CHAPTER 3

# Operators on a Hilbert Space

This chapter is a review of basic results concerning operators on a Hilbert space. The main reference for this chapter is the book of Reed-Simon [**RS**].

Throughout this chapter we let $\mathcal{H}$ be a separable Hilbert space and we denote by $\mathcal{L}(\mathcal{H})$ its $C^*$-algebra of continuous endomorphisms.

## 3.1. Polar Decomposition

DEFINITION 3.1.1. *An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be positive if can be written in the form $T = S^*S$ with $S \in \mathcal{L}(\mathcal{H})$.*

We denote by $\mathcal{L}(\mathcal{H})_+$ the set of positive elements of $\mathcal{L}(\mathcal{H})$.

PROPOSITION 3.1.2. *For $T \in \mathcal{L}(\mathcal{H})$ the following are equivalent:*

(i) *$T$ is positive.*
(ii) *There exists $S \in \mathcal{L}(\mathcal{H})$ selfadjoint such that $T = S^2$.*
(iii) *$T$ is selfadjoint and $\operatorname{Sp} T \subset [0, \infty)$.*
(iv) *$\langle T\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$.*

PROOF. We shall prove the implications (i) $\Rightarrow$ (iv), (ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (iii), which will give the proposition.

• <u>(i) $\Rightarrow$ (iv)</u>: If $T = S^*S$ for some $S \in \mathcal{L}(\mathcal{H})$ then, for all $\xi \in \mathcal{H}$,

$$\langle T\xi, \xi\rangle = \langle S^*S\xi, \xi\rangle = \langle S\xi, S\xi\rangle = \|S\xi\|^2 \geq 0,$$

proving that (i) $\Rightarrow$ (iv).

• <u>(ii) $\Rightarrow$ (i)</u>: If $T = S^2$ with $S \in \mathcal{L}(\mathcal{H})$ selfadjoint, then $T = S^*S$ and hence $T$ is positive. Thus (ii) $\Rightarrow$ (i).

• <u>(iii) $\Rightarrow$ (ii)</u>: Assume that $T$ is selfadjoint and $\operatorname{Sp} T \subset [0, \infty)$. Then the function $\sqrt{t}$ is defined and continuous on $\operatorname{Sp} T$, and so we can define $S := \sqrt{T}$ by functional continuous calculus. Notice that, as $\sqrt{t}$ takes real values on $\operatorname{Sp} T$, it follows from Remark 1.5.4 that $S$ is selfdajoint. Recall that the continuous functional calculus $f \to f(T)$ is a homomorphism of algebrs from $C(\operatorname{Sp} T)$ to $\mathcal{L}(\mathcal{H})$. Since $(\sqrt{t})^2 = t$ on $\operatorname{Sp} T$, it follows that $S^2 = (\sqrt{T})^2 = T$, and hence $T$ satisfies (ii). Thus (iii) $\Rightarrow$ (ii).

• <u>(iv) $\Rightarrow$ (iii)</u>: Assume that $\langle T\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$. Then $T$ is selfadjoint. Indeed, for all $\xi, \eta$ in $\mathcal{H}$, we have

$$4\langle T\xi, \eta\rangle = \langle T(\xi+\eta), \xi+\eta\rangle - \langle T(\xi-\eta), \xi-\eta\rangle - i\langle T(\xi+i\eta), \xi+i\eta\rangle + i\langle T(\xi-i\eta), \xi-i\eta\rangle,$$

from which we see that $\overline{\langle T\xi, \eta\rangle} = \langle T\eta, \xi\rangle$, and hence $\langle T\xi, \eta\rangle = \langle \xi, T\eta\rangle$.

Let us now show that $\operatorname{Sp} T \subset [0, \infty)$. Since $T$ is selfadjoint, and hence $\operatorname{Sp} T \subset \mathbb{R}$ by Proposition 1.3.4, we only need to show that $T - \lambda$ is invertible for all $\lambda < 0$.

Thus let $\lambda \in (-\infty, 0)$ and let $\xi \in \mathcal{H}$. As $\langle T\xi, \xi\rangle \geq 0$ we have

$$|\lambda|\|\xi\|^2 = -\lambda\langle \xi, \xi\rangle \leq \langle (T-\lambda)\xi, \xi\rangle \leq \|(T-\lambda)\xi\|\|\xi\|.$$

Thus,

$$\|(T-\lambda)\xi\| \leq |\lambda|\|\xi\| \qquad \forall \xi \in \mathcal{H}. \tag{3.1}$$

This implies that $\ker(T-\lambda) = \{0\}$. As $T - \lambda$ is selfadjoint we then see that $\overline{\operatorname{im}(T-\lambda)} = (\ker(T-\lambda))^{\perp} = \mathcal{H}$.

In fact, the inequality (3.1) also implies that $\operatorname{im}(T-\lambda)$ is closed. Indeed, if $\eta = \lim (T-\lambda)\xi_n$, then (3.1) implies that the sequence $(\xi_n)_{n\geq 0}$ is Cauchy in $\mathcal{H}$, and hence $\xi_n$ converges to some $\xi$ in $\mathcal{H}$. Then $\eta = \lim(T-\lambda)\xi_n = T\xi$, showing that $\eta$ is contained in $\operatorname{im}(T-\lambda)$. Thus $\ker(T-\lambda) = 0$ and $\operatorname{im}(T-\lambda) = \mathcal{H}$, i.e., $T - \lambda$ is bijective. Recall that by the open mapping theorem any bijective continuous linear map of $\mathcal{H}$ onto $\mathcal{H}$ has a continuous inverse (see [**Fo**, p. 162]), so $T - \lambda$ is an invertible element of $\mathcal{L}(\mathcal{H})$. Thus $\operatorname{Sp} T \subset [0, \infty)$, and hence $T$ satisfies (iii), showing that (iv) implies (iii). The proof is complete. □

