T\xi_n\rangle = \|T\xi_n\|^2.$$

Thus,

$$\|T\|_2^2 = \sum_{n\geq 0} \|T\xi_n\|^2.$$

Finally, the equivalences (4.50) immediately follow from (4.49). The lemma is thus proved. □

LEMMA 4.4.4. *We have*

$$\|S + T\|_2 \leq \|S\|_2 + \|T\|_2 \qquad \forall S, T \in \mathcal{L}(\mathcal{H}). \tag{4.52}$$

PROOF. Let $S, T \in \mathcal{L}(\mathcal{H})$ and let $(\xi_n)_{n\geq 0}$ be an orthonormal basis. Using (4.49) and the Minkowski's inequality for series we get

$$\|S + T\|_2 = \left(\sum_{n\geq 0} \|(S+T)\xi_n\|^2\right)^{\frac{1}{2}} \leq \left(\sum_{n\geq 0} (\|S\xi_n\| + \|T\xi_n\|)^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{n\geq 0} \|S\xi_n\|^2\right)^{\frac{1}{2}} + \left(\sum_{n\geq 0} \|T\xi_n\|^2\right)^{\frac{1}{2}} = \|S\|_2 + \|T\|_2,$$

proving the lemma. □

PROPOSITION 4.4.5. *The following hold.*

(1) *$\mathcal{L}^1$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$.*
(2) *$\|.\|_1$ is a norm on $\mathcal{L}^1$ with respect to which $\mathcal{L}^1$ is a Banach space.*

Proof. It follows from (4.45)–(4.46), (4.47) and (4.52) that $\mathcal{L}^2$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$ and $\|.\|_2$ is a norm on $\mathcal{L}^2$. Moreover, by arguing as in the proof Proposition 4.2.5 we can show that $\mathcal{L}^2$ is complete with respect to the norm $\|.\|_2$, i.e., $\mathcal{L}^2$ is a Banach space for this norm. □

Remark 4.4.6. Combining (4.17) and (4.47) shows that, for any $T \in \mathcal{L}(\mathcal{H})$,

$$\|T\| \leq \|T\|_2 \leq \|T\|_1.$$

Therefore, the inclusions $\mathcal{L}^1 \subset \mathcal{L}^2$ and $\mathcal{L}^2 \subset \mathcal{K}$ are continuous.

In addition, arguing as in the proof of Lemma 4.4.7 yields the following.

Lemma 4.4.7. *The finite-rank operators are dense in $\mathcal{L}^2$.*

Proposition 4.4.8. *Let S and T be Hilbert-Schmidt operators. Then ST and TS are trace-class operators and we have*

$$|\operatorname{Trace}(ST)| \leq \|ST\|_1 \leq \|S\|_2\|T\|_2,$$
$$\operatorname{Trace}(ST) = \operatorname{Trace}(TS).$$

Proof. Let $ST = U|ST|$ be the polar decomposition of ST and let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$. By Proposition 3.1.8 $|ST| = U^*ST$, and so using (4.21) we get

$$\|ST\|_1 = \sum_{n\geq 0}\langle \xi_n, U^*ST\xi_n\rangle = \sum_{n\geq 0}\langle SU\xi_n, T\xi_n\rangle \leq \sum_{n\geq 0}\|SU\xi_n\|\|T\xi_n\|.$$

Using Cauchy-Schwartz Inequality for sequences together with (4.49) we then get

$$\|ST\|_1 \leq \left(\sum_{n\geq 0}\|SU\xi_n\|^2\right)^{\frac{1}{2}}\left(\sum_{n\geq 0}\|T\xi_n\|^2\right)^{\frac{1}{2}} = \|SU\|_2\|T\|_2 \leq \|S\|_2\|U\|\|T\|_2 \leq \|S\|_2\|T\|_2,$$

Notice that by Proposition 3.1.8 and (4.46) we have $\|SU\|_2 \leq \|S\|_2\|U\| \leq \|S\|_2$. Thus,

$$\|ST\|_1 \leq \|S\|_2\|T\|_2. \tag{4.53}$$

This proves that ST is trace-class. Moreover, combining (4.53) with (4.27) yields

$$|\operatorname{Trace}(ST)| \leq \|ST\|_1 \leq \|S\|_2\|T\|_2,$$

proving (4.4.8).

It follows from all this that, if we fix $S \in \mathcal{L}^2$, then both $T \to \operatorname{Trace}(ST)$ and $T \to \operatorname{Trace}(TS)$ are continuous linear forms on $\mathcal{L}^2$. Moreover, as finite-rank operators are trace-class, it follows from (4.29) that these linear forms agree on finite-rank operators. Since the latter are dense in $\mathcal{L}^2$ by Lemma 4.4.7, it follows that $\operatorname{Trace}(ST) = \operatorname{Trace}(TS)$ for all $T \in \mathcal{L}^2$, completing the proof. □

For $S, T \in \mathcal{L}^2$ we define

$$(S, T) := \operatorname{Trace}(ST).$$

This defines a bilinear form on $\mathcal{L}^2$. If $S \in \mathcal{L}^2$, then by (4.4) its adjoint S^* is in $\mathcal{L}^2$ too. Therefore, for $S, T \in \mathcal{L}^2$ we may also define

$$\langle S, T\rangle_{\mathcal{L}^2} := \operatorname{Trace}(S^*T). \tag{4.54}$$

This defines a Hermitian form on $\mathcal{L}^2$. Moreover, as $T^*T = |T|^2$, using (4.49) we get

$$\langle T, T\rangle_{\mathcal{L}^2} = \operatorname{Trace} T^*T = \operatorname{Trace} |T|^2 = \|T\|_2^2.$$

Since $\|.\|_2$ is a norm on $\mathcal{L}^2$ this proves that $\langle ., .\rangle_{\mathcal{L}^2}$ is actually a positive-definite inner product on $\mathcal{L}^2$ whose associated norm is just the Hilbert-Schmidt norm. Combining this with Proposition 4.4.5 we obtain:

PROPOSITION 4.4.9. *$\mathcal{L}^2$ is a Hilbert space with respect to the inner product (4.54).*

For $S \in \mathcal{L}^2$ denote by $\langle S, .\rangle_{\mathcal{L}^2}$ the linear form $T \to \langle S, T\rangle_{L^2}$. Similarly, let us denote by $(S, .)$ the linear form $T \to (S, T)$. By definition of $\langle ., .\rangle_{\mathcal{L}^2}$ we have $(S, .) = \langle S^*, .\rangle$. Since $\langle ., .\rangle$ is a Hilbert-space inner product on $\mathcal{L}^2$. The map $S \to \langle S, .\rangle_{\mathcal{L}^2}$ is an isometric *antilinear* isomorphism from $\mathcal{L}^2$ onto $(\mathcal{L}^2)'$. Since (4.44) implies that $S \to S^*$ is an isometric *antilinear* isomorphism of $\mathcal{L}^2$ we obtain:

PROPOSITION 4.4.10. *The map $\mathcal{L}^2 \ni S \to (S, .) \in (\mathcal{L}^2)'$ is an isometric linear isomorphism from $\mathcal{L}^2$ onto $(\mathcal{L}^2)'$.*

4.5. Integral Operators

Let (X, μ) be a σ-finite measured space such that $L^2_\mu(X)$ is separable. Let $K(x, y) \in L^2_{\mu\otimes\mu}(X \times X)$. For $f \in L^2_\mu(X)$ define

$$T_K f(x) = \int_X K(x, y) f(y) d\mu(y).$$

The function $T_K f(x)$ is measurable and by Cauchy-Schwartz's Inequality,

$$|T_K(x)|^2 \leq \int_X |K(x, y)|^2 d\mu(y) \int_X |f(x)|^2 d\mu(x) = \|f\|^2_{L^2} \int_X |K(x, y)|^2 d\mu(y),$$

and hence

$$\int_X |T_K(x)|^2 d\mu(x) \leq \|f\|^2_{L^2} \int_X \left(\int_X |K(x, y)|^2 d\mu(y) \right) d\mu(x) = \|f\|^2_{L^2} \|K\|^2_{L^2}$$

Therefore the map $T_K : f \to T_K f$ is a continuous endomorphism of $L^2_\mu(X)$. Such an operator is called a *integral operator*.

EXAMPLE 4.5.1. Let φ and ψ be in $L^2_\mu(X)$. Then, for any $f \in L^2_\mu(X)$,

$$(\varphi \otimes \psi^*) f(x) = \langle \psi, f\rangle \varphi(x) = \varphi(x) \int_X \overline{\psi(y)} f(y) d\mu(y) = T_{\varphi\otimes\overline{\psi}} f(x),$$

where $\varphi \otimes \overline{\psi}$ is the element of $L^2_{\mu\otimes\mu}(X \times X)$ defined by

$$(\varphi \otimes \overline{\psi})(x, y) = \varphi(x)\overline{\psi(y)}.$$

This shows that any rank 1 operator is an integral operator, and hence by linearity any finite rank operator is an integral operator.

PROPOSITION 4.5.2. *Let K and K' be in $L^2_{\mu\otimes\mu}(X \times X)$. Then*

$$T_K^* = T_{K^*} \qquad \text{and} \qquad T_K T_{K'} = T_{K*K'},$$

where K^* and $K * K'$ are the functions in $L^2_{\mu\otimes\mu}(X \times X)$ defined by

$$K^*(x, y) = \overline{K(y, x)}, \tag{4.55}$$

$$K * K'(x, y) = \int_X K(x, z) K'(z, y) d\mu(z). \tag{4.56}$$

Proof. It is immediate from its definition that $K^*(x,y)$ is in $L^2_{\mu\otimes\mu}(X\times X)$. Moreover, for any f and g in $L^2_\mu(X)$, we have

$$\begin{aligned}\langle f, T_K g\rangle &= \int_X \overline{f(x)} T_K g(x) d\mu(x) = \int_X \overline{f(x)} \left(\int_X K(x,y) g(y) d\mu(y) \right) d\mu(x) \\ &= \int_X \left(\int_X \overline{K^*(y,x) f(x)} d\mu(x) \right) g(y) d\mu(y) = \int_X \overline{T_{K^*} f(y)} g(y) d\mu(y) \\ &= \langle T_{K^*} f, g\rangle,\end{aligned}$$

that is, T_{K^*} is the adjoint of T_K.

Next, as $K(x,y)$ and $K'(x,y)$ both are in $L^2_{\mu\otimes\mu}(X\times X)$ the function $K*K'(x,y)$ defined by (4.56) is a well defined measurable function. This is in fact an element of $L^2_{\mu\otimes\mu}(X\times X)$, for we have

$$|K*K'(x,y)|^2 \leq \int_X |K(x,z)||K'(z,x)| d\mu(z) \leq \int_X |K(x,z)|^2 d\mu(z) \int_X |K'(z,y)|^2 d\mu(z),$$

and hence

$$\int_{X\times X} |K*K'(x,y)|^2 d\mu(x) d\mu(y) \leq \int_X \int_X |K(x,z)|^2 d\mu(z) d\mu(x) \int_X \int_X |K'(z,y)|^2 d\mu(z) d\mu(y) < \infty.$$

Moreover, for any $f \in L^2_\mu(x)$, we have

$$\begin{aligned}T_K T_{K'} f(x) &= \int_X K(x,z) (T_{K'} f)(z) d\mu(z) = \int_X K(x,z) \left(\int_X K'(z,y) f(y) d\mu(y) \right) d\mu(z) \\ &= \int_X \left(\int_X K(x,z) K'(z,y) d\mu(z) \right) f(y) d\mu(y) = \int_X K*K'(x,y) f(y) d\mu(y) \\ &= T_{K*K'} f(x),\end{aligned}$$

which shows that $T_K T_{K'} = T_{K*K'}$. The proof is complete. □

Proposition 4.5.3. *The following hold.*

(1) *For any $K \in L^2_{\mu\otimes\mu}(X\times X)$ the operator T_K is a Hilbert-Schmidt operator.*
(2) *The map $K \to T_K$ is an isometric isomorphism from $L^2_{\mu\otimes\mu}(X\times X)$ onto $\mathcal{L}^2(L^2_\mu(X))$.*

Proof. Let $K \in L^2_{\mu\otimes\mu}(X\times X)$ and let $(\varphi_n)_{n\geq 0}$ be an orthonormal basis of $L^2_\mu(X)$. By Lemma 4.4.3 we have

$$\|T_K\|_2^2 = \sum_{n\geq 0} \|T_K \varphi_n\|^2 = \sum_{n,m} |\langle \varphi_m, T_K \varphi_n\rangle|^2.$$

Notice that

$$\langle \varphi_m, T_K \varphi_n\rangle = \int_{X\times X} \overline{\varphi_m(x)} \varphi_n(x) K(x,y) d\mu(x) d\mu(y) = \langle \varphi_m \otimes \overline{\varphi_n}, K\rangle_{L^2_{\mu\otimes\mu}(X\times X)}.$$

Since $(\varphi_m \otimes \overline{\varphi_n})_{m,n\geq 0}$ is an orthonormal basis of $L^2_\mu(X\times X)$, it follows that

$$\|T_K\|_2^2 = \sum_{n,m} |\langle \varphi_m \otimes \overline{\varphi_n}, K\rangle_{L^2_{\mu\otimes\mu}(X\times X)}|^2 = \|K\|^2_{L^2(X\times X)}.$$

Thus T_K is a Hilbert-Schmidt operator and $\|T_K\| = \|K\|_{L^2(X\times X)}$.

This shows that $K \to T_K$ is an isometric linear map from $L^2_{\mu\otimes\mu}(X \times X)$ to $\mathcal{L}^2(L^2_\mu(X))$. Lemma 1.1.8 then insures us that this map is an isometric isomorphism from $L^2_{\mu\otimes\mu}(X\times X)$ onto its range which is a closed subspace of $\mathcal{L}^2(L^2_\mu(X))$. It follows from Example 4.5.1 that any finite-rank operator on $L^2_\mu(X)$ is an integral operator. As by Lemma 4.4.7 the finite-rank operators are dense in $\mathcal{L}^2$, we then deduce that the map $K \to T_K$ is onto. Therefore, it realizes an isometric isomorphism $L^2_{\mu\otimes\mu}(X \times X)$ onto $\mathcal{L}^2(L^2_\mu(X))$. The proof is complete. □

4.6. Trace Theorems for Integral Operators

In this section we prove a trace theorem for integral operators. Here we assume that X is a separable metrizable locally compact Hausdorff space (e.g., X is a manifold) and we also assume that μ is a Radon measure on X.

The assumption on the topology of X implies that X is σ-compact, and hence (X, μ) is a σ-finite measured space. This assumption also insures us that, for any compact $K \subset X$, the space $C_K(X)$ is separable. Together with the σ-compacity of X and the density of $C_c(X)$ in $L^2_\mu(X)$ this implies that $L^2_\mu(X)$ is separable.

Let us denote by $\operatorname{supp}\mu$ the support of μ. Recall that $X \setminus (\operatorname{supp}\mu)$ is the union set of all open sets $O \subset X$ such that $\mu(O) = 0$, and hence $\operatorname{supp}\mu$ is a closed subset of X.

Lemma 4.6.1. *The following hold.*

(i) *The open set* $X \setminus (\operatorname{supp}\mu)$ *has measure* 0.
(ii) *The support of* $\mu_{|\operatorname{supp}\mu}$ *is equal to* $\operatorname{supp}\mu$.
(iii) *The support* $\mu \otimes \mu$ *is equal to* $(\operatorname{supp}\mu) \times (\operatorname{supp}\mu)$.

Proof. Set $V = X\setminus(\operatorname{supp}\mu)$. Since μ is a Radon measure and X is σ-compact, μ is a regular measure, and hence

$$\mu(V) = \sup\left\{\mu(K);\ K \subset V,\ K \text{ compact}\right\}. \tag{4.57}$$

Let K be a compact set contained in V, i.e., K is covered by the family of open subsets of measure 0. Since K is compact, there exist finitely many such open sets $O_1, \ldots, O_k$ that cover K, and hence K has measure zero. Thus any compact contained in V has measure zero. Combining this with (4.57) shows that $\mu(V) = 0$.