Corollary 3.1.3. *The set of positive operators $\mathcal{L}(\mathcal{H})_+$ is a positive cone of $\mathcal{L}(\mathcal{H})$, i.e.,*

$$\lambda_1 T_1 + \lambda_2 T_2 \in \mathcal{L}(\mathcal{H})_+ \qquad \forall T_j \in \mathcal{L}(\mathcal{H})_+ \ \forall \lambda_j \geq 0.$$

Proof. For $j = 1, 2$ let $T_j \in \mathcal{L}(\mathcal{H})_+$ and let $\lambda_j \in [0, \infty)$. Using the characterization (iv) of Proposition 3.1.2, we see that, for all $\xi \in \mathcal{H}$,

$$\langle (\lambda_1 T_1 + \lambda_2 T_2)\xi, \xi\rangle = \lambda_1\langle T_1\xi, \xi\rangle + \lambda_2\langle T_2\xi, \xi\rangle \geq 0,$$

proving that $\lambda_1 T_1 + \lambda_2 T_2$ is a positive operator. □

Corollary 3.1.4. *Let $T \in \mathcal{L}(\mathcal{H})$ be normal and let $f \in C(\operatorname{Sp} T)$ be non-negative. Then the operator $f(T)$ is positive.*

Proof. Since $f$ is real-valued it follows from Remark 1.5.4 that $f(T)$ is selfadjoint. Moreover, by (1.13) we have $\operatorname{Sp} f(T) = f(\operatorname{Sp} T) \subset [0, \infty)$, so it follows from Proposition 3.1.2 (iii) that $f(T)$ is positive. □

Let $T \in \mathcal{L}(\mathcal{H})$. Then $T^*T$ is a positive operator, so by the previous proposition $T^*T$ is selfadjoint and its spectrum is contained in $[0, \infty)$. Therefore, by continuous functional calculus we can define its square root $\sqrt{T^*T}$ as an element of $\mathcal{L}(\mathcal{H})$. It follows from Corollary 3.1.4 that $|T|$ is a positive operator.

Definition 3.1.5. *For all $T \in \mathcal{L}(\mathcal{H})$ the operator $\sqrt{T^*T}$ is denoted $|T|$ and is called the modulus of $T$.*

Lemma 3.1.6. *Let $T \in \mathcal{L}(\mathcal{H})$. Then*

(i) *$|T|$ is the unique element of $\mathcal{L}(\mathcal{H})_+$ whose square is $T^*T$.*

(ii) *We have*

$$\||T|\xi\| = \|T\xi\| \qquad \forall \xi \in \mathcal{H}, \tag{3.2}$$

*and hence* $\ker|T| = \ker T$.

Proof. The continuous functional calculus $f \to f(T^*T)$ is a $*$-homomorphism from $C(\operatorname{Sp} T^*T)$ to $\mathcal{L}(\mathcal{H})$. As $(\sqrt{t})^2 = t$ on $[0, \infty)$ it follows that $|T|^2 = (\sqrt{T^*T})^2 = T^*T$.

Let $S \in \mathcal{L}(\mathcal{H})_+$ be such that $S^2 = T$. Since $S$ is positive, by Proposition 3.1.2 its spectrum is contained in $[0, \infty)$, and hence $\sqrt{t^2} = t$ on $\operatorname{Sp} S$. Therefore, by continuous functional calculus $S = \sqrt{S^2} = \sqrt{T^*T} = |T|$. Thus $|T|$ is the unique element of $\mathcal{L}(\mathcal{H})_+$ whose square is $T^*T$.

As $|T|$ is selfadjoint $|T|^*|T| = |T|^2 = T^*T$. Therefore, for all $\xi \in \mathcal{H}$,

$$\||T|\xi\|^2 = \langle |T|\xi, |T||\xi| \rangle = \langle |T|^*|T|\xi, \xi \rangle = \langle T^*T\xi, \xi \rangle = \langle T\xi, T\xi \rangle = \|T\xi\|^2,$$

proving (3.2). This immediately implies that $|T|$ and $T$ have same kernel. □

PROPOSITION 3.1.7 (Polar Decomposition). *Let $T \in \mathcal{L}(\mathcal{H})$. Then there exists a unique $U \in \mathcal{L}(\mathcal{H})$, called the phase of $T$, such that*

(i) $T = U|T|$;
(ii) $\ker U = \ker |T|$.

PROOF. Since $|T$ is selfadjoint $(\ker |T|)^\perp = \overline{\operatorname{im} |T|}$, and hence $|T|$ is a bijection from $\operatorname{im} |T|$ onto itself. Let $|T|^{-1} : \operatorname{im} |T| \to \operatorname{im} |T|$ be its inverse and denote by $U$ the linear map $T|T|^{-1} : \operatorname{im} |T| \to \mathcal{H}$. Then, for any $\xi \in \operatorname{im} |T|$, we have

$$\begin{aligned} \|U\xi\|^2 = \langle T|T|^{-1}\xi, T|T|^{-1}\xi \rangle = \langle T^*T|T|^{-1}\xi, |T|^{-1}\xi \rangle = \langle |T|^2|T|^{-1}\xi, |T|^{-1}\xi \rangle \\ = \langle |T|\xi, |T|^{-1}\xi \rangle = \langle \xi, |T||T|^{-1}\xi \rangle = \|\xi\|^2. \end{aligned}$$

Thus $U$ uniquely extends to an isometric linear map $U : \overline{\operatorname{im} |T|} \to \mathcal{H}$. Extending $U$ to be 0 on $\ker |T| = (\operatorname{im} |T|)^\perp$ we then get a continuous endomorphism $U : \mathcal{H} \to \mathcal{H}$ whose null space is $\ker |T|$.

If $\xi \in \operatorname{im} |T|$, then $U|T|\xi = T|T|^{-1}|T|\xi = \xi$, and hence $T = U|T|$ on $\overline{\operatorname{im} |T|}$ by continuity. Moreover, as by Lemma 3.1.6 $\ker |T| = \ker T$, we have $U|T|\xi = 0 = T\xi$ for all $\xi$ in $\ker |T| = (\operatorname{im} |T|)^\perp$. Therefore, we see that $T = U|T|$ on $\mathcal{H}$, showing that $U$ satisfies the conditions (i) and (ii) of the proposition.

Let $V \in \mathcal{L}(\mathcal{H})$ be such that $T = V|T|$ and $\ker V = \ker |T|$. If $\xi \in \ker |T|$, then obviously $V\xi = 0 = U\xi$. If $\xi \in \operatorname{im} |T|$, then $V\xi = V|T||T|^{-1}\xi = T|T|^{-1}\xi = U\xi$, so by continuity $V = U$ on $\overline{\operatorname{im} |T|}$. It follows from this that $V = U$ on $\mathcal{H}$, and hence $U$ is the unique element of $\mathcal{L}(\mathcal{H})$ such that $T = U|T|$ and $\ker U = \ker |T|$, giving the proposition. □

PROPOSITION 3.1.8. *Let $T \in \mathcal{L}(\mathcal{H})$ have polar decomposition $T = U|T|$ and denote by $\Pi_0(T)$ (resp. $\Pi_0(T^*)$) the orthogonal projection onto $\ker T$ (resp. $\ker T^*$).*

(i) *The range of $U$ is $\overline{\operatorname{im} T}$.*
(ii) *We have*

$$U^*U = 1 - \Pi_0(T) \qquad \text{and} \qquad UU^* = 1 - \Pi_0(T^*),$$

*so that $U$ is a partial isometry and has norm 1 unless $T = 0$.*

(iii) *If $T$ is injective and has dense range, then $U$ is unitary.*
(iv) *We have*

$$|T| = U^*T, \qquad T^* = UTU^*, \qquad |T^*| = TU^* = U|T|U^*.$$

(v) *The phase of $T^*$ is $U^*$.*

PROOF. As $U$ vanishes on $\ker |T|$ we see that $\operatorname{im} U = U(\ker |T|)$. Notice that by its construction in the proof of Proposition 3.1.7 the operator $U$ is isometric on $(\ker |T|)^\perp$ and agrees with $T|T|^{-1}$ on $\operatorname{im} |T|$. In particular, it follows from Lemma 1.1.8 that $U((\ker |T|)^\perp)$ is closed and $U$ induces a unitary operator from

$(\ker|T|)^{\perp}$ onto $U((\ker|T|)^{\perp}) = \operatorname{im} U$. As $(\ker|T|)^{\perp} = \overline{\operatorname{im} T}$ we then see that $\operatorname{im} U = \overline{U(\operatorname{im}|T|)}$. Since $U = T|T|^{-1}$ on $\operatorname{im}|T|$ we have $U(\operatorname{im}|T|) \subset \operatorname{im} T$, and hence $\operatorname{im} U \subset \overline{\operatorname{im} T}$. However, as $T = U|T|$ we also have $\operatorname{im} T \subset \operatorname{im} U$, and as $\operatorname{im} U$ is closed we see that $\overline{\operatorname{im} T}$ is contained is $\operatorname{im} U$, and so the range of $U$ is $\overline{\operatorname{im} T}$.

As abovementioned $U$ induces a unitary operator from $(\ker|T|)^{\perp}$ onto $\operatorname{im} U = \overline{\operatorname{im} T}$. Since by Lemma 3.1.6 $\ker|T| = \ker T$ this shows that $U$ is a unitary endomorphism of $\mathcal{H}$ when $T$ is injective and has dense range. In any case, as we have $\overline{\operatorname{im} T} = (\ker T^*)^{\perp}$ we see that $U$ induces a unitary operator from $(\ker T)^{\perp}$ onto $(\ker T^*)^{\perp}$. Since $\ker U = \ker|T| = \ker T$ we then deduce that $U^*U$ is the orthogonal projection onto $(\ker T)^{\perp}$ and $UU^*$ is the orthogonal projection onto $(\ker T)^{\perp}$, that is, $U^*U = 1 - \Pi_0(T)$ and $UU^* = 1 - \Pi_0(T^*)$. In particular, if $T \neq 0$ then $\|U\|^2 = \|U^*U\| = \|1 - \Pi_0(T)\| = 1$, i.e., $\|U\| = 1$.

Notice that, as $\ker|T| = \ker T$, we have $|T|(1 - \Pi_0(T)) = (1 - \Pi_0(T))|T| = |T|$, and hence $U^*T = U^*U|T| = (1 - \Pi_0(T))|T| = |T|$. Moreover,

$$(U|T|U^*)^2 = U|T|U^*U|T|U^* = U|T|(1 - \Pi_0(T))|T|U^* = (U|T|)(U|T|)^* = TT^*.$$

As $|T|$ is positive, for any $\xi \in \mathcal{H}$, we have $\langle U|T|U^*\xi, \xi\rangle = \langle |T|U^*\xi, U^*\xi\rangle \geq 0$, and hence $U|T|U^*$ is positive. Thus $U|T|U^*$ is a positive operator whose square is equal to $TT^*$, so using Lemma 3.1.6 we see that $U|T|U^* = |T^*|$. Since $T = U|T|$ this also shows that $|T^*| = TU^*$.

Notice that $\ker U^* = (\operatorname{im} U)^{\perp} = (\operatorname{im} T)^{\perp} = \ker T^* = \ker|T^*|$. Moreover,

$$U^*|T^*| = U^*U|T|U^* = (1 - \Pi_0(T))|T|U^* = |T|U^* = (U|T|)^* = T^*,$$

so by Proposition 3.1.7 the phase of $T^*$ is $U^*$. Notice that the above equalities include the equality $T^* = |T|U^*$. As $|T| = U^*T$ we see that $T^* = U^*TU^*$, completing the proof. □

### 3.2. Spectral Theorem and Borel Functional Calculus

Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator (i.e., $T^*T = TT^*$) and set $S = \operatorname{Sp} T$.