Set $E = \operatorname{supp}\mu$ and let O be an open subset of E such that $\mu(O) = 0$. Then there exists an open $O' \subset X$ such that $O = O' \cap E$. Then $O' \subset O \cup V$. As both O and V have measure zero, it follows that so does O'. Therefore O' is contained in V, and hence O must be the empty set. This shows that the only open subset of E that has measure zero is the empty set. Therefore the support of $\mu_{|E}$ is equal to E.

Set $W = X \times X \setminus \operatorname{supp}(\mu\otimes\mu)$. As $O \times X$ and $X \times O$ are open subsets of $X \times X$ on which $\mu \otimes \mu$ vanishes, we see that they both are contained in W. Conversely, if $(x, y) \in W$, then there exist an open neighborood O_1 of x in X and an open neighborhood O_2 of y in X such that $O_1 \times O_2 \subset W$. Then, using (i), we see that

$$\mu(O_1)\mu(O_2) = (\mu \otimes \mu)(O_1 \times O_2) \leq \mu(W) = 0,$$

Thus $\mu(O_1)$ or $\mu(O_1)$ must be zero, that is, O_1 or O_2 must be contained in V, and hence (x, y) is contained in $(V \times X) \cup (X \times V)$. It follows from all this that $W = (V \times X) \cup (X \times V)$, so taking complements shows that $\operatorname{supp}(\mu \otimes \mu)$ is equal to $(\operatorname{supp}\mu) \times (\operatorname{supp}\mu)$ agree. The proof is complete. □

THEOREM 4.6.2 (Duflo [**Du**]). *Let $K(x,y) \in L^2_{\mu\otimes\mu}(X\times X)\cap C(X\times X)$ and assume that the operator T_K is trace-class. Then the function $K(x,x)$ is in $L^1_\mu(X)$ and we have*

$$\operatorname{Trace} T_K = \int_X K(x,x)d\mu(x).$$

PROOF. Let $T_K = U|T_K|$ be the polar decomposition of T_K and let $(\xi_n)_{n\geq 0}$ be an orthonormal family in $L^2_\mu(X)$ such that $|T_K|\xi_n = \mu_n(T)\xi_n$ for all $n\in\mathbb{N}$. For $n\in\mathbb{N}$ set $\eta_n = U\xi_n$. As T_K is trace-class and the L^2-norms of the functions ξ_n are equal to 1, we have

$$\sum_{n\geq 0}\int_X \mu_n(T_K)|\xi_n(x)|^2 d\mu(x) = \sum_{n\geq 0}\mu_n(T_K) < \infty,$$

that is, the series $\sum_{n\geq 0}\mu_n(T)|\xi_n(x)|^2$ converges normally in $L^1_\mu(X)$. Therefore, its sum is finite almost everywhere, i.e, the series $\sum_{n\geq 0}\mu_n(T)|\xi_n(x)|^2$ converges almost evrywhere. Likewise, the series $\sum_{n\geq 0}\mu_n(T)|\eta_n(x)|^2$ converges almost everywhere.

CLAIM. Let L be a compact subset of X. For any $\epsilon > 0$ there exists a compact $L'\subset L$ such that

(i) $\mu(L\setminus L') < \epsilon$.
(ii) For all $n\in\mathbb{N}_0$ the functions ξ_n and η_n are continuous on L'.
(iii) The series $\sum_{n\geq 0}\mu_n(T)|\xi_n(x)|^2$ and $\sum_{n\geq 0}\mu_n(T)|\eta_n(x)|^2$ converge uniformly on L'.

PROOF OF THE CLAIM. By Lusin's theorem (see, e.g., [**Fo**]), for any $n\in\mathbb{N}_0$, there exists a Borel set $E_n\subset L$ such that $\mu(L\setminus E_n)\leq 2^{-(n+1)}\epsilon$ and the functions $\xi_n(x)$ and $\eta_n(x)$ are continuous on E_n. Set $E=\bigcap_{n\geq 0}E_n$. Then E is a Borel set of L such that $\mu(L\setminus E)\leq\epsilon$ and *all* the functions $\xi_n(x)$ and $\eta_n(x)$ are continuous on E.

As the series $\sum_{n\geq 0}\mu_n(T)|\xi_n(x)|^2$ converges almost everywhere on E, Egoroff's theorem (see, e.g., [**Fo**]) implies that there exists a Borel set $F\subset E$ such that $\mu(E\subset F)<\epsilon$ and the series $\sum_{n\geq 0}\mu_n(T)|\xi_n(x)|^2$ and $\sum_{n\geq 0}\mu_n(T)|\eta_n(x)|^2$ converge uniformly on F.

Since μ is a regular measure, there exists a compact subset L' of L such that $\mu(F\setminus L')\leq\epsilon$. As $L\setminus L' = (L\setminus E)\cup(E\setminus F)\cup(F\setminus\tilde{L})$, we then get

$$\mu(L\setminus L')\leq\mu(L\setminus E)+\mu(E\setminus F)+\mu(F\setminus L')\leq 3\epsilon.$$

In addition, as L' is contained in E and in F, all the functions $\xi_n(x)$ and $\eta_n(x)$ are continuous on L' and the series $\sum_{n\geq 0}\mu_n(T)|\xi_n(x)|^2$ and $\sum_{n\geq 0}\mu_n(T)|\eta_n(x)|^2$ converge uniformly on L'. The claim is thus proved. □

Since X is σ-compact, there exists an increasing sequence $(L_j)_{j\geq 0}$ of compact sets such that $X=\bigcup_{j\geq 0}L_n$. For every $j\in\mathbb{N}_0$ the above claim insures us the existence of a compact $L_j'\subset L_j$ satisfies the conditions (i), (ii) and (iii) above with $L=L_j$, $L'=L_j'$ and $\epsilon=\frac{1}{j+1}$.

Let $j\in\mathbb{N}_0$. Set $\tilde{Y}_j=\bigcup_{k\leq j}L_k'$ and $Y_j=\operatorname{supp}\mu_{|Y_j}$. It follows from Lemma 4.6.1 that $\mu(\tilde{Y}_j\setminus Y_j)=0$ and $\operatorname{supp}\mu_{|Y_j}=Y_j$. Moreover, as

$$L_j\setminus Y_j = (L_j\setminus\tilde{Y}_j)\cup(\tilde{Y}_j\setminus Y_j)\subset(L_j\setminus L_j')\cup(\tilde{Y}_j\setminus Y_j),$$

we see that

$$\mu(L_j \setminus Y_j) \leq \mu(L_j \setminus L'_j) + \mu(\tilde{Y}_j \setminus Y_j) \leq \frac{1}{j+1}. \tag{4.58}$$

In addition, as Y_j is contained in $\bigcup_{k\leq j} L'_k$ the following hold:

(a) All the functions $\xi_n(x)$ and $\eta_n(x)$ are continuous on Y_j.
(b) The series $\sum_{n\geq 0} \mu_n(T)|\xi_n(x)|^2$ and $\sum_{n\geq 0} \mu_n(T)|\eta_n(x)|^2$ converge uniformly on Y_j.

Set $Y = \bigcup_{j\geq 0} Y_j$. As $X \setminus Y = \bigcup_{j\geq 0}(X_j \setminus Y)$ and the sequence $(X_j \setminus Y)_{j\geq 0}$ is increasing, we have $\mu(X \setminus Y) = \lim_{j\to 0} \mu(X_j \setminus Y)$. As Y contains Y_j, using (4.58) we see that $\mu(X_j \setminus Y) \leq \mu(X_j \setminus Y_j) \leq \frac{1}{j+1}$. It then follows that $\mu(X \setminus Y) = 0$.

Claim. For all $(x, y) \in Y \times Y$, we have

$$K(x, y) = \sum_{n\geq 0} \mu_n(T)\eta_n(x)\overline{\xi_n(y)}. \tag{4.59}$$

Proof. Let $j \in \mathbb{N}_0$. For all $p, q \in \mathbb{N}_0$ we have

$$\sum_{p\leq n\leq q} \mu_n(T)|\eta_n(x)\overline{\xi_n(y)}| \leq \Big(\sum_{p\leq n\leq q} \mu_n(T)|\xi_n(x)|^2\Big)^{\frac{1}{2}} \Big(\sum_{p\leq n\leq q} \mu_n(T)|\eta_n(x)|^2\Big)^{\frac{1}{2}}.$$

Therefore, using the property (b) above we see that the series $\sum_{n\geq 0} \mu_n(T)\eta_n(x)\overline{\xi_n(y)}$ converges uniformaly on $Y_j \times Y_j$.

For all $(x, y) \in Y_j \times Y_j$ set

$$K'(x, y) = \sum_{n\geq 0} \mu_n(T)\eta_n(x)\overline{\xi_n(y)}.$$

Since by the property (a) all the functions $\eta_n(x)\overline{\xi_n(y)}$ are continuous on $Y_j \times Y_j$ and the above series converges uniformly on $Y_j \times Y_j$, this defines a continuous function on $Y_j \times Y_j$. Moreover, as $Y_j \times Y_j$ is compact, and hence has finite measure, since $\mu \otimes \mu$ is a Radon measure, the uniform convergence implies the convergence in L^2-norm.

On the other hand, as T_K is trace-class, the proof of Lemma 4.3.3 shows that

$$T = \sum_{n\geq 0} (U\xi_n) \otimes \xi_n^* = \sum_{n\geq 0} \eta_n \otimes \xi_n^*, \tag{4.60}$$

Since by Remark 4.4.6 the inclusion of $\mathcal{L}^2$ into $\mathcal{L}^1$ is continuous, it follows that the above series converges in $\mathcal{L}^2$-norm too. Combining this with Proposition 4.5.3 we deduce that the series of the corresponding kernel functions converge to $K(x, y)$ in L^2-norm. As shown in Example 4.5.1, for every $n \in \mathbb{N}_0$, the kernel function of $\eta_n \otimes \xi_n^*$ is equal to $\eta_n(x)\overline{\xi_n(y)}$. Therefore, the series $\sum_{n\geq 0} \mu_n(T)\eta_n(x)\overline{\xi_n(y)}$ converges to $K(x, y)$ in L^2-norm. Since we already know that it converges to $K'(x, y)$ in L^2-norm on $Y_j \times Y_j$, it follows that

$$K(x, y) = K'(x, y) \qquad \text{almost everywhere on } Y_j \times Y_j.$$

Observe that both $K(x, y)$ and $K'(x, y)$ are continuous functions on $Y_j \times Y_j$. Therefore $W := \{(x, y) \in Y_j \times Y_j; K(x, y) \neq K'(x, y)\}$ is an open subset $Y_j \times Y_j$ of measure zero, and hence it is contained $V = X \setminus (\operatorname{supp} \mu \otimes \mu_{|Y_j \times Y_j})$. By construction $\operatorname{supp} \mu_{|Y_j} = Y_j$, so using Lemma 4.6.1 we see that the support of $(\mu \otimes \mu)_{Y_j \times Y_j} =$

$(\mu_{|Y_j}) \otimes (\mu_{|Y_j})$ is equal to $Y_j \times Y_j$. Therefore, V must be the empty set, and hence $K(x,y) = K'(x,y)$ for all $(x,y) \in Y_j \times Y_j$. Thus,

$$K(x,y) = \sum_{n \geq 0} \mu_n(T) \eta_n(x) \overline{\xi_n(y)} \qquad \forall (x,y) \in Y_j \times Y_j,$$

where the series converges uniformly on $Y_j \times Y_j$.

All this shows that the equality (4.59) holds on all the products $Y_j \times Y_j$. Since $Y \times Y = \bigcup_{j \geq 0} Y_j \times Y_j$ the claim follows. □

Since $\mu(X \setminus Y) = 0$, it follows from (4.59) that, almost everywhere on X,

$$K(x,x) = \sum_{n \geq 0} \mu_n(T) \eta_n(x) \overline{\xi_n(x)}. \tag{4.61}$$

Moreover, as

$$\int_X |\eta_n(x) \overline{\xi_n(x)}| d\mu(x) \leq \left(\int_X |\eta_n(x)|^2 d\mu(x) \right)^{\frac{1}{2}} \left(\int_X |\xi_n(x)|^2 d\mu(x) \right)^{\frac{1}{2}} \leq 1,$$

we see that

$$\sum_{n \geq 0} \int_X \mu_n(T) |\eta_n(x) \overline{\xi_n(x)}| d\mu(x) \leq \sum_{n \geq 0} \mu_n(T) < \infty.$$

Therefore, the series in (4.61) converges in L^1-norm. This implies that $K(x,x)$ is contained in $L^1_\mu(X)$ and we have

$$\int_X K(x,x) d\mu(x) = \sum_{n \geq 0} \mu_n(T) \int_X \eta_n(x) \overline{\xi_n(x)} d\mu(x) = \sum_{n \geq 0} \mu_n(T) \langle \xi_n, \eta_n \rangle. \tag{4.62}$$

On the other hand, as explained in Example 4.62, for every n, the trace of the projection $\eta_n \otimes \xi_n^*$ is equal to $\langle \xi_n, \eta_n \rangle$. Since in (4.60) the series converges in $\mathcal{L}^1$-norm and the functional $T \to \operatorname{Trace}(T)$ is continuous with respect to that norm, we deduce that

$$\operatorname{Trace} T_K = \sum_{n \geq 0} \mu_n(T) \operatorname{Trace}(\eta_n \otimes \xi_n^*) = \sum_{n \geq 0} \mu_n(T) \langle \xi_n, \eta_n \rangle.$$

Combining this with (4.62) proves that

$$\operatorname{Trace} T_K = \int_X K(x,x) d\mu(x).$$

The proof is complete. □

Remark 4.6.3. If X is compact then the sole continuity of $K(x,y)$ insures us that $K(x,y)$ is square-integrable on $X \times X$ and $K(x,x)$ is integrable on X. However, in general this is not enough to insure us that T_K is trace-class (see, e.g., [**GK**, §10.3]). Thus in Theorem 4.6.2 we cannot remove the assumption on T_K being trace-class (unless T_K is positive; see below).

When T_K we don't need to assume T_K to be trace-class, because we can make use of Mercer's theorem to prove:

Theorem 4.6.4 ([**Du**]). *Let $K(x,y) \in L^2_{\mu \otimes \mu}(X \times X) \cap C(X \times X)$ be such that T_K is positive. Then*

(1) *$K(x,x) \geq 0$ for all $x \in X$.*

(2) *We have*

$$\operatorname{Trace} T_K = \int_X K(x,x) d\mu(x).$$

Thus,

$$T_K \in \mathcal{L}^1 \iff K(x,x) \in L^1_\mu(X).$$

REMARK 4.6.5. We refer to [**Br**] for generalizations of Duflo's theorems where the assumptions on the continuity of $K(x,y)$ are relaxed.

4.7. Banach Ideals

In the remainder of the chapter we shall present a detailed account on the theory of Calkin and Gohberg-Krein of operator ideals and operator ideals associated to symmetric norms. As we shall see these ideals play an important role in noncommutative geometry.

Most of the material that follows is taken from [**GK**] and [**Si**] (see also [**Co**, Chap. 4, Appendix C], [**GVF**, Section 7.C]).

This section is devoted to presenting the primary definitions and properties of Banach ideals. We start with basic facts about two-sided ideals in $\mathcal{L}(\mathcal{H})$.

PROPOSITION 4.7.1. *Let $\mathcal{I}$ be a two-sided ideal of $\mathcal{L}(\mathcal{H})$.*

(1) *For any $T \in \mathcal{L}(\mathcal{H})$,*

$$T \in \mathcal{I} \Longleftrightarrow |T| \in \mathcal{I} \Longleftrightarrow T^* \in \mathcal{I}.$$

(2) *Any $T \in \mathcal{I}$ can be written as*

$$T = T_1 - T_2 + i(T_3 - T_4) \qquad \text{with } T_j \in \mathcal{I} \cap \mathcal{L}(\mathcal{H})_+.$$

PROOF. Once (1) is proved the proof of (2) follows along the same lines as that of the proof of Lemma 4.3.3. Thus, we only have to prove (1).