THEOREM 3.2.1 (Spectral Theorem; see [**RS**, Thm. VII.3]). *There exist a finite measure space $(X, \mu)$, a unitary operator $U : \mathcal{H} \to L^2_\mu(X)$, and a bounded measurable function $f$ on $X$, in such way that*

$$UTU^*\xi = f\xi \qquad \forall \xi \in L^2_\mu(X). \tag{3.3}$$

For $F \in \mathcal{L}^\infty(X)$ denote by $T_F$ the multiplication operator by $F$, i.e., the operator $T_F \in \mathcal{L}(L^2_\mu(X))$ defined by

$$T_F\xi = F\xi \qquad \xi \in L^2_\mu(X).$$

For instance, Eq. (3.3) says that $UTU^* = T_f$.

The *essential range* of $F$ consists of all $\lambda \in \mathbb{C}$ such that

$$\mu(\lambda - \epsilon < F < \lambda + \epsilon) > 0 \qquad \forall \epsilon > 0.$$

It can be shown that $\operatorname{Sp} T_F$ agrees with the essential range of $F$. In particular, we see that the essential range of $T_f$ is $S$. Thus, without any loss of generality, we may assume that the range of $f$ is $S$.

We endow $L^\infty(X)$ with its usual norm, i.e.,

$$\|F\|_{L^\infty(X)} = \{\lambda;\ \lambda \text{ in the essential range of } |F|\} \qquad \forall F \in L^\infty(\mathbb{R}).$$

We also endow $L^\infty(X)$ with the involution $F \to \overline{F}$ given by complex conjugation. This turns $L^\infty(X)$ into a commutative unital $C^*$-algebra. Then it is not difficult to check that the map $F \to T_F$ is a $*$-homomorphism from $L^\infty(X)$ to $\mathcal{L}(L^2_\mu(X))$. Moreover, as $T_F$ is a normal operator, we have

$$\|T_F\| = \sup_{\lambda \in \operatorname{Sp} T_F} |\lambda| = \|F\|_{L^\infty(S)}.$$

If $g \in C(S)$ then $g \circ f$ is again a bounded measurable function on $X$. In fact, the continuous functional calculus for $f$ is just $g \to g \circ f$. Since the maps $F \to T_F$ (resp., $F \to U^* T_F U$) is an isometric $*$-homomorphisms from $L^\infty(X)$ to $\mathcal{L}(L^2_\mu(X))$ (resp., $\mathcal{L}(\mathcal{H})$), it follows that, for all $g \in C(S)$,

$$g(T) = g(U^* T_f U) = U^* g(T_F) U = U^* T_{g \circ f} U. \tag{3.4}$$

We then can extend the definition of $g(T)$ for any any bounded Borel function $g$ on $\operatorname{Sp} T$ by letting

$$g(T) := U^* T_{g\circ f} U.$$

This defines a bounded operator on $\mathcal{H}$.

Theorem 3.2.2 (Borel Functional Calculus; see [**RS**, Thm. VII.2]). *The following holds.*

(1) *The map $g \to g(T)$ is a $*$-homomorphism from $L^\infty(S)$ to $\mathcal{L}(\mathcal{H})$ such that*

$$\|g(T)\| \leq \|g\|_{L^\infty(S)} \qquad \forall g \in L^\infty(S). \tag{3.5}$$

(2) *If $(g_n)_{n\geq 0}$ is a bounded sequence in $L^\infty(S)$ such that $g_n \to g$ a.e., then $g_n(T) \to g(T)$ strongly (i.e., $g_n(T)\xi \to g(T)\xi$ for all $\xi \in \mathcal{H}$).*

(3) *If $g \in L^\infty(S)$ is real-valued (resp., non-negative), then $g(T)$ is selfadjoint (resp., positive).*

### 3.3. Unbounded Operators

The spectral theorem and the Borel functional calculus can be extended to unbounded operators as follows.

Definition 3.3.1. *An (unbounded) operator on $\mathcal{H}$ is a linear operator $T : D(T) \to \mathcal{H}$, where the domain $D(T)$ is a subspace of $\mathcal{H}$. $T$*

An operator $T$ on $\mathcal{H}$ is said to be *densily defined* when its domain $D(T)$ is a dense subspace of $\mathcal{H}$. An operator $S$ is said to be an extension of $T$, and we write $T \subset S$, if $D(T) \subset D(S)$ and $S$ agrees with $T$ on $D(T)$.

The *graph* of an operator $T$ is defined to be

$$G(T) = \{(\xi, \eta) \in \mathcal{H} \oplus \mathcal{H};\ \eta = T\xi\}.$$

The graph of $T$ is a subspace of $\mathcal{H} \oplus \mathcal{H}$. We say that $T$ is *closed* when $G(T)$ is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$. We say that $T^*$ is *closable* when $\overline{G(T)}$ is the graph of an operator $\overline{T}$. In this case we call $\overline{T}$ the *closure* of $T$. This is the smallest closed extension of $T$.

If $T$ is densily defined, then its *adjoint* is the operator $T^*$ with graph

$$G(T^*) = \{(\xi, \eta) \in \mathcal{H} \oplus \mathcal{H};\ \langle T\zeta, \xi\rangle = \langle \zeta, \eta\rangle\ \forall \zeta \in D(T)\}.$$

Since $G(T^*)$ is a closed subspace, we see that $T^*$ is always a closed operator.

The *resolvent set* of a closed operator $T$ consists of all $\lambda \in \mathbb{C}$ such that

(i) $T - \lambda : D(T) \to \mathcal{H}$ is a bijection.

(ii) The inverse $(T-\lambda)^{-1} : \mathcal{H} \to D(T)$ is bounded.

The *spectrum* of $T$, denoted $\operatorname{Sp} T$, is the complement of the resolvent set.

It can be shown that $\operatorname{Sp} T$ is a closed subset of $\mathbb{C}$ (which may be empty) and the resolvent $\lambda \to (T-\lambda)^{-1}$ is analytic from $\mathbb{C} \setminus \operatorname{Sp} T$ to $\mathcal{L}(\mathcal{H})$ (provided we regard the inverses $(T-\lambda)^{-1}$ as elements of $\mathcal{L}(\mathcal{H})$).