Let $T \in \mathcal{L}(\mathcal{H})$ have polar decomposition $T = U|T|$. If $|T|$ is in $\mathcal{I}$ then, as $\mathcal{I}$ is an ideal, $T = U|T|$ is in $\mathcal{I}$ too. Since by Proposition 3.1.8 $|T| = U^*T$ we also see that if $|T|$ is in $\mathcal{I}$, then so is T.

It also follows from Proposition 3.1.8 that $T^* = U^*TU$, and $T = (U^*TU^*)^* = UT^*U$. Therefore T is in $\mathcal{I}$ if and only if is T^* in $\mathcal{I}$. The proof is complete. ☐

PROPOSITION 4.7.2. *Let $\mathcal{I}$ be a two-sided ideal of $\mathcal{L}(\mathcal{H})$.*

(1) *If $\mathcal{I} \supsetneq \{0\}$, then every finite-rank operator is contained in $\mathcal{I}$.*
(2) *If $\mathcal{I} \subsetneq \mathcal{L}(\mathcal{H})$, then every operator in $\mathcal{I}$ is compact.*

PROOF. Assume $\mathcal{I} \supsetneq \{0\}$. Since the finite-rank operators are linear combinations of rank 1 operators $\xi \otimes \eta^*$, $\xi, \eta \neq 0$, in order to prove (1) it is enough to show that any such projection is contained in $\mathcal{I}$.

Let $\xi, \eta \in \mathcal{H} \setminus \{0\}$ and let $T \in \mathcal{I} \setminus \{0\}$. Since $T \neq 0$ there exists $\xi' \in \mathcal{H} \setminus \{0\}$ such that $\eta' := T\xi' \neq 0$. Set $A = \xi \otimes \xi^{'*}$ and $B = \eta' \otimes \eta^*$. Then the operator ATB is contained in $\mathcal{I}$ and is equal to $(\xi \otimes \eta^{'*})T(\xi \otimes \eta^*) = \langle \eta', T\xi' \rangle (\xi \otimes \eta^*) = \|\eta'\|^2 (\xi \otimes \eta^*)$. Since $\eta' \neq 0$ it follows that $\xi \otimes \eta^*$ is contained in $\mathcal{I}$, proving (1).

Suppose now that $\mathcal{I}$ contains a non-compact operator T. By Proposition 3.4.4 and Proposition 4.7.1 the operator $|T|$ too is non-compact and contained in $\mathcal{I}$. Therefore, possibly by replacing T by $|T|$, we may assume T positive. For $\lambda > 0$ set $\Pi_\lambda = 1_{[\lambda,\infty)}(T)$. If $g(t) := t^{-1} 1_{[\lambda,\infty)}$, then $g(T)$ is a bounded operator. As $\Pi_\lambda = Tg(T)$ it follows that Π_λ is contained in $\mathcal{I}$.

As $\|T - T\Pi_\lambda(T)\| = \|1_{[0,\lambda)}(T)\| \leq \lambda$, we see that $T\Pi_\lambda(T)$ converges to T in norm as $\lambda \to 0^+$. Since T is non-compact, it follows there is at least one $\lambda > 0$ such that $T\Pi_\lambda$ does not have finite rank. Then Π_λ does not have finite rank.

Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$, let $(\eta_n)_{n\geq 0}$ be an orthonormal basis of $\operatorname{im}\Pi_\lambda$, and let $V \in \mathcal{L}(\mathcal{H})$ be such that $V\xi_n = \eta_n$. As $V^*\eta_n = \xi_n$ for all $n \in \mathbb{N}$, we see that $V^*\Pi_\lambda V = 1$. Thus $1 \in \mathcal{I}$, which implies that $\mathcal{I} = \mathcal{L}(\mathcal{H})$. Therefore, if $\mathcal{I} \subsetneq \mathcal{L}(\mathcal{H})$, then $\mathcal{I}$ cannot contain any non-compact operator, i.e., T is contained in $\mathcal{K}$. The proof is complete. □

The previous proposition shows that, among non-trivial ideals of $\mathcal{L}(\mathcal{H})$, the ideal of finite-rank operators is minimal and the ideal of compact operators is maximal. Since the former is the closure of the latter in $\mathcal{L}(\mathcal{H})$ we obtain:

Corollary 4.7.3. *The only closed non-trivial two-sided ideal of $\mathcal{L}(\mathcal{H})$ is $\mathcal{K}$.*

Definition 4.7.4. *A Banach ideal is a two-ideal $\mathcal{I}$ of $\mathcal{L}(\mathcal{H})$ which is equipped with a norm $\|.\|_{\mathcal{I}}$ such that*

(i) *$\mathcal{I}$ is a Banach space for $\mathcal{I}$.*

(ii) *We have*

$$\|ATB\|_{\mathcal{I}} \leq \|A\|\|T\|_{\mathcal{I}}\|B\| \qquad \forall T \in \mathcal{I} \quad \forall A, B \in \mathcal{L}(\mathcal{H}). \tag{4.63}$$

Example 4.7.5. $\mathcal{L}(\mathcal{H})$ and $\mathcal{K}$ are Banach ideals for the operator norm $\|.\|$.

Example 4.7.6. It follows from (4.20) and Proposition 4.2.5 that $\mathcal{L}^1$ is a Banach ideal for the norm $\|.\|_1$. Likewise, using (4.46) and Proposition 4.4.5, we see that $\mathcal{L}^2$ is a Banach ideal for the Hilbert-Schmidt norm $\|.\|_2$.

In the sequel we let $\mathcal{I}$ be a Banach ideal with norm $\|.\|_{\mathcal{I}}$. We assume $\mathcal{I}$ non-trivial, so by Proposition 4.7.2 all the finite-rank operators are contained in $\mathcal{I}$ and all the elements of $\mathcal{I}$ are compact operators.

Lemma 4.7.7. *Let $T \in \mathcal{I}$ and let $S \in \mathcal{K}$.*

(i) *If $\mu_n(S) \leq \mu_n(T)$ $\forall n \in \mathbb{N}_0$, then $S \in \mathcal{I}$ and $\|S\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}$.*

(ii) *If $\mu_n(S) = \mu_n(T)$ $\forall n \in \mathbb{N}_0$, then $S \in \mathcal{I}$ and $\|S\|_{\mathcal{I}} = \|T\|_{\mathcal{I}}$.*

Proof. We only have to prove (i), since it implies (ii). Thus, let us assume that $\mu_n(S) \leq \mu_n(T)$ $\forall n \in \mathbb{N}_0$, and let $T = U|T|$ and $S = V|S|$ be the respective polar decompositions of T and S. Let $(\xi_n)_{n\geq 0}$ and $(\eta_n)_{n\geq 0}$ be orthonormal families in $\mathcal{H}$ such that $|T|\xi_n = \mu_n(T)\xi_n$ and $|S|\eta_n = \mu_n(S)\eta_n$ for all $n \in \mathbb{N}_0$. Let $C \in \mathcal{L}(\mathcal{H})$ be such that $C = 0$ on $\ker|S|$ and $C\eta_n = (\sqrt{\mu_n(S)}/\sqrt{\mu_n(T)})\xi_n$ for all $n \in \mathbb{N}_0$ such that $\mu_n(T) > 0$ (i.e., ξ_n is in $\operatorname{im}|S| = (\ker|S|)^\perp$. This defines bounded operator of norm ≤ 1,since by assumption $\mu_n(S) \leq \mu_n(T)$ for all $n \in \mathbb{N}_0$. As $C^*\xi_n = (\sqrt{\mu_n(S)}/\sqrt{\mu_n(T)})\eta_n$ for all $n \in \mathbb{N}_0$, we see that $C^*|T|C = |S|$.

By Proposition 3.1.8 we know that $|T| = U^*T$, so we have

$$VC^*U^*TC = VC^*|T|C = V|S| = S.$$

Therefore S is contained in $\mathcal{I}$ and, by (4.63), we have $\|S\|_{\mathcal{I}} \leq \|V\|\|C^*\|\|U\|\|T\|_{\mathcal{I}}\|C\|$. Since the operator norms of U, V and C are ≤ 1, it follows that $\|S\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}$, as claimed. □

Combining this lemma with (4.4) and (4.6) we see that, for any $T \in \mathcal{I}$,

$$\|T\|_{\mathcal{I}} = \||T|\|_{\mathcal{I}} = \|T^*\|_{\mathcal{I}},$$
$$\|U^*TU\|_{\mathcal{I}} = \|T\|_{\mathcal{I}} \qquad \forall U \in \mathcal{L}(\mathcal{H}),\ U \text{ unitary}.$$

Proposition 4.7.8. *There exists a constant $c > 0$ such that*

$$\|T\|_{\mathcal{I}} = c\|T\| \qquad \forall T \in \mathcal{R}_1. \tag{4.64}$$

Furthermore, we have

$$c\|T\| \leq \|T\|_{\mathcal{I}} \qquad \forall T \in \mathcal{I}. \tag{4.65}$$

Proof. Let $R \in \mathcal{R}_1$ be such that $\|R\| = 1$ and set $c = \|R\|_{\mathcal{I}}$. It follows from (4.2) that $\mu_0(R) = \|R\| = 1$ and $\mu_n(R) = 0$ for $n \geq 1$. Likewise, if $S \in \mathcal{R}_1$, then $\mu_0(S) = \|S\|$ and $\mu_n(S) = 0$ for $n \geq 1$, so the operators S and $\|S\|R$ have the same singular values. Lemma 4.7.7 then implies that $\|S\|_{\mathcal{I}} = \|S\|\|R\|_{\mathcal{I}} = c\|S\|$.

Let $T \in \mathcal{I}$. Then $\mu_0(T) = \|T\| = \mu_0(\|T\|R)$ and $\mu_n(T) \geq 0 = \mu_n(\|T\|R)$ for $n \geq 1$, so by Lemma 4.7.7 we have $\|T\|_{\mathcal{I}} \geq \|(\|T\|R)\|_{\mathcal{I}} = c\|T\|$, as claimed. □

Because the norm $\|.\|_{\mathcal{I}}$ on rank-one operators is constant, we sometimes require the normalization,

$$\|T\|_{\mathcal{I}} = \|T\| \quad \text{for any operator } T \text{ of rank 1.} \tag{4.66}$$

In this case, the inequality (4.65) holds with $c = 1$.

Proposition 4.7.9. *Any other Banach norm on $\mathcal{I}$ satisfying (4.63) is equivalent to $\|.\|_{\mathcal{I}}$.*

Proof. Let $\|.\|'_{\mathcal{I}}$ be another Banach norm on $\mathcal{I}$ satisfying (4.63) and let $|.|_{\mathcal{I}}$ be the norm on $\mathcal{I}$ defined by

$$|T|_{\mathcal{I}} := \sup\{\|T\|_{\mathcal{I}}, \|T\|'_{\mathcal{I}}\} \qquad \forall T \in \mathcal{I}.$$

Let $(T_n)_{n\geq 0}$ be a Cauchy sequence in $(\mathcal{I}, |.|_{\mathcal{I}})$, i.e., it is a Cauchy sequence both in $(\mathcal{I}, \|.\|_{\mathcal{I}})$ and $(\mathcal{I}, \|.\|'_{\mathcal{I}})$. It thus converges in $(\mathcal{I}, \|.\|_{\mathcal{I}})$ and in $(\mathcal{I}, \|.\|'_{\mathcal{I}})$. The limits may be different. However, using (4.65) we see that $(T_n)_{n\geq 0}$ is a Cauchy sequence in $(\mathcal{L}(\mathcal{H}), \|.\|)$ and its limit in $(\mathcal{L}(\mathcal{H}), \|.\|)$ agrees with the limits in $(\mathcal{I}, \|.\|_{\mathcal{I}})$ and $(\mathcal{I}, \|.\|'_{\mathcal{I}})$. Thus, the last two limits are equal and $(T_n)_{n\geq 0}$ converges in $(\mathcal{I}, |.|_{\mathcal{I}})$. This shows that $(\mathcal{I}, |.|_{\mathcal{I}})$ is a Banach space.

Notice that the identity map is continuous from $(\mathcal{I}, |.|_{\mathcal{I}})$ to $(\mathcal{I}, \|.\|_{\mathcal{I}})$. Since this is a bijection and both $(\mathcal{I}, |.|_{\mathcal{I}})$ to $(\mathcal{I}, \|.\|_{\mathcal{I}})$ are Banach spaces, the open mapping theorem insures us that its inverse is continuous. Therefore $|.|_{\mathcal{I}}$ and $\|.\|_{\mathcal{I}}$ are equivalent norms. Likewise, the norms $|.|_{\mathcal{I}}$ and $\|.\|'_{\mathcal{I}}$ are equivalent, so $\|.\|_{\mathcal{I}}$ and $\|.\|'_{\mathcal{I}}$ are equivalent norms, proving the proposition. □

As we shall now see the separability of the topology of $\mathcal{I}$ defined by the norm $\|.\|_{\mathcal{I}}$ is intimately related to the density of finite-rank operators.

Definition 4.7.10. *$\mathcal{I}^0$ is the closure in $\mathcal{I}$ of the the ideal $\mathcal{R}_\infty$ of finite-rank operators.*

Since $\mathcal{R}_\infty$ is a two-sided ideal, $\mathcal{I}^0$ can easily be seen to be a Banach ideal for the norm of $\mathcal{I}$.

Let $T \in \mathcal{K}$ have polar decomposition $T = U|T|$ and let $(\xi_n)_{n\geq 0}$ be an orthonormal family such that $|T|\xi_n = \mu_n(T)\xi_n$ for all $n \in \mathbb{N}_0$. Then, by Proposition 4.1.8,

$$T = \sum_{n\geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*, \tag{4.67}$$

where the series converges in $\mathcal{K}$. Any series of the form (4.67) is called a *Schmidt series* for T.

Lemma 4.7.11. *Let $T \in \mathcal{K}$. Then the following are equivalent:*

(i) *T is contained in $\mathcal{I}^0$.*
(ii) *Any Schmidt series for T converges in $\mathcal{I}$ to T.*
(iii) *There is a Schmidt series for T which converges in $\mathcal{I}$.*

Proof. It is clear that (ii) implies (iii). Moreover, if there is a Schmidt series for T converging in $\mathcal{I}$ then, as it converges to T in $\mathcal{K}$, using (4.65) we see that its sum is equal to T. Thus T is contained in $\mathcal{I}$ and is the sum of a series of finite-rank operators, hence T is an element of $\mathcal{I}^0$.

Suppose now that T is in $\mathcal{I}$. For any $N \in \mathbb{N}$ set

$$T_N := T - \sum_{n<N} \mu_n(T)(U\xi_n) \otimes \xi_n^* = \sum_{n \geq N} \mu_n(T)(U\xi_n) \otimes \xi_n^*.$$

As $\sum_{n<N} \mu_n(T)(U\xi_n) \otimes \xi_n^*$ has rank $\geq N$, it is immediate that

$$\|T_N\|_{\mathcal{I}} \geq \inf\left\{ \|T - R\|_{\mathcal{I}};\ R \in \mathcal{R}_N \right\}. \tag{4.68}$$

It follows from Proposition 3.1.8 the operator U^*U is the orthogonal projection onto $(\ker T)^\perp = (\ker |T|)^\perp = \operatorname{im} |T|$, so $U^*U\xi_n = \xi_n$ if $\mu_n(T) \neq 0$. Thus,

$$(T_N)^* T_N = \sum_{n \geq N} \mu_n(T)^2 \xi_n \otimes \xi_n^* \qquad \text{and} \qquad |T_N| = \sum_{n \geq N} \mu_n(T) \xi_n \otimes \xi_n^*.$$

Using the min-max principle we then see that

$$\mu_n(T_N) = \mu_{n+N}(T) \qquad \forall n \in \mathbb{N}. \tag{4.69}$$

Let $R \in \mathcal{R}_N$. Then (4.2) implies that $\mu_N(R) = 0$, so using (4.7) we get

$$\mu_n(T_N) = \mu_{n+N}(T) \leq \mu_n(T - R) + \mu_N(R) = \mu_n(T - R).$$

Therefore, applying Lemma 4.7.7 we see that $\|T_N\|_{\mathcal{I}} \leq \|T - R\|_{\mathcal{I}}$ for all $R \in \mathcal{R}_N$. Combining this with (4.68) then shows that

$$\left\| \sum_{n \geq N} \mu_n(T)(U\xi_n) \otimes \xi_n^* \right\|_{\mathcal{I}} = \inf\left\{ \|T - R\|_{\mathcal{I}}; R \in \mathcal{R}_N \right\}.$$

This implies that T is a limit of finite-rank operators in $\mathcal{I}$ (i.e., T is in $\mathcal{I}^0$) if and only if the Schmidt series $\sum_{n \geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*$ converges to T in $\mathcal{I}$. The proof is complete. ∎

Lemma 4.7.12. *The Banach ideal $\mathcal{I}^0$ is separable.*

Proof. Without any loss of generality we may assume that in (4.64)–(4.65) the constant c is equal to 1. Let $(\zeta_k)_{k \geq 0}$ be a countable dense subset of $\mathcal{H}$. Let $\xi, \eta \in \mathcal{H}$. For any $\epsilon > 0$ there exist $k, l \in \mathbb{N}_0$ such that $\|\xi - \zeta_k\| \leq \epsilon$ and $\|\eta - \zeta_l\| \leq \epsilon$. Then using (4.64) we get

$$\begin{aligned} \|\xi \otimes \eta^* - \zeta_k \otimes \zeta_l^*\|_{\mathcal{I}} &\leq \|(\xi - \zeta_k) \otimes \eta^*\|_{\mathcal{I}} + \|\zeta_k \otimes (\eta - \zeta_l)^*\|_{\mathcal{I}} \\ &\leq \|(\xi - \zeta_k) \otimes \eta^*\| + \|\zeta_k \otimes (\eta - \zeta_l)^*\| \\ &\leq \|\xi - \zeta_k\|\|\eta\| + \|\zeta_k\|\|\eta - \zeta_l\| \leq \epsilon\|\eta\| + (\epsilon + \|\xi\|)\epsilon. \end{aligned} \tag{4.70}$$

Let $\mathcal{D}$ be the set of operators of the form,

$$\sum_{(k,l) \in K \times L} \zeta_k \otimes \zeta_l^*,$$

where K and L range over all finite subsets of $\mathbb{N}_0$. Then $\mathcal{D}$ is a countable subset of $\mathcal{R}_\infty$. As any operator in $\mathcal{R}_\infty$ is a finite sum of rank one operators $\xi \otimes \eta^*$, it follows from (4.70) that, for any $T \in \mathcal{R}_\infty$ and for any $\epsilon > 0$, there exists $R \in \mathcal{D}$ such that $\|T - R\|_{\mathcal{I}} < \epsilon$. Combining this with the density of $\mathcal{R}_\infty$ in $\mathcal{I}^0$ we deduce that $\mathcal{D}$ is dense in $\mathcal{I}^0$. Since $\mathcal{D}$ is countable, this proves that $\mathcal{I}^0$ is separable. □

Proposition 4.7.13. *The following are equivalent:*

(1) *The finite-rank operators are dense in $\mathcal{I}$, i.e., $\mathcal{I} = \mathcal{I}_0$.*
(2) *$\mathcal{I}$ is separable.*

Proof. It immediately follows from Lemma 4.7.12 that if $\mathcal{I} = \mathcal{I}^0$ then $\mathcal{I}$ is separable.

Conversely, suppose that $\mathcal{I}_0 \subsetneq \mathcal{I}$. Let $T \in \mathcal{I} \setminus \mathcal{I}_0$. Since $\mathcal{I}$ and $\mathcal{I}_0$ both are ideals, using Proposition 4.7.1 we see that $|T|$ is in $\mathcal{I}$, but is not in $\mathcal{I}$. Therefore, possibly by replacing T by $|T|$ we may assume T positive.

Since T is in $\mathcal{I} \backslash \mathcal{I}_0$ and is positive, Lemma 4.7.11 implies that there is a Schmidt series $\sum_{n \geq 0} \mu_n(T) \xi_n \otimes \xi_n^*$ which does not converges in $\mathcal{I}$. As $\mathcal{I}$ is a Banach space, this implies that the series does not satisfy Cauchy's criterion, so there exists $\delta > 0$ and an increasing sequence $(n_k)_{k \geq 0} \subset \mathbb{N}_0$ such that

$$\Big\| \sum_{n_k \leq n < n_{k+1}} \mu_n(T) \xi_n \otimes \xi_n^* \Big\|_{\mathcal{I}} \geq \delta \qquad \forall k \in \mathbb{N}_0. \tag{4.71}$$

For any sequence $a = (a_k)_{k \geq 0} \in \{0,1\}^{\mathbb{N}_0}$ we set

$$T_a := \sum_{k=0}^{\infty} a_k \Big(\sum_{n_k \leq n < n_{k+1}} \mu_n(T) \xi_n \otimes \xi_n^* \Big) = \sum_{a_k \neq 0}^{\infty} \sum_{n_k \leq n < n_{k+1}} \mu_n(T) \xi_n \otimes \xi_n^*.$$

If we let Π_a be the orthogonal projection onto the closure of the vector space spanned by $\bigcup_{a_k \neq 0} \{\xi_n;\ n_k \leq n < n_{k+1}\}$, then $T_a = \Pi_a T$. Therefore, the operator T_a is in $\mathcal{I}$.

Let $b = (b_k)_{k \geq 0} \in \{0,1\}^{\mathbb{N}_0}$ be such that $b \neq a$, i.e., there exists $k \in \mathbb{N}_0$ such that $b_k \neq a_k$. Set $\Pi_k = \sum_{n=n_k}^{n_{k+1}-1} \xi_n \otimes \xi_n^*$. Then

$$\Pi_k (T_b - T_a) = (b_k - a_k) \sum_{n_k \leq n < n_{k+1}} \mu_n(T) \xi_n \otimes \xi_n^* = \pm \sum_{n_k \leq n < n_{k+1}} \mu_n(T) \xi_n \otimes \xi_n^*.$$

Combining this with (4.71) we get

$$\delta \leq \|\Pi_k (T_b - T_a)\|_{\mathcal{I}} \leq \|\Pi_k\| \|T_b - T_a\|_{\mathcal{I}} = \|T_b - T_a\|_{\mathcal{I}}.$$

Since $\{0,1\}^{\mathbb{N}}$ is not countable, it follows that no countable subset of $\mathcal{I}$ can be dense, so $\mathcal{I}$ is not separable if $\mathcal{I}_0 \subsetneq \mathcal{I}$. Equivalently, if $\mathcal{I}$ is separable, then $\mathcal{I} = \mathcal{I}_0$. The proof is complete. □

The following result hows that, among the non-trivial Banach ideals, the ideal $\mathcal{L}^1$ of trace-class operators is minimal.

Proposition 4.7.14. *There is a continuous inclusion,*

$$\mathcal{L}^1 \subset \mathcal{I}^0.$$

In fact, if the normalization (4.66) holds, then

$$\|T\|_{\mathcal{I}} \leq \|T\|_1 \qquad \forall T \in \mathcal{I}. \tag{4.72}$$

Proof. We may assume that the normalization (4.66) holds, so that we can take $c = 1$ in (4.65). Let $T \in \mathcal{L}^1$ and let $\sum_{n\geq 0} \mu_n(T)(U\xi_n)\otimes\xi_n^*$ be a Schmidt series for T as in (4.67). Using (4.63) and (4.65)) we see that, for all N and p in $\mathbb{N}$, we have

$$(4.73)\quad \Big\| \sum_{N\leq n\leq N+p} \mu_n(T)(U\xi_n)\otimes\xi_n^* \Big\|_{\mathcal{I}} \leq \sum_{N\leq n\leq N+p} \mu_n(T)\|U\|\|\xi_n\otimes\xi_n^*\|_{\mathcal{I}}$$
$$\leq \sum_{N\leq n\leq N+p} \mu_n(T)\|\xi_n\otimes\xi_n^*\| \leq \sum_{N\leq n\leq N+p} \mu_n(T).$$

Since $\sum_{n\geq 0} \mu_n(T) < \infty$ it follows that the series $\sum_{n\geq 0} \mu_n(T)(U\xi_n)\otimes\xi_n^*$ converges in $\mathcal{I}$. Lemma 4.7.11 then insures us that T is contained in $\mathcal{L}^0$ and the Schmidt series converges to T in $\mathcal{I}$. Therefore, using (4.73), we get

$$\|T\|_{\mathcal{I}} = \Big\| \sum_{n\geq 0} \mu_n(T)(U\xi_n)\otimes\xi_n^* \Big\|_{\mathcal{I}} \leq \sum_{n\geq 0} \mu_n(T) = \|T\|_1.$$

This proves (4.72) when T is in $\mathcal{L}^1$ and shows there is a continuous inclusion of $\mathcal{L}^1$ in $\mathcal{I}^0$. In addition, if $T \in \mathcal{I}\setminus\mathcal{L}^1$, then $\|T\|_1 = \infty$ and (4.72) holds trivially, so (4.72) holds for all $T \in \mathcal{I}$. The proof is complete. □

4.8. Symmetric norms

In the sequel we denote by l_f the vector space of sequences $a = (a_n)_{n\geq 0}$ of complex numbers that have finite support (i.e., $a_n = 0$ for n large enough). We denote by l_0 the space of sequences $(a_n)_{n\geq 0}$ of complex numbers such that

$$\lim_{n\to\infty} a_n = 0.$$

For any sequence $a = (a_n)_{n\geq 0}$ in l_0 we denote by $\sigma(a) = (\sigma_N(a))_{N\geq 1})$ the sequence defined by

$$\sigma_N(a) := \sum_{n<N} a_n \qquad \forall N \in \mathbb{N}.$$

In addition, for any $a \in l_0$ we denote by $a^* = (a_n^*)_{n\geq 0}$ the sequence defined by

$$a_n^* = \inf_{\substack{J\subset\mathbb{N}_0 \\ |J|=n}} \sup_{j\in J} |a_j| \qquad \forall n \in \mathbb{N}_0.$$

In other word, the sequence $(a_n^*)_{n\geq 0}$ is the sequence obtained by re-ordering the sequence $(|a_n|)_{n\geq 0}$ into a non-increasing sequence. In particular, for any $N \in \mathbb{N}$, we always have

$$|\sigma_N(a_n)| \leq \sum_{n<N} |a_n| \leq \sigma_N(a^*).$$

It can also be shown (see [**Si**, Lem. 1.8]) that, for all $a, b \in l_f$,

$$(4.74)\qquad |\sum_{n<N} a_n b_n| \leq \sum_{n<N} a_n^* b_n^* \qquad \forall N \in \mathbb{N}.$$

Definition 4.8.1. *Let Φ be a norm on l_f. We say that Φ is symmetric when*

$$\Phi(a) = \Phi(a^*) \qquad \forall a \in l_f.$$

Remark 4.8.2. It is not difficult to check that a norm Φ on l_f is symmetric if and only if it satisfies the following two conditions:

(i) For any sequence $(a_n)_{n\geq 0}$ in l_f and any bijection $\sigma : \mathbb{N}_0 \to \mathbb{N}_0$, we have

$$\Phi\left((a_{\sigma(n)})_{n\geq 0}\right) = \Phi\left((a_n)_{n\geq 0}\right).$$

(ii) For any sequence $(a_n)_{n\geq 0}$ in l_f and any sequence $(\theta_n)_{n\geq 0} \subset [0, 2\pi)$, we have

$$\Phi\left((e^{i\theta_n} a_n)_{n\geq 0}\right) = \Phi\left((a_n)_{n\geq 0}\right).$$

Example 4.8.3. For $p \in [1,\infty)$ the p-norm Φ_p on l_f is defined by

$$\Phi_p(a) = \left(\sum_{n\geq 0} |a_n|^p\right)^{\frac{1}{p}} \qquad \forall a = (a_n)_{n\geq 0} \in l_f.$$

For $p = \infty$ we define the Φ_∞-norm by

$$\Phi_\infty(a) = \sup_{n\geq 0} |a_n| \qquad \forall a = (a_n)_{n\geq 0} \in l_f. \tag{4.75}$$

All the p-norms are symmetric norms on l_f.

Let Φ be a symmetric norm on l_f.

Lemma 4.8.4 (Markus; see [**GK**, Lem. 3.1], [**Si**, Thm. 1.9]). *Let $a, b \in l_f$. Then*

$$\left(\sigma_N(a^*) \leq \sigma_N(b^*) \quad \forall N \in \mathbb{N}\right) \Longrightarrow \Phi(a) \leq \Phi(b). \tag{4.76}$$

If follows from Markus' lemma that if $a_n^* \leq b_n^*$ for all $n \in \mathbb{N}_0$, then $\Phi(a) \leq \Phi(b)$. In particular, if $a = (a_n)_{\geq 0}$ is a sequence in l_0, then

$$\Phi(a_0, \ldots, a_{N-1}, 0, 0, \ldots) \leq \Phi(a_0, \ldots, a_N, 0, 0, \ldots) \qquad \forall N \in \mathbb{N}.$$

This means that $(\Phi(a_0, \ldots, a_N, 0, 0, \ldots))_{N\geq 0}$ is a non-decreasing sequence of non-negative numbers, so it admits a limit as $N \to \infty$. We then set

$$\Phi(a) = \lim_{N\to\infty} \Phi(a_0, \ldots, a_N, 0, 0, \ldots) = \sup_{N\geq 1} \Phi(a_0, \ldots, a_N, 0, 0, \ldots) \tag{4.77}$$

This extends Φ to a function $\Phi : l_0 \to [0, \infty]$.

It is not hard to check that

$$\Phi(a) = 0 \Longrightarrow a = 0, \tag{4.78}$$

$$\Phi(\lambda a) = |\lambda|\Phi(a) \qquad \forall a \in l_0 \ \forall \lambda \in \mathbb{C}, \tag{4.79}$$

$$\Phi(a + b) \leq \Phi(a) + \Phi(b) \qquad \forall a, b \in l_0. \tag{4.80}$$

In addition, we have

Proposition 4.8.5 (see [**Si**, Thm. 1.16]). *Let $a, b \in l_0$. Then*

$$\Phi(a) = \Phi(a^*),$$

$$\left(\sigma_N(a^*) \leq \sigma_N(b^*) \quad \forall N \in \mathbb{N}\right) \Longrightarrow \Phi(a) \leq \Phi(b).$$

It follows from Proposition 4.8.5 that, for any $a, b \in l_0$,

$$\left(a_n^* \leq b_n^* \quad \forall n \in \mathbb{N}_0\right) \Longrightarrow \Phi(a) \leq \Phi(b).$$

In the sequel, we denote by l_f^+ the positive cone of l_f consisting of non-increasing sequences of non-negative numbers with finite supports.