A densily defined operator $T$ is said to be *symmetric* is $T \subset T^*$. It is said to be *selfadjoint* if $T = T^*$. Note that any symmetric operator is closable. We also say that $T$ is *essentially selfadjoint* when $T$ is symmetric and closable.

PROPOSITION 3.3.2 ([**RS**, Thm. VIII.3]). *Let $T$ be a symmetric operator on $\mathcal{H}$. Then the following are equivalent:*

(i) *$T$ is selfadjoint.*
(ii) *$T$ is closed and* $\ker(T \pm i) = \{0\}$.
(iii) $\operatorname{im}(T \pm i) = \mathcal{H}$.

COROLLARY 3.3.3 (see [**RS**]). *Let $T$ be a symmetric operator on $\mathcal{H}$. Then the following are equivalent:*

(i) *$T$ is essentially selfadjoint.*
(ii) $\ker(T \pm i) = \{0\}$.
(iii) $\operatorname{im}(T \pm i) = \mathcal{H}$.

Let $T$ be a selfadjoint (unbounded) operator on $\mathcal{H}$.

THEOREM 3.3.4 (Spectral Theorem; see [**RS**, Thm. VIII.4]). *There exist a measured space $(X, \mu)$ with $\mu(X) < \infty$, a unitary operator $U : \mathcal{H} \to L^2_\mu(X)$, and a measurable real-valued function $f$ on $X$ such that*

$$U\left(D(T)\right) = \{\xi \in L^2_\mu(X);\ f\xi \in L^2_\mu(X)\},$$
$$UTU^*\xi = f\xi \qquad \forall \xi \in U\left(D(T)\right).$$

If $g$ is a bounded Borel function on $\mathbb{R}$, then we define $g(T)$ as the bounded operator on $\mathcal{L}(\mathcal{H})$ given by

$$g(T) := U^* T_{g\circ f} U.$$

THEOREM 3.3.5 (Borel Functional Calculus; [**RS**, Thm. VIII.5]). *The following holds.*

(1) *The map $g \to g(T)$ is a $*$-homomorphism from $L^\infty(\mathbb{R})$ to $\mathcal{L}(\mathcal{H})$ such that*

$$\|g(T)\| \leq \|g\|_{L^\infty(\mathbb{R})} \qquad \forall g \in L^\infty(\mathbb{R}). \tag{3.6}$$

(2) *If $(g_n)_{n\geq 0}$ is a bounded sequence in $L^\infty(\mathbb{R})$ such that $g_n \to g$ a.e., then $g_n(T) \to g(T)$ strongly.*
(3) *If $g \in L^\infty(\mathbb{R})$ is real-valued (resp., non-negative), then $g(T)$ is selfadjoint (resp., positive).*

More generally, if $g$ is a possibly unbounded Borel function on $\mathbb{R}$, then $g(T)$ makes sense as an unbounded operator as follows. The domain of $g(T)$ is

$$D(g(T)) = U^*\Big(\{\xi \in L^2_\mu(X);\ (g\circ f)\xi \in L^2_\mu(X)\}\Big),$$

and we define $g(T)$ by the formula

$$g(T)\xi := U^*\left((g\circ f)U\xi\right) \qquad \forall \xi \in D(g(T)).$$

For instance, if $g(t)=t$, then $g(T)=T$. In addition, it is not hard to check that, if $(g_n)$ be a sequence of Borel functions on $\mathbb{R}$ such that $g_n\rightarrow g$ and $|g_n|\leq |g|$, then as $n\rightarrow\infty$ we have

$$g_n(T)\xi \longrightarrow g(T)\xi \qquad \forall \xi\in D(g(T)).$$

EXAMPLE 3.3.6. Let $\Delta=-(\partial_{x_1}^2+\ldots+\partial_{x_n}^2)$ be the (positive) Laplacian on $\mathbb{R}^n$. We shall regard $\Delta$ as an operator on the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions on $\mathbb{R}^n$. Denoting by $u\rightarrow \hat{u}$ the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$, we have

$$(\Delta u)^{\wedge}=|\xi|^2\hat{u} \qquad \forall u\in\mathcal{S}'(\mathbb{R}^n). \tag{3.7}$$

Thus, we can regard $\Delta$ as an unbounded operator on the Hilbert space $\mathcal{H}=L^2(\mathbb{R}^n)$ with domain,

$$D(\Delta)=\{u\in L^2(\mathbb{R}^n);\ |\xi|^2\hat{u}\in L^2(\mathbb{R}^n)\}.$$

Notice that $D(\Delta)$ agrees with the Sobolev space $W^{2,2}(\mathbb{R}^n)$. As such $\Delta$ is a selfadjoint operator with spectrum $[0,\infty)$.

Denote by $U$ the unitary operator on $L^2(\mathbb{R}^n)$ defined by

$$Uv=(2\pi)^{-\frac{n}{2}}\hat{v} \qquad \forall v\in L^2(\mathbb{R}^n).$$

Then it follows from (3.7) that

$$\Delta v=U^*T_{|\xi|^2}Uv \qquad \forall v\in W^{2,2}(\mathbb{R}^n).$$

Therefore, we see that in this example the spectral theorem follows from elementary Fourier-analytic considerations.

If $g$ is a bounded Borel function on $\mathbb{R}$, then $g(\Delta)$ is given by

$$g(\Delta)v(x)=(U^*T_{g(|\xi|^2)}Uv)(x)=(2\pi)^{-n}\int e^{ix.\xi}g(|\xi|^2)\hat{u}(\xi)d\xi \quad \forall v\in L^2(\mathbb{R}^n).$$

The following operators are of special interest:

- The heat semigroup $e^{-t\Delta}$, $t\geq 0$.
- The wave group $e^{it\Delta}$, $t\in\mathbb{R}$.
- The complex powers $\Delta^z$, $\Re z\geq 0$.