Lemma 4.8.6 ([**GK**, Lem. 3.2]). *Let* $\Phi : l_f^+ \rightarrow [0,\infty)$ *be a function such that*

$$\Phi(a) = 0 \Longrightarrow a = (0,0,\ldots), \tag{4.81}$$

$$\Phi(\lambda a) = \lambda \Phi(a) \qquad \forall a \in l_f^+ \ \forall \lambda \geq 0, \tag{4.82}$$

$$\Phi(a+b) \leq \Phi(a) + \Phi(b) \qquad \forall a, b \in l_f^+, \tag{4.83}$$

$$\Big(\sigma_N(a) \leq \sigma_N(b) \quad \forall N \in \mathbb{N}\Big) \Longrightarrow \Phi(a) \leq \Phi(b). \tag{4.84}$$

Then Φ *can be uniquely extend into a symmetric norm on* l_f *by letting*

$$\Phi(a) := \Phi(a^*) \qquad \forall a \in l_f.$$

Finally, let $\Phi' : l_f \rightarrow [0,\infty)$ be the function defined by

$$\Phi'(a) := \sup\left\{ \left| \sum_{n\geq 0} a_n b_n \right| ;\ b \in l_f,\ \Phi(b) \leq 1 \right\} \tag{4.85}$$

This is a norm on l_f called the *dual norm* of Φ. Using (4.74) and the fact that Φ is symmetric, we can check that

$$\Phi'(a) = \sup\left\{ \sum_{n\geq 0} a_n^* b_n ;\ b \in l_f^+,\ \Phi(b) \leq 1 \right\}, \tag{4.86}$$

from which it follows that Φ' is a symmetric norm. It also implies that

$$\sum_{n\geq 0} a_n^* b_n^* \leq\leq \Phi'(a)\Phi(b) \qquad \forall a, b \in l_0. \tag{4.87}$$

Lemma 4.8.7 ([**GK**, Thm. 1.11]). *The dual norm of* Φ' *is equal to* Φ*, i.e.,* $(\Phi')' = \Phi$.

Remark 4.8.8. Two norms Φ and Ψ on l_f are equivalent when there exists $c > 0$ such that

$$c^{-1}\Phi(a) \leq \Psi(a) \leq c\Phi(a) \qquad \forall a \in l_f. \tag{4.88}$$

It is not hard to see that Φ and Ψ are equivalent if and only if their dual norms are equivalent.

Example 4.8.9. Let $p \in [1,\infty]$ and let $p' \in [1,\infty]$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then the dual norm of Φ_p is equal to $\Phi_{p'}$. This follows from the following simple facts:

- The Hölder inequality,

$$\left| \sum_{n\geq 0} a_n b_n \right| \leq \Phi_{p'}(a)\Phi_p(b) \qquad \forall a, b \in l_f.$$

- If $p > 1$, the Hölder inequality is an equality if $b_n = \frac{\overline{a_n}}{|a_n|}|a_n|^{\frac{p'}{p}} = \frac{\overline{a_n}}{|a_n|}|a_n|^{p'-1}$ when $a_n \neq 0$ and $b_n = 0$ otherwise.
- If $p = 1$ the the Hölder inequality is an equality if $b_n = \frac{a_{n_0}}{a_{n_0}}$ for $n = n_0$ and $b_n = 0$ for $n \neq n_0$, where n_0 is such that $|a_{n_0}| = \Phi_\infty(a)$.

4.9. Banach ideals associated to symmetric norms

Let Φ be a symmetric norm on l_f. We shall also denote by Φ its extension to l_0 given by (4.77).

For any operator $T \in \mathcal{K}$, the sequence of singular values $\mu(T) := (\mu_n(T))_{n\geq 0}$ is an element if l_0. Therefore, we can set

$$\|T\|_\Phi := \Phi(\mu(T)).$$

We then define

$$\mathcal{I}_\Phi := \Big\{T \in \mathcal{K};\ \|T\|_\Phi < \infty\Big\}.$$

For $T \in \mathcal{K}$ and $N \in \mathbb{N}$ we define

$$\mu^N(T) := (\mu_0(T), \ldots, \mu_{N-1}(T), 0, 0, \ldots) \in l_f.$$

Then by (4.77) we have

$$\|T\|_\Phi = \lim_{N\to\infty} \Phi(\mu^N(T)) = \sup_{N\geq 1} \Phi(\mu^N(T)). \tag{4.89}$$

In addition, we set

$$\sigma_N(T) := \sum_{n<N} \mu_n(T) = \sigma_N(\mu(T)).$$

For $T \in \mathcal{L}(\mathcal{H})$ and $N \in \mathbb{N}$ we set

$$\sigma_N(T) = \sum_{n<N} \mu_n(T). \tag{4.90}$$

Using the properties (4.1) and (4.5) of singular values we get

$$\sigma_N(cT) = |c|\sigma_N(T) \qquad \forall c \in \mathbb{C}, \tag{4.91}$$

$$\sigma_N(ATB) \leq \|A\|\sigma_N(T)\|B\| \qquad \forall A, B \in \mathcal{L}(\mathcal{H}). \tag{4.92}$$

As $\mu_n(T) \leq \mu_0(T) = \|T\|$ for all $n \in \mathbb{N}$, we also see that

$$\|T\| \leq \sigma_N(T) \leq N\|T\|. \tag{4.93}$$

In the sequel if E is a closed subspace of $\mathcal{H}$ we denote by Π_E the orthogonal projection onto E.

Lemma 4.9.1. *Let $T \in \mathcal{K}$. For any $N \in \mathbb{N}$, we have*

$$\sigma_N(T) = \sup\{\|T\Pi_E\|_1;\ \dim E = N\}, \tag{4.94}$$

$$= \sup\{|\operatorname{Trace}(T\Pi_E)|;\ \dim E = N\} \quad \textit{(if } T \textit{ is positive).} \tag{4.95}$$

Proof. It follows from (4.4) that $\sigma_N(T) = \sigma_N(|T|)$. Moreover, if E is a closed subspace of $\mathcal{H}$, then $||T|\Pi_E| = |T\Pi_E|$, for $||T|\Pi_E|$ is a positive operator such that

$$||T|\Pi_E|^2 = (|T|\Pi_E)^*(|T|\Pi_E) = \Pi_E|T|^2\Pi_E = \Pi_E T^*T\Pi_E = (T\Pi_E)^*(T\Pi_E).$$

Therefore, upon replacing T by $|T|$ we may assume T positive.

Notice also that if E is a closed subspace of $\mathcal{H}$, then by (4.27)

$$|\operatorname{Trace}(T\Pi_E)| \leq \|T\Pi_E\|_1,$$

and hence

$$\sup\{|\operatorname{Trace}(T\Pi_E)|;\ \dim E = N\} \leq \sup\{\|T\Pi_E\|_1;\ \dim E = N\}. \tag{4.96}$$

Let $(\xi_n)_{n\geq 0} \subset \mathcal{H}$ be an orthonormal family such that $T\xi_n = \mu_n(T)$ for all $n \in \mathbb{N}_0$. Then by Proposition (4.1.8) we have

$$T = \sum_{n\geq 0} \mu_n(T)\xi_n \otimes \xi_n^*,$$

where the series converges in norm. Let E_N be N-dimensional subspace spanned by $\xi_0, \ldots, \xi_{N-1}$; this is a subspace of dimension N. Then $\Pi_{E_N} = \sum_{n<N} \xi_n \otimes \xi_n^*$, and hence

$$T\Pi_{E_N} = \sum_{n<N} \mu_n(T)\xi_n \otimes \xi_n^*.$$

Thus the $(n+1)$'th eigenvalue of $T\Pi_N$ counted with multiplicity is equal to $\mu_n(T)$ if $n < N$ and is zero if $N \geq 0$. Therefore, using (4.31) we get

$$\mathrm{Trace}(T\Pi_{E_N}) = \sum_{n<N} \mu_n(T) = \sigma_n(T).$$

Since $\dim E_N = N$, it follows that

$$\sigma_N(T) \leq \sup\{|\,\mathrm{Trace}(T\Pi_E)|;\ \dim E = N\}. \tag{4.97}$$

Let E be an N-dimensional subspace of $\mathcal{H}$. Then $T\Pi_E$ has rank $\leq N$, and so using Proposition 4.1.2 we see that $\mu_n(T\Pi_E) = 0$ for $n \geq N$. Thus,

$$\|T\Pi_E\|_1 = \sum_{n\geq 0} \mu_n(T\Pi_E) = \sum_{n<N} \mu_n(T\Pi_E). \tag{4.98}$$

Thanks to (4.5) and the fact that Π_E is an orthogonal projection we have

$$\mu_n(T\Pi_E) \leq \mu_n(T)\|\Pi_E\| \leq \mu_n(T).$$

Combining this with (4.98) we get

$$\|T\Pi_E\|_1 \leq \sum_{n<N} \mu_n(T) = \sigma_N(T),$$

and hence

$$\sup\{\|T\Pi_E\|_1;\ \dim E = N\} \leq \sigma_N(T).$$

Combining this with (4.96) and (4.97) proves the lemma. □

Notice that for every subspace E of dimension N the function $T \to \|T\Pi_E\|_1$ is a semi-norm on $\mathcal{K}$, so as a supremum of all such semi-norms σ_N is a semi-norm on $\mathcal{K}$. In particular, we have:

LEMMA 4.9.2. *Let $N \in \mathbb{N}$. Then*

$$\sigma_N(S+T) \leq \sigma_N(S) + \sigma_N(T) \qquad \forall S, T \in \mathcal{K}. \tag{4.99}$$

Granted this lemma we shall prove:

LEMMA 4.9.3. *The following hold.*

(1) *Let $T \in \mathcal{K}$. Then*

$$\|T\|_\Phi = 0 \Longrightarrow T = 0, \tag{4.100}$$

$$\|\lambda T\|_\Phi = |\lambda|\|T\|_\Phi \qquad \forall \lambda \in \mathbb{C}, \tag{4.101}$$

$$\|ATB\|_\Phi \leq \|A\|\|T\|_\Phi\|B\| \qquad \forall A, B \in \mathcal{L}(\mathcal{H}). \tag{4.102}$$

(2) *Let $S, T \in \mathcal{K}$. Then*

$$\|S+T\|_\Phi \leq \|S\|_\Phi + \|T\|_\Phi.$$

(3) *If* $\Phi(1,0,0,\ldots)=1$, *then*

$$\|T\|_\Phi = \|T\| \qquad \forall T\in\mathcal{R}_1, \tag{4.103}$$

$$\|T\| \leq \|T\|_\Phi \qquad \forall T\in\mathcal{K}. \tag{4.104}$$

PROOF. The implication (4.100) is due to (4.78) and the fact that $\mu_0(T)=\|T\|$. We obtain (4.101) by using (4.1) and (4.79). The inequality (4.102) follows by combining (4.5) and (4.8).

Let $S,T\in\mathcal{K}$. Then (4.99) shows that $\sigma_N(\mu(S+T))\leq\sigma_N(\mu(S)+\mu(T))$ for all $N\in\mathbb{N}$, so using Proposition 4.8.5 and (4.80) we get

$$\|S+T\|_\Phi=\Phi(\mu(S+T))\leq\Phi(\mu(S)+\mu(T))\leq\Phi(\mu(S))+\Phi(\mu(T))=\|S\|_\Phi+\|T\|_\Phi.$$

Suppose now that $\Phi(1,0,0,\ldots)=1$ and let $T\in\mathcal{K}$. As $\mu^0(T)=\|T\|(1,0,0,\ldots)$ we see that $\Phi(\mu^0(T))=\|T\|\Phi(1,0,0,\ldots)=\|T\|$. Since $\mu_n(T)\geq\mu_n^0(T)$ for all $n\in\mathbb{N}_0$, using (4.8) we see that $\|T\|_\Phi\geq\Phi(\mu^0(T))=\|T\|$. Moreover, if $\operatorname{rk}T=1$ then $\mu(T)=\mu^0(T)$, and hence $\|T\|_\Phi=\Phi(\mu^0(T))=\|T\|$. The lemma is proved. □

PROPOSITION 4.9.4. *$\mathcal{I}_\Phi$ is a Banach ideal for $\|.\|_\Phi$, i.e., $\mathcal{I}_\Phi$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$ and $\|.\|_\Phi$ is a Banach norm on $\mathcal{I}_\Phi$ satisfying (4.63).*

PROOF. It follows from Lemma 4.9.3 that $\mathcal{I}_\Phi$ is a a two-sided ideal of $\mathcal{L}(\mathcal{H})$ and $\|.\|_\Phi$ is a norm on $\mathcal{I}_\Phi$ satisfying (4.63). It remains to check that $\mathcal{I}_\Phi$ is complete for the norm $\|.\|_\Phi$.

Let $(T_n)_{n\geq0}$ be a Cauchy sequence in $(\mathcal{I},\|.\|_\mathcal{I})$ and let us show that it converges in $(\mathcal{I},\|.\|_\mathcal{I})$. Then (4.104)implies that $(T_n)_{n\geq0}$ is a Cauchy sequence in $\mathcal{K}$, hence converges in $\mathcal{K}$ to some operator T.

Let $\epsilon>0$. There exists $n_0\in\mathbb{N}$ such that $\|T_p-T_q\|_\Phi<\epsilon$ when p and p are greater than n_0. Let $N\in\mathbb{N}$ and denote by $e^N=(e^N)_{n\geq0}$ the sequence such that $e_n^N=1$ for $n<N$ and $e_n^N=0$ for $n\geq N$. Let p and q be integers greater than n_0. It follows from (4.8) that, for all $n\in\mathbb{N}_0$,

$$\mu_n^N(T-T_p))\leq\mu_n^N(T_p-T_q)+\|T-T_q\|e_n^N.$$

Therefore, using (4.80) and (4.8) we see that

$$\begin{aligned}\Phi(\mu^N(T-T_p)&\leq\Phi(\mu^N(T_p-T_q))+\Phi(\|T-T_q\|e_n^N)\\&\leq\|T_p-T_q\|_\Phi+\|T-T_q\|\Phi(e^N)\leq\epsilon+\|T-T_q\|\Phi(e^N).\end{aligned}$$

Letting $q\to\infty$ shows that

$$\Phi(\mu^N(T-T_p)\leq\epsilon+\|T-T_q\|\Phi(e^N) \qquad \forall N\in\mathbb{N}.$$

Combining this with (4.77) we get

$$\|T-T_p\|_\Phi\leq\epsilon \qquad \forall p>n_0. \tag{4.105}$$

This implies that T is contained in $\mathcal{I}_\Phi$ and the sequence $(T_n)_{n\geq0}$ converges to T in $(\mathcal{I}_\Phi,\|.\|_\Phi)$, proving that $(\mathcal{I}_\Phi,\|.\|_\Phi)$ is a Banach space. The proof is complete. □

REMARK 4.9.5. If Ψ is another symmetric norm on l_f then $\mathcal{I}_\Phi=\mathcal{I}_\Psi$ if and only if Φ and Ψ are equivalent in the sense of (4.88). Furthermore, it is immediate that in that case the norms $\|.\|_\Phi$ and $\|.\|_\Psi$ are equivalent on $\mathcal{I}_\Phi=\mathcal{I}_\Psi$.

LEMMA 4.9.6. *Let $T\in\mathcal{I}_\Phi$. Then T is contained in $\mathcal{I}_\Phi^0$ if and only if*

$$\lim_{N\to\infty}\Phi(\mu_N(T),\mu_{N+1}(T),\ldots)=0. \tag{4.106}$$

Proof. Lemma 4.7.11 says that T is contained in $\mathcal{I}_\Phi^0$ if and only if any Schmidt series (4.67) for T converges to T in $\mathcal{I}_\Phi$. Let $T = U|T|$ be the polar decomposition of T and let $(\xi_n)_{n\geq 0}$ be an orthonormal family such that $|T|\xi_n = \mu_n(T)\xi_n$ for all $n \in \mathbb{N}_0$. It immediately follows from (4.69) that

$$\Big\| \sum_{n\geq N} \mu_n(T)(U\xi_n)\otimes \xi_n^* \Big\|_\Phi = \Phi(\mu_N(T), \mu_{N+1}(T), \ldots).$$

Thus T is contained in $\mathcal{I}_\Phi^0$ if and only if (4.106) holds. □

Combining this lemma with Proposition 4.7.13 we obtain:

Proposition 4.9.7. *The following are equivalent.*

(1) *The Banach ideal $\mathcal{I}_\Phi$ is separable.*
(2) *The finite-rank operators are dense in $\mathcal{I}_\Phi$, i.e., $\mathcal{I}_\Phi^0 = \mathcal{I}_\Phi$.*
(3) *For any $a \in l_0$,*

$$\Phi(a) < \infty \Longrightarrow \lim_{N\to\infty} \Phi(a_N, a_{N+1}, \ldots) = 0. \tag{4.107}$$

Proposition 4.9.8. *Let $\mathcal{I}$ be a Banach ideal with norm $\|.\|_\mathcal{I}$. Then*

(1) *There exists a unique symmetric norm on l_f such that*

$$\mathcal{I} \subset \mathcal{I}_\Phi \qquad and \qquad \|T\|_\mathcal{I} = \|T\|_\Phi \quad \forall T \in \mathcal{R}_\infty. \tag{4.108}$$

(2) *The Banach ideals $\mathcal{I}^0$ and $\mathcal{I}_\Phi^0$ coincide.*

Proof. Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$. For any $a \in l_0$ set

$$T_a = \sum_{n\geq 0} a_n \xi_n \otimes \xi_n^*.$$

As $\lim_{n\to\infty} a_n = 0$ the above series converges in $\mathcal{K}$, i.e., T_a is a compact operator. Observe also that $\mu_n(T_a) = a_n^*$ $\forall n \in \mathbb{N}_0$. In addition, if $T \in \mathcal{K}$, then $\mu(T_{\mu(T)}) = \mu(T)$.