These operators can be similarly defined by Borel functional calculus for any selfadjoint unbounded operators with nonnegative spectrum.

## 3.4. Compact Operators

In the sequel for any $r>0$ we denote by $B(0,r)$ the (open) unit ball of $\mathcal{H}$ of radius $r$ about the origin.

DEFINITION 3.4.1. *An operator $T\in\mathcal{L}(\mathcal{H})$ is said to be compact when $\overline{T(B(0,1)}$ is compact in $\mathcal{H}$.*

We denote by $\mathcal{K}$ the space of compact operators.

The following lemma will be useful to study compact operators.

PROPOSITION 3.4.2. *Let $T\in\mathcal{L}(\mathcal{H})$. Then the following are equivalent:*

(i) *$T$ is a compact operator.*
(ii) *For any bounded sequence $(\xi_n)_{n\geq 0}\subset\mathcal{H}$ there is a subsequence $(\xi_{n_k})_{k\geq 0}$ such that the sequence $(T\xi_{n_k})_{k\geq 0}$ converges in norm.*
(iii) *For any sequence $(\xi_n)_{n\geq 0}\subset\mathcal{H}$ converging weakly to $0$ the sequence $(T\xi_n)_{n\geq 0}$ converges to $0$ in norm.*

(iv) *There is an orthonormal basis* $(\xi_n)_{n\geq 0}$ *of* $\mathcal{H}$ *such that*

$$\lim_{N\to\infty} \|T_{|E_N^\perp}\| = 0,$$

*where we have denoted by* $E_N$ *the span of* $\xi_0, \ldots, \xi_{N-1}$.

(v) *For any orthonormal basis* $(\xi_n)_{n\geq 0}$ *of* $\mathcal{H}$,

$$\lim_{N\to\infty} \|T_{|E_N^\perp}\| = 0,$$

*where we have denoted by* $E_N$ *the span of* $\xi_0, \ldots, \xi_{N-1}$.

Proof. Taking into account that the implication (v) $\Rightarrow$ (iv) is immediate, to prove the proposition we only need to prove the implications (ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (v).

• <u>(ii) $\Rightarrow$ (i)</u>: Assume that $T$ sastifies the condition (ii). Then for any sequence $(\xi_n)_{n\geq 0} \subset B(0,1)$ the sequence $(T\xi_n)_{n\geq 0}$ admits a convergent subsequence. By virtue of the Bolzano-Weierstrass criterion this proves that $\overline{T(B(0,1))}$ is compact, i.e., $T$ is a compact operator. Thus (ii) implies (i).

• <u>(iii) $\Rightarrow$ (ii)</u>: Suppose that $T$ satisfies the condition (iii). Let $(\xi_n)_{n\geq 0} \subset \mathcal{H}$ be a bounded sequence, i.e., there exists $r > 0$ such that $\xi_n \in B(0,r)$ for all $n$. By Alaoglu theorem (see [**Fo**, pp. 169-170]) the ball $B(0,r)$ is precompact with respect to the weak topology and, as $\mathcal{H}$ is separable, the weak topology is metrizable (see [**Fo**]), so by the Bolzano-Weierstrass criterion there is a subsequence $(\xi_{n_k})_{k\geq 0}$ converging weakly to some $\xi$ in $\mathcal{H}$. Then $\xi_{n_k} - \xi$ converges weakly to 0, so by (iii) the sequence $T(\xi_{n_k} - \xi)$ converges to 0 in norm, i.e., $T\xi_{n_k} \to T\xi$ in norm. This shows that (iii) implies (ii).

• <u>(iv) $\Rightarrow$ (iii)</u>: Suppose that $T$ satisfies (iv). Thus there exists an orthonormal basis $(\xi_n)_{n\geq 0}$ of $\mathcal{H}$ such that if, for any $N \in \mathbb{N}$, we denote by $E_N$ the span of $\xi_0, \ldots, \xi_{N-1}$ then $\|T_{|E_N^\perp} \to 0$ as $N \to \infty$. In addition, we denote by $\Pi_N$ the orthogonal projection onto $E_N$.

Let $(\eta_k)_{k\geq 0} \subset \mathcal{H}$ be a sequence converging weakly to 0. In particular $(\eta_k)_{k\geq 0}$ is weakly bounded, and hence is bounded in norm by the uniform boundedness principle and the fact $\mathcal{H}$ is isometrically isomorphic to its dual (see [**Fo**, pp. 163, 174–175 ]). Thus there exists $C > 0$ such that $\|\eta_k\| \leq C$ for all $k \in \mathbb{N}_0$.

Let $\epsilon > 0$. Since $\Pi_N \eta_k = \sum_{n<N} \langle \xi_n, \eta_k\rangle \xi_n$ we have

(3.8)
$$\begin{aligned}\|T\eta_k\| \leq \|T\Pi_N\eta_k\| + \|T(1-\Pi_N)\eta_k\| &\leq \sum_{n<N} |\langle \xi_n, \eta_k\rangle| \|T\xi_n\| + \|T(1-\Pi_N)\|\|\eta_k\| \\ &\leq \|T\| \sum_{n<N} |\langle \xi_n, \eta_k\rangle| + C\|T_{|E_N^\perp}\|.\end{aligned}$$

Since $\|T_{|E_N^\perp}\| \to 0$ as $N \to \infty$ by choosing $N$ large enough we have

$$\|T(1-\Pi_N)\| < \epsilon. \tag{3.9}$$

As $\eta_k$ converges weakly to 0 we see that $\sum_{n<N} |\langle \xi_n, \eta_k\rangle|$ goes to 0 as $k \to \infty$, and hence there exists $k_0 \in \mathbb{N}_0$ such that, for any $k \geq k_0$,

$$\sum_{n<N} |\langle \xi_n, \eta_k\rangle| < \epsilon. \tag{3.10}$$

Combining (3.8), (3.9) and (3.10) we see that, for all $k \geq k_0$, we have

$$\|T\eta_k\| \leq (\nu_0 + C)\epsilon.$$

This shows that $T\eta_k \to 0$ in norm as $k \to \infty$. Thus (iv) implies (iii).

• (i) ⇒ (v): Suppose that $T$ is a compact operator. Let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$. For any $N \in \mathbb{N}$ we denote by $E_N$ the span of $\xi_0, \ldots, \xi_{N-1}$. Assume that the sequence $(\|T_{|E_N^\perp}\|)_{N\geq 1}$ does not converge to 0 as $N \to 0$. Since this is a non-increasing sequence of non-negative numbers there is $c > 0$ such that $\|T_{|E_N^\perp}\| > c$ for all $N \in \mathbb{N}$. Therefore, for every $N \in \mathbb{N}$, there is a unit vector $\eta_N \in E_N^\perp$ such that $\|T\eta_N\| > c$.

Let $\xi \in \mathcal{H}$. As $\eta_N$ is contained in $E_N^\perp$, and hence $\eta_N = \sum_{n\geq N} \langle \xi_n, \eta_N\rangle \xi_n$, we have

$$|\langle \eta_n, \xi\rangle| \leq \sum_{n\geq N} |\langle \eta_N, \xi_n\rangle\langle \xi, \xi_n\rangle| \leq \left(\sum_{n\geq N} |\langle \eta_N, \xi_n\rangle|^2\right)^2 \left(\sum_{n\geq N} |\langle \xi, \xi_n\rangle|^2\right)^2.$$

Since $\sum_{n\geq N} |\langle \eta_N, \xi_n\rangle|^2 = \|\eta_N\|^2 = 1$ and $\sum_{n\geq N} |\langle \xi, \xi_n\rangle|^2 \to 0$ as $N \to \infty$, it follows that $\langle \eta_n, \xi\rangle \to 0$ as $N \to \infty$. Thus $\eta_N$ converges weakly to 0 as $N \to \infty$. As $T$ is continuous with respect to the weak topology it follows that $T\eta_N$ converges weakly to 0 as $N \to \infty$.

On the other hand, the sequence $(T\eta_N)_{n\geq 1}$ is contained in the image by $T$ of the unit sphere, which is precompact since $T$ is compact. Therefore, by the Bolzano-Weierstrass criterion there is a subsequence $(\eta_{N_k})_{k\geq 0}$ such that $T\eta_{N_k}$ converges in norm to some $\zeta$ as $k \to \infty$. As $T\eta_{N_k}$ converges weakly to 0 we must have $\zeta = 0$, i.e., $T\eta_{N_k}$ converges to 0 in norm. This contradicts the fact that $\|T\eta_N\| > c$ for all $N \in \mathbb{N}$. Therefore, it is not possible for the sequence $(\|T_{|E_N^\perp}\|)_{N\geq 1}$ to not converge to 0. This proves that $\|T_{|E_N^\perp}\| \to 0$ as $N \to \infty$. Thus (i) implies (v). The proof is complete. □

PROPOSITION 3.4.3. *$\mathcal{K}$ is a closed two-sided ideal of $\mathcal{L}(\mathcal{H})$.*

PROOF. Let $(\xi_n)_{n\geq 0}$ be any sequence converging weakly to 0.

For $j = 1, 2$ let $T_j \in \mathcal{K}$ and $\lambda_j \in \mathbb{C}$. By Proposition 3.4.2 (iii) the sequences $(T_1\xi_n)_{n\geq 0}$ and $(T_2\xi_n)_N$ converge to 0 in norm, and so $(\lambda_1 T_1 + \lambda_2 T_2)\xi_n)_{n\geq 0}$ too converges to 0 in norm. It then follows from Proposition 3.4.2 (iii) that $\lambda_1 T_1 + \lambda_2 T_2$ is a compact operator. Thus $\mathcal{K}$ is a subspace of $\mathcal{L}(\mathcal{H})$.

Let $T \in \mathcal{K}$ and let $A, B \in \mathcal{L}(\mathcal{H})$. Since $B$ is continuous with respect to the weak topology the sequence $(B_{\xi_n})_{n\geq 0}$ converges weakly to 0. Since $T$ is compact Proposition 3.4.2 (iii) insures us that $(TB\xi_n)_{n\geq 0}$ converges to 0 in norm. Then $(ATB\xi_n)_{n\geq 0}$ converges to 0 in norm too. Thanks to Proposition 3.4.2 (iii), this shows that $ATB$ is a compact operator. Therefore, we see that $\mathcal{K}$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$.

It remains to show that $\mathcal{K}$ is closed. Thus, let $(T_k)_{k\geq 0} \subset \mathcal{K}$ be a sequence such that $T_k \to T$ in $\mathcal{L}(\mathcal{H})$ and let us show that $T$ is compact. Let $\epsilon > 0$. Then for $k$ large enough $\|T - T_k\| < \epsilon$. Since $T_k$ is compact by Proposition 3.4.2 (iii) the sequence $(T_k\xi_n)_{n\geq 0}$ converges to 0 in norm, and hence there exists $N \in \mathbb{N}$, such that $\|T_k\xi_n\| < \epsilon$ for all $n \geq N$. Then, for any $n \geq N$, we have

$$\|T\xi_n\| \leq \|T_k\xi_n\| + \|(T - T_k)\xi_n\| \leq \|T_k\xi_n\| + \|T - T_k\|\|\xi_n\| < 2\epsilon.$$

Thus $(T\xi_n)_{n\geq 0}$ converges to 0 in norm, which by Proposition 3.4.2 (iii) shows that $T$ is compact. This proves that $\mathcal{K}$ is closed, completing the proof. $\square$

PROPOSITION 3.4.4. *For any $T \in \mathcal{L}(\mathcal{H})$,*

$$T \in \mathcal{K} \Longleftrightarrow T^* \in \mathcal{K} \Longleftrightarrow |T| \in \mathcal{K}. \tag{3.11}$$

PROOF. Let $T \in \mathcal{L}(\mathcal{H})$ and let $T = U|T|$ be its polar decomposition. The fact that $\mathcal{K}$ is a two-sided ideal then implies that if $|T|$ is compact, then so is $T$. Likewise, as by Proposition 3.1.8 we have $|T| = U^*T$, if $T$ is compact, then so is $|T|$.