Let $\Phi : l_f \to [0,\infty)$ be the function function defined by

$$\Phi(a) = \|T_a\|_\mathcal{I} \qquad \forall a \in l_f.$$

It is not hard to check that Φ is a norm on l_f. Moreover, as $\mu(T_a) = a^* = \mu(T_{a^*})$, using Lemma 4.7.7 we see that $\|T_a\|_\mathcal{I} = \|T_{a^*}\|_\mathcal{I}$, i.e., $\Phi(a) = \Phi(a^*)$. Thus Φ is a symmetric norm on l_f. Let $\mathcal{I}_\Phi$ be the associated Banach ideal with norm $\|.\|_\Phi$.

Let $T \in \mathcal{I}$ and let $N \in \mathbb{N}$. As $\mu_n(T_{\mu^N(T)}) = \mu_n^N(T) \leq \mu_n(T)$, it follows from Lemma 4.7.7 that

$$\Phi(\mu^N(T)) = \|T_{\mu^N(T)}\|_\mathcal{I} \leq \|T\|_\mathcal{I}.$$

Thus,

$$\|T\|_\Phi = \sup_{N\geq 1} \Phi(\mu^N(T)) \leq \|T\|_\mathcal{I} < \infty,$$

that is, the operator T is contained in $\mathcal{I}_\Phi$. If in addition T has finite-rank then, as $\mu(T) = \mu(T_{\mu(T)})$, Lemma 4.7.7 insures us that

$$\|T\|_\mathcal{I} = \|T_{\mu(T)}\|_\mathcal{I} = \Phi(\mu(T)) = \|T\|_\Phi.$$

Therefore $\|.\|_\mathcal{I}$ and $\|.\|_\Phi$ agree on finite-rank operators.

Let Ψ be another symmetric norm on l_f such that $\mathcal{I} \subset \mathcal{I}_\Psi$ and $\|.\|_\Psi = \|.\|_\mathcal{I}$ on $\mathcal{R}_\infty$. Let $a \in l_f$. As Ψ is symmetric and $\mu(T_a) = a^*$, we have

$$\Psi(a) = \Psi(a^*) = \Psi(\mu(T_a)) = \|T_a\|_\Psi = \|T_a\|_\mathcal{I} = \Phi(a).$$

Therefore Ψ and Φ agrees, so Φ is the unique symmetric norm on l_f satisfying (4.108).

Let $T \in \mathcal{K}$. As a Schmidt series for T is a series of finite-rank operators, it follows from (4.108) that a Schmidt series for T satisfies Cauchy's criterion for $\|.\|_{\mathcal{I}}$ if and and only it satisfies it for $\|.\|_{\Phi}$. Using Lemma 4.7.11 we then deduce that T is contained in $\mathcal{I}^0$ if and only if it is contained in $\mathcal{I}^0_{\Phi}$. Thus, as sets, $\mathcal{I}^0$ and $\mathcal{I}^0_{\Phi}$ agree.

Let T be in $\mathcal{I}^0 = \mathcal{I}^0_{\Phi}$ and let $\sum_{n\geq 0} \mu_n(T)(U\eta_n) \otimes \eta_n^*$ be a Schmidt series for T. As this series converges to T both in $\mathcal{I}$ and in $\mathcal{I}_{\Phi}$, using (4.108) we get

$$\|T\|_{\mathcal{I}} = \lim_{N\to\infty} \Big\| \sum_{n<N} \mu_n(T)(U\eta_n) \otimes \eta_n^* \Big\|_{\mathcal{I}} = \lim_{N\to\infty} \Big\| \sum_{n<N} \mu_n(T)(U\eta_n) \otimes \eta_n^* \Big\|_{\Phi} = \|T\|_{\Phi}.$$

Thus $\|.\|_{\mathcal{I}}$ and $\|.\|_{\Phi}$ agrees on $\mathcal{I}^0$. This proves that the Banach ideals $\mathcal{I}^0$ and $\mathcal{I}^0_{\Phi}$ coincide. The proof is complete. □

Combining Proposition 4.9.7 and Proposition 4.9.8 we obtain:

PROPOSITION 4.9.9. *A Banach ideal $\mathcal{I}$ is separable if and only if there exists a symmetric norm Φ on l_f such that $\mathcal{I}$ coincides with the Banach ideal $\mathcal{I}^0_{\Phi}$.*

Next, let us denote by Φ' the dual symmetric norm of Φ as defined in (4.85). We can relate the Banach ideal $\mathcal{I}_{\Phi'}$ to the dual of $\mathcal{I}_{\Phi}$ as follows.

LEMMA 4.9.10 (Horn Inequality; see [**Si**, Thm. 1.15]). *Let $S, T \in \mathcal{K}$. Then*

$$\sum_{n<N} \mu_n(ST) \leq \sum_{n<N} \mu_n(S)\mu_n(T) \qquad \forall N \in \mathbb{N}.$$

PROPOSITION 4.9.11. *The following hold.*

(1) *For all $S, T \in \mathcal{K}$, we have*

$$\|ST\|_1 \leq \|S\|_{\Phi'} \|T\|_{\Phi}.$$

(2) *Let $S \in \mathcal{I}_{\Phi'}$ and $T \in \mathcal{I}_{\Phi}$. Then the operator ST is trace-class and*

$$|\operatorname{Trace}(ST)| \leq \|S\|_{\Phi'} \|T\|_{\Phi}. \tag{4.109}$$

(3) *For all $S \in \mathcal{K}$, we have*

$$\|S\|_{\Phi'} = \sup_{\substack{\|T\|_{\Phi}=1 \\ T\in\mathcal{R}_\infty}} |\operatorname{Trace}(ST)|. \tag{4.110}$$

PROOF. Let $S, T \in \mathcal{K}$. Then using Horn's inequality and (4.87) we see that, for any $N \in \mathbb{N}$, we have

$$\sum_{n<N} \mu_n(ST) \leq \sum_{n\geq 0} \mu_n^N(S)\mu^N(T) \leq \Phi'(\mu^N(S))\Phi(\mu^N(T)) \leq \|S\|_{\Phi'}\|T\|_{\Phi}.$$

Thus,

$$\|ST\|_1 = \sum_{n\geq 0} \mu_n(ST) \leq \|S\|_{\Phi'}\|T\|_{\Phi}.$$

Therefore, if $S \in \mathcal{I}_{\Phi'}$ and $T \in \mathcal{I}_{\Phi}$, then ST is trace-class and, using (4.27), we get

$$|\operatorname{Trace}(ST)| \leq \|ST\|_1 \leq \|S\|_{\Phi'}\|T\|_{\Phi}. \tag{4.111}$$

Let $S \in \mathcal{K}$. Then (4.111) implies that

$$\|S\|_{\Phi'} \geq \sup_{\substack{\|T\|_\Phi = 1 \\ T \in \mathcal{R}_\infty}} |\operatorname{Trace}(ST)|. \tag{4.112}$$

Let $A \in (0, \|S\|_\Phi)$. In view of (4.89) we can find $N \in \mathbb{N}$ large enough such that $A < \Phi'(\mu^N(S))$. Using (4.87) we see that there exists a sequence $b = (b_n)_{n\geq 0}$ in l_f^+ with same support as μ^N such that $\Phi'(b) = 1$ and

$$A < \sum_{n\geq 0} \mu_n^N(S) b_n \leq \sum_{n\geq 0} \mu_n(S) b_n. \tag{4.113}$$

Let $S = U|S|$ be the polar decomposition of S and let $(\xi_n)_{n\geq 0} \subset \mathcal{H}$ be an orthonormal family such that $|S|\xi_n = \mu_n(S)\xi_n$ $\forall n \in \mathbb{N}_0$. Set

$$T = \sum_{n\geq 0} b_n (\xi_n \otimes \xi_n^*) U^*.$$

The operator T has finite rank, since the support of b is finite.

By Proposition 3.1.8, the operator U^*U is the orthogonal projection onto $(\ker S)^\perp = (\ker |S|)^\perp$, so $U^*U\xi_n = \xi_n$ whenever $\mu_n(S) \neq 0$. Therefore, we can check that

$$T^*T = \sum_{n\geq 0} b_n^2 \xi_n \otimes \xi_n^* \quad \text{and} \quad |T| = \sum_{n\geq 0} b_n \xi_n \otimes \xi_n^*$$

Using the min-max principle we then deduce that $\mu_n(T) = b_n$ for all $n \in \mathbb{N}_0$. Thus,

$$\|T\|_\Phi = \Phi(\mu(T)) = \Phi(b) = 1.$$

We also have

$$ST = \Big(\sum_{n\geq 0} \mu_n(S) U(\xi_n \otimes \xi_n^*)\Big)\Big(\sum_{n\geq 0} b_n (\xi_n \otimes \xi_n^*) U^*\Big) = \sum_{n\geq 0} \mu_n(S) b_n U(\xi_n \otimes \xi_n^*) U^*,$$

Thus $\operatorname{Trace}(ST)$ is equal to

$$\sum_{n\geq 0} \mu_n(S) b_n \operatorname{Trace}\left(U(\xi_n \otimes \xi_n^*)U^*\right) = \sum_{n\geq 0} \mu_n(S) b_n \operatorname{Trace}\left(U^*U(\xi_n \otimes \xi_n^*)\right) = \sum_{n\geq 0} \mu_n(S) b_n.$$

In view of (4.113) this implies that $\operatorname{Trace}(ST) > A$. Since $\|T\|_\Phi = 1$ we deduce that

$$A < \sup_{\substack{\|T\|_\Phi = 1 \\ T \in \mathcal{R}_\infty}} |\operatorname{Trace}(ST)| \qquad \forall A \in (0, \|S\|_\Phi).$$

Combining this with (4.112) yields (4.110). The proof is complete. □

Recall that by Proposition 4.3.4 we have an isometric isomorphism from $\mathcal{L}(\mathcal{H})$ onto $(\mathcal{L}^1)'$ given by

(4.114)

$$\mathcal{L}(\mathcal{H}) \ni S \longrightarrow (S, .) \in (\mathcal{L}^1)', \qquad (S, .) : \mathcal{L}(\mathcal{H}) \ni T \longrightarrow (S, T) := \operatorname{Trace}(ST).$$

It follows from Proposition 4.9.11 that we also have linear map,

$$\mathcal{I}_{\Phi'} \ni S \longrightarrow (S, .) \in (\mathcal{I}_\Phi^0)', \qquad (S, .) : \mathcal{L}(\mathcal{H}) \ni T \longrightarrow (S, T) := \operatorname{Trace}(ST). \tag{4.115}$$

and the density of $\mathcal{L}^1$ in $\mathcal{I}_\Phi^0$ show that if $S \in \mathcal{I}_{\Phi'}$ then $(S, .)$ uniquely extends to a continuous linear map on $\mathcal{I}_\Phi^0$, which we shall continue to denote $(S, .)$.

Proposition 4.9.12. *If $\mathcal{I}_\Phi \supsetneq \mathcal{L}^1$, then (4.115) yields an isometric isomorphism,*

$$\mathcal{I}_{\Phi'} \simeq (\mathcal{I}_\Phi^0)'.$$

Proof. Let us denote by $\mathcal{T}_\Phi$ the linear map (4.115). It follows from (4.109) and (4.110) that, for all $S \in \mathcal{I}_{\Phi'}$,

$$\|S\|_{\Phi'} = sup_{\substack{\|T\|_\Phi=1 \\ T\in\mathcal{R}_\infty}} |\operatorname{Trace}(ST)| \leq \sup_{\substack{\|T\|_\Phi=1 \\ T\in\mathcal{R}_\infty}} |\operatorname{Trace}(ST)| \leq \|S\|_{\Phi'}.$$

Therefore $\mathcal{T}_\Phi$ is an isometry, so it follows from Lemma 1.1.8 that for proving that $\mathcal{T}_\Phi$ is an isomorphism it is enough to show that it is onto.

Consider the following subspace of $\mathcal{L}(\mathcal{H})$,

$$\mathcal{I} := \mathcal{T}^{-1}((\mathcal{I}_\Phi^0)') = \left\{ S \in \mathcal{L}(\mathcal{H});\ \sup_{T\in\mathcal{R}_\infty\setminus 0} \frac{|\operatorname{Trace}(ST)|}{\|T\|_\Phi} < \infty \right\}.$$

Let $S \in \mathcal{I}$, let $A, B \in \mathcal{L}(\mathcal{H})$ and $C := \sup_{T\in\mathcal{R}_\infty\setminus 0} \frac{|\operatorname{Trace}(ST)|}{\|T\|_\Phi}$. Then, for all $T \in \mathcal{R}_\infty$,

$$|\operatorname{Trace}(ASBT)| = |\operatorname{Trace}(SBTA)| \leq C\|BTA\|_\Phi \leq C\|B\|\|T\|_\Phi\|A\|,$$

hence ASB is contained in $\mathcal{I}$. This shows that $\mathcal{I}$ is a two-sided ideal.

Suppose that $\mathcal{I}$ is not contained in $\mathcal{K}$. As $\mathcal{I}$ is a two-sided ideal, Proposition 4.7.2 then insures us that $\mathcal{I} = \mathcal{L}(\mathcal{H})$ and $\mathcal{K} = \mathcal{I} \cap \mathcal{K}$. Since (4.110) shows that $\mathcal{I} \cap \mathcal{K} = \mathcal{I}_{\Phi'}$, we see that $\mathcal{I}_{\Phi'} = \mathcal{K}$. Observe that $\mathcal{K}$ is the Banach ideal $\mathcal{I}_{\Phi_\infty}$ associated to the norm Φ_∞ in (4.75). Using Remark 4.9.5 we see that Φ' and Φ_∞ are equivalent norms on l_f, so by Remark 4.8.8 their dual norms. By Lemma 4.8.7 the dual norm of Φ' is Φ and, as shown in Example 4.8.9, the dual norm of Φ_∞ is the norm Φ_1. Combining this with Remark 4.9.5, we see that $\mathcal{I}_\Phi = \mathcal{I}_{\Phi_1} = \mathcal{L}^1$. This contradicts the assumption that $\mathcal{I}_\Phi$ does not coincide with $\mathcal{L}^1$, so $\mathcal{I}$ must be contained in $\mathcal{K}$. As $\mathcal{I} \cap \mathcal{K} = \mathcal{I}_{\Phi'}$, this proves that $\mathcal{I} = \mathcal{I}_{\Phi'}$.

Let $\varphi \in (\mathcal{I}_\Phi^0)'$. By Proposition 4.7.14 the inclusion $\mathcal{L}^1 \subset \mathcal{L}_\Phi^0$ is continuous, so φ induces a continuous linear map on $\mathcal{L}^1$. The isomorphism (4.114) insures us that there exists $S \in \mathcal{L}(\mathcal{H})$ such that

$$\langle \varphi, T \rangle = \operatorname{Trace}(ST) \qquad \forall T \in \mathcal{L}^1. \tag{4.116}$$

Then, for all $T \in \mathcal{R}_\infty$, we have

$$|\operatorname{Trace}(ST)| = |\langle \varphi, T \rangle| \leq \|\varphi\|_{\mathcal{I}_\Phi'}\|T\|_\Phi.$$

This shows that S is contained in $\mathcal{I} = \mathcal{I}_{\Phi'}$. Observe that (4.116) shows that φ and $\mathcal{T}_\Phi$ agree on $\mathcal{L}^1$. In particular, they agree on finite-rank operators. As finite-rank operators are dense in $\mathcal{I}_\Phi^0$, it follows that φ and $\mathcal{T}_\Phi$ agrees on all $\mathcal{I}_\Phi^0$, i.e., $\mathcal{T}_\Phi(S) = \varphi$. This proves that $\mathcal{T}_\Phi$ is onto, completing the proof. ☐

4.10. Schatten Ideals

Let $p \in [0, \infty]$. The Schatten ideal $\mathcal{L}^p$ is the Banach ideal $\mathcal{L}_{\Phi_p}$ associated to the p-norm Φ_p. Thus, if for any $T \in \mathcal{K}$, we set

$$\|T\|_p := \Phi_p\left((\mu_n(T))_{n\geq 0}\right) = \left(\sum_{n\geq 0} \mu_n(T)^p\right)^{\frac{1}{p}},$$

then

$$\mathcal{L}^p = \left\{T \in \mathcal{K}; \|T\|_p < \infty\right\}.$$

For $p = 1$ (resp. $p = 2$) we recover the Banach ideal of trace-class operators (resp. Hilbert-Schmidt operators). As alluded to in the proof of Proposition 4.9.12, for $p = \infty$ the Banach ideal $\mathcal{L}^\infty = \mathcal{I}_{\Phi_\infty}$ is the whole Banach ideal of compact operators.

As shown in Example 4.1.9 we have $\mu_n(|T|^p) = \mu_n(T)^p$ $\forall n \in \mathbb{N}$, so we have

$$\sum_{n\geq 0} \mu_n(T)^p = \operatorname{Trace}|T|^p \qquad \forall T \in \mathcal{L}(\mathcal{H}).$$

Thus,

$$T \in \mathcal{L}^p \Longleftrightarrow \operatorname{Trace}|T|^p < \infty \Longleftrightarrow |T|^p \in \mathcal{L}^1.$$

PROPOSITION 4.10.1. *Let $q \in (p, \infty)$. Then*

$$\|T\|_q \leq \|T\|_p \qquad \forall T \in \mathcal{K}.$$

In particular, there is a continuous inclusion,

$$\mathcal{L}^q \subset \mathcal{L}^p.$$

PROOF. Let $T \in \mathcal{K}$. Observe that, for all $n \in \mathbb{N}_0$,

$$\mu_n(T)^q = \mu_n(T)^{q-p}.\mu_n(T)^p \leq (\|T\|_p)^{q-p}.\mu_n(T)^p.$$

Therefore, we get

$$\|T\|_q = \left(\sum_{n\geq 0} \mu_n(T)^q\right)^{\frac{1}{q}} \leq (\|T\|_p)^{\frac{q-p}{q}} \left(\sum_{n\geq 0} \mu_n(T)^p\right)^{\frac{1}{q}} = (\|T\|_p)^{\frac{q-p}{q}} (\|T\|_p)^{\frac{p}{q}} = \|T\|_p,$$

proving the lemma. □

It can be easily seen that, for $p < \infty$, the symmetric norm Φ_p satisfies the condition (4.107). Therefore, Proposition 4.9.7 gives

PROPOSITION 4.10.2. *The Schatten ideal $\mathcal{L}^p$ is separable and the finite-rank operators are dense in $\mathcal{L}^p$.*

Assume $p > 1$ and let $p' \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then the dual norm of Φ_p is the p'-norm $\Phi_{p'}$. This follows from the Hölder inequality,

$$\left|\sum_{n\geq 0} a_n b_n\right| \leq \left(\sum_{n\geq 0} |a_n|^{p'}\right)^{\frac{1}{p'}} \left(\sum_{n\geq 0} |b_n|^p\right)^{\frac{1}{p}} \qquad \forall a, b \in l_f,$$

and the fact that we actually have an equality when $b_n = \frac{\overline{a_n}}{|a_n|}|a_n|^{\frac{p'}{p}}$. Therefore, using Proposition 4.9.11 and Proposition 4.9.12 we obtain:

PROPOSITION 4.10.3. *The following hold.*

(1) *For all $S, T \in \mathcal{K}$, we have*

$$\|ST\|_1 \leq \|S\|_{p'}\|T\|_p.$$

(2) *If $S \in \mathcal{L}^{p'}$ and $T \in \mathcal{L}^p$, then the operator ST is trace-class and*

$$|\operatorname{Trace}(ST)| \leq \|S\|_q\|T\|_p.$$

(3) *The linear map (4.115) gives rise to an isometric isomorphism,*

$$\mathcal{L}^{p'} \simeq (\mathcal{L}^p)'.$$

The Horn inequality admits the following generalization (see [**Si**, Thm. 1.15]). Let $r \in [1,\infty)$ and $N \in \mathbb{N}$. Then

$$\sum_{n<N} \mu_n(ST)^r \leq \sum_{n<N} \mu_n(S)^r \mu_n(T)^r \qquad \forall S,T \in \mathcal{K}.$$

Using this generalization we can show that, if $\frac{1}{p_1} + \ldots + \frac{1}{p_k} = \frac{1}{r}$, then

$$\|T_1 T_2 \cdots T_k\|_1 \leq \|T_1\|_{p_1} \|T_2\|_{p_2} \cdots \|T_1\|_{p_k} \qquad \forall T_j \in \mathcal{K}.$$

In particular, if for every $j = 1, \ldots, k$ the operator T_j is in $\mathcal{L}^{p_j}$, then $T_1 T_2 \cdots T_k$ is a trace-class operator and we have

$$|\operatorname{Trace}(T_1 T_2 \cdots T_k)| \leq \|T_1\|_{p_1} \|T_2\|_{p_2} \cdots \|T_1\|_{p_k}.$$

4.11. Banach ideals associated to divergent series

Following [**GK**] we can produce a large class of non-separable ideals as follows.

Let $\pi = (\pi_n)_{n\geq 0}$ be a non-increasing sequence of positive real numbers satisfying the following two conditions:

$$\lim_{n\to\infty} \pi_n = 0 \qquad \text{and} \qquad \sum_{n\geq 0} \pi_n = \infty. \tag{4.117}$$

Examples of sequences satisfying all these conditions are provided by the sequences

$$\pi^{(p)} := ((n+1)^{\frac{1}{p}})_{n\geq 0}, \qquad p \geq 1. \tag{4.118}$$

Using Lemma 4.8.6 it is not difficult to check that we define a symmetric norm on l_f by letting

$$\Phi_\pi(a) := \sup_{N\geq 1} \frac{\sigma_N(a^*)}{\sigma_N(\pi)}.$$

We denote by $\mathcal{I}_{\Phi_\pi}$ the associated Banach ideal. In particular,

$$\mathcal{I}_{\Phi_\pi} = \Big\{ T \in \mathcal{K};\ \sigma_N(T) = \mathrm{O}(\sigma_N(\pi)) \Big\}. \tag{4.119}$$

Lemma 4.11.1. *Let $a = (a_n)_{n\geq 0}$ be a non-increasing sequence in l_0 such that $\Phi_\pi(a) < \infty$. Then*

$$\lim_{N\to\infty} \Phi(a_N, a_{N+1}, \ldots) = \limsup_{N\to\infty} \frac{\sigma_N(a)}{\sigma_N(\pi)}.$$

Proof. For $N \in \mathbb{N}$ set $a^N = a_N, a_{N+1}, \ldots) \in l_0$. Let $n \in \mathbb{N}_0$. As the sequence $(a_n)_{n\geq 0}$ is non-decreasing, we have

$$\sigma_n(a^N) = \sum_{j<N} a_{j+N} = \sigma_{n+N}(a) - \sigma_n \geq \sigma_n(a) - \sigma_N(a).$$

Thus,

$$\frac{\sigma_n(a)}{\sigma_n(\pi)} \leq \frac{\sigma_n(a^N)}{\sigma_n(\pi)} + \frac{\sigma_N(a)}{\sigma_n(\pi)} \leq \Phi_\pi(a^N) + \frac{\sigma_N(a)}{\sigma_n(\pi)}.$$

Since (4.117) implies that $\lim_{n\to\infty} \sigma_n(\pi) = \sum_{j\geq 0} a_j = \infty$, we deduce that

$$\limsup_{n\to\infty} \frac{\sigma_n(a)}{\sigma_n(\pi)} \leq \Phi_\pi(a^N) \qquad \forall N \in \mathbb{N}.$$

Thus,

$$\limsup_{n\to\infty} \frac{\sigma_n(a)}{\sigma_n(\pi)} \leq \lim_{N\to\infty} \Phi_\pi(a^N). \tag{4.120}$$

Let N and m be positive integers. As the sequence $(a_n)_{n\geq 0}$ is non-decreasing, for any $n \in \mathbb{N}$, we have

$$\sigma_n(a^N) = \sum_{j<n} a_{N+j} \leq \sum_{j<n} a_j = \sigma_n(a).$$

Therefore, for all $n \geq m$, we have

$$\frac{\sigma_n(a^N)}{\sigma_n(\pi)} \leq \frac{\sigma_n(a)}{\sigma_n(\pi)} \leq \sup_{p\geq m} \frac{\sigma_p(a)}{\sigma_p(\pi)}.$$

Notice also that, for all $n \leq m-1$,

$$\frac{\sigma_n(a^N)}{\sigma_n(\pi)} \leq \frac{na_0^N}{\pi_0} = \frac{m}{\pi_0} a_N.$$

Therefore, we have

$$\Phi_\pi(a^N) = \sup_{n\geq 1} \frac{\sigma_n(a^N)}{\sigma_n(\pi)} \leq \sup\left\{ \sup_{p\geq m} \frac{\sigma_p(a)}{\sigma_p(\pi)}, \frac{m}{\pi_0} a_N \right\}.$$

Since $\lim_{N\to\infty} a_N = 0$, it follows that, ,

$$\lim_{N\to\infty} \Phi_\pi(a^N) \leq \sup_{p\geq m} \frac{\sigma_p(a)}{\sigma_p(\pi)} \qquad \forall m \in \mathbb{N}.$$

Thus,

$$\lim_{N\to\infty} \Phi_\pi(a^N) \leq \lim_{m\to\infty} \sup_{p\geq m} \frac{\sigma_p(a)}{\sigma_p(\pi)} = \limsup_{n\to\infty} \frac{\sigma_n(a)}{\sigma_n(\pi)}.$$

Combining this with (4.120) yields the lemma. □

Proposition 4.11.2. *The Banach ideal $\mathcal{I}_{\Phi_\pi}$ is not separable and*

$$\mathcal{I}^0_{\Phi_\pi} = \left\{ T \in \mathcal{K}; \sigma_N(T) = \mathrm{o}(\sigma_N(\pi)) \right\}. \tag{4.121}$$

Proof. It follows from Lemma 4.9.6 and Lemma 4.11.1 that an operator $T \in \mathcal{K}$ is in $\mathcal{I}^0_{\Phi_\pi}$ if and only if $\sigma_N(T) = \mathrm{o}(\sigma_N(\pi))$.

Thanks to (4.117) the sequence $\pi = (\pi_n)_{n\geq 0}$ is contained in l_0. It is immediate that $\Phi_\pi(\pi) = 1$. Moreover, using Lemma 4.11.1, we see that

$$\lim_{N\to\infty} \Phi_\pi(\pi_N, \pi_{N+1}, \ldots) = \limsup_{N\to\infty} \frac{\sigma_N(\pi)}{\sigma_N(\pi)} = 1.$$

Therefore (4.107) does hold, so using Proposition 4.9.7 we see that the Banach ideal $\mathcal{I}_{\Phi_\pi}$ is not separable. The proof is complete. □

Proposition 4.11.3. *Suppose that $\sigma_n(\pi) = \mathrm{O}(n\pi_n)$. Then*

$$\mathcal{I}_{\Phi_\pi} = \left\{ T \in \mathcal{K}; \sigma_N(T) = \mathrm{O}(\pi_n) \right\} \quad \textit{and} \quad \mathcal{I}^0_{\Phi_\pi} = \left\{ T \in \mathcal{K}; \sigma_N(T) = \mathrm{o}(\pi_n) \right\}.$$

Proof. Let $T \in \mathcal{K}$. Let $m \in \mathbb{N}$ and set $a_m = \sup_{n \geq m} \frac{\mu_n(T)}{\pi_n}$. For $n \geq m$ we have $\mu_n(T) = \frac{\mu_n(T)}{\pi_n}\pi_n \leq a_m \pi_n$, so for all $N > m$ we get

$$\sigma_N(T) = \sigma_m(T) + \sum_{m \leq n < N} \mu_n(T) \leq \sigma_m(T) + a_m \sum_{m \leq n < N} \pi_n \leq \sigma_m(T) + a_m \sigma_N(T).$$

Since $\lim_{N \to \infty} \sigma_N(\pi) = \infty$, we deduce that

$$\limsup_{N \geq \infty} \frac{\sigma_N(T)}{\sigma_N(\pi)} \leq a_m \qquad \forall m \in \mathbb{N}.$$

Thus,

$$\limsup_{N \to \infty} \frac{\sigma_N(T)}{\sigma_N(\pi)} \leq \lim_{m \to \infty} a_m = \limsup_{n \to \infty} \frac{\mu_n(T)}{\pi_n}. \tag{4.122}$$

Set $C := \sup_{n \geq 1} \frac{\sigma_n(\pi)}{n\pi_n}$. Since the sequence $(\mu_n(T))_{n \geq 0}$ is non-decreasing, for any $n \in \mathbb{N}$, we have $\sigma_n(T) = \sum_{j<n} \mu_j(T) \geq n\mu_n(T)$, and hence

$$\frac{\mu_n(T)}{\pi_n} \leq \frac{\sigma_n(T)}{n\pi_n} = \frac{\sigma_n(\pi)}{n\pi_n} . \frac{\sigma_n(T)}{\sigma_n(\pi)} \leq C \frac{\sigma_n(T)}{\sigma_n(\pi)}.$$

Thus,

$$\limsup_{n \to \infty} \frac{\mu_n(T)}{\pi_n} \leq C \limsup_{n \to \infty} \frac{\sigma_n(T)}{\sigma_n(\pi)}. \tag{4.123}$$

Combining (4.122)–(4.123) with (4.119) and (4.121) yields the proposition. □

Let Φ'_π be the dual norm of Φ_π as defined in (4.85). This is a symmetric norm on l_f.

Lemma 4.11.4. *We have*

$$\Phi'_\pi(a) = \sum_{n \geq 0} \pi_n a_n^* \qquad \forall a \in l_f. \tag{4.124}$$

Proof. Let $\tilde{\Phi}$ be the function on l_f^+ defined by

$$\tilde{\Phi}(a) := \sum_{n \geq 0} \pi_n a_n \qquad \forall a \in l_f^+.$$

In view of Lemma 4.8.6, in order to prove that $\tilde{\Phi}$ agrees with Φ'_π it is enough to show that $\tilde{\Phi}$ satisfies the conditions (4.81)–(4.84) and agrees with Φ'_π on l_f^+.

Clearly, $\tilde{\Phi}$ satisfies (4.81)–(4.83). As, for all $a \in l_f$, we have

$$\sum_{n \geq 0} \pi_n a_n = \pi_0 \sigma_0 + \sum_{n \geq 1} \pi_n(\sigma_{n+1}(a) - \sigma_n(a)) = \sum_{N \geq 1} (\pi_{N-1} - \pi_N)\sigma_N(a), \tag{4.125}$$

we see that $\tilde{\Phi}$ satisfies (4.84) as well.

It remains to prove that $\tilde{\Phi}$ agrees with Φ'_π on l_f^+. Let $a \in l_f^+$. Then by (4.86) we have

$$\Phi'_\pi(a) = \sup \left\{ \sum_{n \geq 0} a_n b_n;\ b \in l_f^+,\ \Phi_\pi(b) = 1 \right\}.$$

Notice that, for any $N \in \mathbb{N}$, the sequence $(\pi_0, \ldots, \pi_N, 0, 0, \ldots)$ belongs to l_f^+ and we can check that $\Phi_\pi(\pi_0, \ldots, \pi_N, 0, 0, \ldots) = 1$. Therefore, if N is large enough so that $a_n = 0$ for $n \geq N$, then

$$\Phi'_\pi(a) \geq \sum_{n \leq N} a_n \pi_n = \sum_{n \geq 0} a_n \pi_n = \tilde{\Phi}(a). \tag{4.126}$$

Let $b \in l_f^+$ be such that $\Phi_\pi(b) = 1$. Then $\sigma_N(b) \leq \sigma_N(\pi)$ for all $N \in \mathbb{N}$. Therefore, arguing as in (4.125), we get

$$\sum_{n \geq 0} a_n b_n = \sum_{N \geq 1} (a_N - a_{N-1}) \sigma_N(b) \leq \sum_{N \geq 1} (a_N - a_{N-1}) \sigma_N(\pi) = \sum_{n \geq 0} a_n \pi_n.$$

It then follows that $\Phi'_\pi(a) \leq \tilde{\Phi}(a)$. Combining this with (4.126) proves that $\tilde{\Phi}$ and Φ'_π agree on l_f^+. The proof is complete. ☐

It follows from Lemma 4.11.4 that

$$\mathcal{I}_{\Phi'_\pi} = \Big\{ T \in \mathcal{K};\ \sum_{n \geq 0} \pi_n \mu_n(T) < \infty \Big\} \quad \text{and} \quad \|T\|_{\Phi'_\pi} = \sum_{n \geq 0} \pi_n \mu_n(T) \quad \forall T \in \mathcal{K}.$$

Using (4.124) it is not hard to check that the symmetric norm Φ'_π satisfies (4.107). Therefore, from Proposition 4.9.7 we get

Proposition 4.11.5. *The Banach ideal $\mathcal{I}_{\Phi'_\pi}$ is separable and the finite-rank operators are dense in it, i.e., $\mathcal{I}^0_{\Phi'_\pi} = \mathcal{I}_{\Phi'_\pi}$.*

Using Proposition 4.9.12, Lemma 4.8.7 and the fact that $\mathcal{I}^0_{\Phi'_\pi} = \mathcal{I}_{\Phi'_\pi}$ we get

Proposition 4.11.6. *The linear map (4.115) gives rise to isometric isomorphisms,*

$$\mathcal{I}_{\Phi'_\pi} \simeq (\mathcal{I}^0_{\Phi_\pi})' \qquad \textit{and} \qquad \mathcal{I}_{\Phi_\pi} \simeq (\mathcal{I}_{\Phi'_\pi})'.$$

4.12. The Banach ideals $\mathcal{L}^{(p,\infty)}$ and $\mathcal{L}^{(p,1)}$

Let $p \in (1, \infty)$. We denote by $\mathcal{L}^{(p,\infty)}$ the Banach ideal $\mathcal{I}_{\Phi_{(p,\infty)}}$ associated to the symmetric norm $\Phi_{(p,\infty)}$ on l_f defined by

$$\Phi_{(p,\infty)}(a) := \sup_{N \geq 1} \frac{\sigma_N(a)}{N^{1-\frac{1}{p}}} \qquad \forall a \in l_f.$$

Thus,

$$\mathcal{L}^{(p,\infty)} = \Big\{ T \in \mathcal{K};\ \sigma_N(T) = \mathrm{O}(N^{1-\frac{1}{p}}) \Big\},$$

and $\mathcal{L}^{(p,\infty)}$ is a Banach ideal for the norm,

$$\|T\|_{(p,\infty)} := \|T\|_{\Phi_{(p,\infty)}} = \sup_{N \geq 1} \frac{\sigma_N(T)}{N^{1-\frac{1}{p}}}.$$

Since $\sum_{n<N} (n+1)^{-\frac{1}{p}} \sim \frac{1}{1-\frac{1}{p}} N^{1-\frac{1}{p}}$ as $N \to \infty$, we see that $\Phi_{(p,\infty)}$ is equivalent to the symmetric norm $\Phi_{\pi^{(p)}}$ associated to the sequence $\pi^{(p)}$ in (4.118). Therefore, the Banach ideals $\mathcal{L}^{(p,\infty)}$ and $\mathcal{I}_{\Phi_{\pi^{(p)}}}$ have same underlying sets and their norms are equivalent. Since for $p > 1$ we have $\sigma_n(\pi^{(p)}) = \mathrm{O}(n\pi_n^{(p)})$, using Proposition 4.11.2 and Proposition 4.11.3 we obtain:

Proposition 4.12.1. *Let $p \in (1, \infty)$. Then*

(1) *We have*

$$\mathcal{L}^{(p,\infty)} = \Big\{ T \in \mathcal{K};\ \mu_n(T) = \mathrm{O}(n^{-\frac{1}{p}}) \Big\}.$$

(2) *The Banach ideal* $\mathcal{L}^{(p,\infty)}$ *is not separable and the closure of finite-rank operators in* $\mathcal{L}^{(p,\infty)}$ *is*

$$\begin{aligned}\mathcal{L}_0^{(p,\infty)} &= \Big\{ T \in \mathcal{K};\ \sigma_N(T) = \mathrm{o}(N^{1-\frac{1}{p}}) \Big\} \\ &= \Big\{ T \in \mathcal{K};\ \mu_n(T) = \mathrm{o}(n^{-\frac{1}{p}}) \Big\}.\end{aligned}$$

For $p = 1$ we let $\mathcal{L}^{(1,\infty)}$ be the Banach ideal $\mathcal{I}_{\Phi_{(1,\infty)}}$ associated to the symmetric norm $\Phi_{(p,\infty)}$ on l_f defined by

$$\Phi_{(1,\infty)}(a) := \sup_{N \geq 2} \frac{\sigma_N(a)}{\log N} \qquad \forall a \in l_f.$$

Thus,

$$\mathcal{L}^{(1,\infty)} = \Big\{ T \in \mathcal{K};\ \sigma_N(T) = \mathrm{O}(\log N) \Big\},$$

and the norm of $\mathcal{L}^{(p,\infty)}$ is

$$\|T\|_{(1,\infty)} := \|T\|_{\Phi_{(1,\infty)}} = \sup_{N \geq 2} \frac{\sigma_N(T)}{\log N}.$$

This Banach ideal is sometimes called the *Dixmier ideal*, since this is the natural domain of the Dixmier trace (cf. Chapter **??**).

As in the case $p > 1$, the symmetric norms $\Phi_{(1,\infty)}$ and $\Phi_{\pi^{(1)}}$ are equivalent, so the Banach ideals $\mathcal{L}^{(1,\infty)}$ and $\mathcal{I}_{\Phi_{\pi^{(1)}}}$ have same underlying sets and their norms are equivalent. Therefore, using Proposition 4.11.2 we get:

PROPOSITION 4.12.2. *The Banach ideal* $\mathcal{L}^{(1,\infty)}$ *is not separable and the closure of finite-rank operators in* $\mathcal{L}^{(1,\infty)}$ *is*

$$\mathcal{L}_0^{(1,\infty)} = \Big\{ T \in \mathcal{K};\ \sigma_N(T) = \mathrm{o}(\log N) \Big\}.$$

REMARK 4.12.3. Unlike in the case $p > 1$, we have a strict inclusion,

$$\mathcal{L}^{(1,\infty)} \supsetneq \Big\{ T \in \mathcal{K};\ \mu_n(T) = \mathrm{O}(n^{-1}) \Big\}.$$

Clearly, if $T \in \mathcal{K}$ is such that $\mu_n(T) = \mathrm{O}(n^{-1})$, then $\sigma_N(T) = \mathrm{O}(\log N)$, and hence T is contained in $\mathcal{L}^{(1,\infty)}$.

To show that the inclusion is strict we only have to exhibit a non-increasing sequence of positive numbers $(a_n)_{n \geq 0}$ such that $\sigma_N(a) = \mathrm{O}(\log N)$ and na_n is not bounded. An example of such a sequence is obtained as follows.

For any $k \in \mathbb{N}$ set $n_k = k^k$ and let $(a_n)_{n \geq 0}$ be the sequence defined by

$$a_0 = a_1 = 1 \qquad \text{and} \qquad a_n = \frac{1 + \log k}{n_k} \quad \text{for } n_{k-1} < n \leq n_k.$$

As $n_k a_{n_k} = 1 + \log k \to \infty$ as $k \to \infty$, we see that a_n is not a $\mathrm{O}(n^{-1})$.

Furthermore, for $k \geq 3$, we have

$$\sum_{2\leq n\leq n_k} a_n = \sum_{2\leq j\leq k} \frac{1+\log j}{n_j}(n_j - n_{j-1}) \leq \sum_{2\leq j\leq k} (1+\log j)$$
$$\leq \int_1^k (1+\log x)dx = k\log k = \log n_k.$$

Therefore, if $n_{k-1} \leq N \leq n_k$, then

$$\sum_{2\leq n<N} a_n \leq \sum_{2\leq n\leq n_k} a_n \leq \frac{\log n_k}{\log n_{k-1}} \log n_{k-1} \leq \frac{k\log k}{(k-1)\log(k-1)} \log N \leq C\log N,$$

where we have set $C := \sup_{k\geq 3} \frac{k\log k}{(k-1)\log(k-1)}$. This shows that $\sigma_N(a) = \mathrm{O}(\log N)$, concluding the remark.

Next, let $p \in (1,\infty]$ and denote by $\mathcal{L}^{(p,1)}$ the Banach ideal associated to the symmetric norm $\Phi_{(1,\infty)}$ on l_f defined by

$$\Phi_{(p,1)}(a) := \sum_{n\geq 0} (n+1)^{\frac{1}{p}-1} a_n \qquad \forall a \in l_f.$$

In other words,

$$\mathcal{L}^{(p,1)} = \Big\{ T \in \mathcal{K};\ \sum_{n\geq 0} (n+1)^{\frac{1}{p}-1} \mu_n(T) < \infty \Big\},$$

and the norm of $\mathcal{L}^{(p,1)}$ is given by

$$\|T\|_{(p,1)} := \sum_{n\geq 0} (n+1)^{\frac{1}{p}-1} \mu_n(T).$$

When $p=\infty$ the Banach ideal $\mathcal{L}^{(\infty,1)}$ is called the *Macaev ideal*.

Let $p' \in [1,\infty)$ be such that $\frac{1}{p}+\frac{1}{p'}=1$. Since Lemma 4.11.4 shows that $\Phi_{(p,1)}$ is the dual norm $\Phi'_{\pi^{(p')}}$, we see that $\mathcal{L}^{(p,1)}$ is the Banach ideal $\mathcal{I}_{\Phi'_{\pi^{(p')}}}$. Therefore, Proposition 4.11.5 and Proposition 4.11.6 yield:

Proposition 4.12.4. *Let $p \in (1,\infty]$ and $p' \in [1,\infty)$ be such that $\frac{1}{p}+\frac{1}{p'}=1$.*

1. *The Banach ideal $\mathcal{L}^{(p,1)}$ is separable and agrees with the closure of finite-rank operators.*
2. *The linear map (4.115) yields isomorphisms,*

$$\mathcal{L}^{(p,1)} \simeq (\mathcal{L}_0^{(p',\infty)})' \qquad \text{and} \qquad \mathcal{L}^{(p',\infty)} \simeq (\mathcal{L}^{(p,1)})'$$

Finally, the ideals $\mathcal{L}^{(p,\infty)}$ and $\mathcal{L}^{(p,1)}$ can be compared to the Schatten ideals.

Proposition 4.12.5. *We have continuous inclusions,*

$$\mathcal{L}^p \subset \mathcal{L}_0^{(p,\infty)} \quad \text{and} \quad \mathcal{L}^{(p,\infty)} \subset \mathcal{L}^q, \qquad 1 \leq p < q < \infty, \tag{4.127}$$

$$\mathcal{L}^q \subset \mathcal{L}^{(p,1)}, \qquad 1 \leq q < p \leq \infty, \tag{4.128}$$

$$\mathcal{L}^{(p,1)} \subset \mathcal{L}^p, \qquad 1 \leq p < \infty. \tag{4.129}$$

Proof. In view of Proposition 4.12.4 it is enough to prove that we have continuous inclusions,

$$\mathcal{L}^p \subset \mathcal{L}_0^{(p,\infty)}, \qquad 1 \leq p < \infty,$$
$$\mathcal{L}^q \subset \mathcal{L}^{(p,1)}, \qquad 1 \leq q < p \leq \infty,$$

since the other continuous inclusions would follow by duality.

Let $p \in (1,\infty]$ and let $q \in [1,p)$. Let $T \in \mathcal{L}^q$. Let p' and q' be such that $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. The Hölder inequality gives

$$\|T\|_{(p,1)} = \sum_{n\geq 0}(n+1)^{-\frac{1}{p'}}\mu_n(T) \leq \left(\sum_{n\geq 0}(n+1)^{-\frac{q'}{p'}}\right)^{\frac{1}{q'}} \|T\|_q.$$

The fact that $q<p$ insures us that $q'>p'$, so the series $\sum_{n\geq 0}(n+1)^{-\frac{q'}{p'}}$ is convergent, so we see that $\mathcal{L}^q$ is contained in $\mathcal{L}^{(p,1)}$ and the inclusion is continuous.

Let $p \in [1,\infty)$ and let $T \in \mathcal{L}^p$. Using the Hölder inequality we see that, for any $N \in \mathbb{N}$,

$$\sigma_N(T) = \sum_{n<N}\mu_n(T) \leq \left(\sum_{n<N} 1^{p'}\right)^{\frac{1}{p'}} \left(\sum_{n<N}\mu_n(T)^p\right)^{\frac{1}{p}} \leq N^{1-\frac{1}{p}}\|T\|_p.$$

In view of the definition of the norm of $\mathcal{L}^{(p,\infty)}$ this implies that

$$\|T\|_{(p,\infty)} \leq \|T\|_p \qquad \forall T \in \mathcal{L}^p.$$

Thus $\mathcal{L}^p$ is contained in $\mathcal{L}^{(p,\infty)}$ and the inclusion is continuous. Since by Proposition 4.10.2 the finite-rank operators are dense in $\mathcal{L}^p$, it follows that $\mathcal{L}^p$ is contained the closure of finite-rank operators in $\mathcal{L}^{(p,\infty)}$, that is, the ideal $\mathcal{L}_0^{(p,\infty)}$. Therefore, we actually have a continuous inclusion of $\mathcal{L}^p$ in $\mathcal{L}_0^{(p,\infty)}$. The proof is complete. □

Remark 4.12.6. Let $T \in \mathcal{K}$ be such that

$$\mu_n(T) = (n+1)^{-\frac{1}{p}}\left(\log(n+2)\right)^{-\alpha} \qquad \forall n \in \mathbb{N}_0.$$

The following observations hold:

- If $\alpha = \frac{1}{p}$ and $p \in (1,\infty)$, then T is not in $\mathcal{L}^p$ and Proposition 4.12.1 and Proposition 4.12.2 insure us that T is in $\mathcal{L}_0^{(p,\infty)}$.
- If $\alpha = -1$ and $p \in [1,\infty)$, then T is contained in every ideal $\mathcal{L}^q$ with $q>p$. If $p>1$, then using Proposition 4.12.1 we see that T is not in $\mathcal{L}^{(p,\infty)}$. Likewise, when $p=1$ the operator T is not in $\mathcal{L}^{(1,\infty)}$.
- If $\alpha = 1$ and $p \in (1,\infty)$, then T is in $\mathcal{L}^{(p,1)}$, but it is not in any ideal $\mathcal{L}^q$ with $q>p$.
- If $p \in (1,\infty]$ and $\alpha \in (p^{-1},1)$, then T is in $\mathcal{L}^p$, but not in $\mathcal{L}^{(p,1)}$.

This shows that all the inclusions in (4.127)–(4.129) are strict.

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