Proposition 3.1.8 also tells us that $T^* = UTU^*$. Therefore, we also see that if $T$ is compact then so is $T^*$. Upon substituting $T^*$ for $T$, we see that if $T^*$ is compact, then $T$ is compact too. The proof is complete. $\square$

PROPOSITION 3.4.5. *$\mathcal{K}$ is a sub-$C^*$-algebra of $\mathcal{L}(\mathcal{H})$, and hence is a $C^*$-algebra.*

PROOF. Since Proposition 3.4.3 tells us that $\mathcal{K}$ is a closed two-sided ideal, we see that $\mathcal{K}$ is a closed subalgebra of $\mathcal{L}(\mathcal{H})$. As by Proposition 3.4.4 $\mathcal{K}$ is closed under the involution of $\mathcal{L}(\mathcal{H})$ it follows that $\mathcal{K}$ is a sub-$C^*$-algebra of $\mathcal{L}(\mathcal{H})$. $\square$

THEOREM 3.4.6 (Riesz-Schauder; see [**RS**, Thm. VI.15]). *Let $T \in \mathcal{K}$.*

(1) *$T$ always contains 0 in its spectrum.*
(2) *If $\lambda \in \operatorname{Sp} T \setminus 0$, then $\lambda$ is an eigenvalue with finite multiplicity.*
(3) *$\operatorname{Sp} T$ is either finite or consists of a sequence of complex numbers converging to 0.*

In the sequel, for any vectors $\xi$ and $\eta$ in $\mathcal{H}$, we denote by $\xi \otimes \eta^*$ the element of $\mathcal{L}(\mathcal{H})$ defined by

$$(\xi \otimes \eta^*)\zeta := \langle \eta, \xi' \rangle \xi \qquad \forall \zeta \in \mathcal{H}. \tag{3.12}$$

Thus in ketbra notation $\xi \otimes \eta^*$ is just the operator $|\xi\rangle\langle\eta|$. This is an operator of rank 1. If $\xi$ is a unit vector then $\xi \otimes \xi^*$ is the orthogonal projection onto $\mathbb{C}\xi$.

THEOREM 3.4.7 (Hilbert-Schmidt; see [**RS**, Thm. VI.16]). *Let $T \in \mathcal{K}$ be normal. Then $T$ diagonalizes in an orthonormal basis, i.e., there exists an orthonormal basis $(\xi_n)_{n\geq 0}$ of $\mathcal{H}$ and a sequence $(\lambda_n)_{n\geq 0} \subset \mathbb{C}$ such that*

$$T\lambda_n = \lambda_n \xi_n \qquad \forall n \in \mathbb{N}_0. \tag{3.13}$$

This result allows us to reinterpret the Borel functional calculus for normal compact operators as follows.

PROPOSITION 3.4.8. *Let $T \in \mathcal{K}$ be normal and let $(\xi_n)_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$ respect to which $T$ is diagonal, i.e., $T\lambda_n = \lambda_n \xi_n$ for all $n \in \mathbb{N}_0$. In addition, let $f$ be bounded function on $\operatorname{Sp} T = \{\lambda_n;\ n \in \mathbb{N}_0\}$.*

(1) *We have*

$$f(T) = \sum_{n\geq 0} f(\lambda_n)(\xi_n \otimes \xi_n^*), \tag{3.14}$$

*where the series converges strongly.*

(2) *$f(T)$ is a compact operator if and only if*

$$\lim_{n\to\infty} f(\lambda_n) = 0.$$

*Furthermore, in this case the series (3.14) converges in norm.*

PROOF. For any $n \in \mathbb{N}_0$ we have $T\xi_n = \lambda_n \xi_n$. Since $(\xi_n)_{n\geq 0}$ is an orthonormal basis we deduce that $T^*\xi_n = \overline{\lambda_n}\xi_n$ for all $n \in \mathbb{N}_0$. Let $n \in \mathbb{N}_0$. For any polynomial $f(z) = \sum a_{jk} z^j \overline{z}^k$, we have

$$f(T) = \sum a_{jk} T^j (T^*)^k \xi_n = \sum a_{jk} \lambda_n^j (\overline{\lambda_n})^k \xi_n = f(\lambda_n)\xi_n.$$

It then follows from (1.14) that, for any $f \in C(\operatorname{Sp} T)$, we have

$$f(T)\xi_n = \lambda_n \xi_n \qquad \forall n \in \mathbb{N}_0. \tag{3.15}$$

Since $\operatorname{Sp} T = \{\lambda_n; n \in \mathbb{N}_0\}$ any function on $\operatorname{Sp} T$ is a Borel function. Moreover, by Theorem 3.4.6 0 is always in $\operatorname{Sp} T$ and the non-zero eigenvalues of $T$ have finite multiplicities, i.e., each non-zero eigenvalue $\lambda$ appears at most finitely many times in the sequence $(\lambda_n)_{n\geq 0}$. Thus $\lambda_n \to 0$ as $n \to \infty$, and hence a function $f(\lambda)$ on $\operatorname{Sp} T$ is continuous iff $\lim_{n\to\infty} f(\lambda_n) = f(0)$. Let $f$ be a bounded function on $\operatorname{Sp} T$ and, for $N \in \mathbb{N}$, let $f_N$ be the function on $\operatorname{Sp} T$ defined by

$$f_N(\lambda) = \begin{cases} f(\lambda_n) & \text{if } \lambda = \lambda_n \text{ with } n < N, \\ f(0) & \text{if } \lambda = 0 \text{ or } \lambda \neq \lambda_n \text{ for all } n < N. \end{cases}$$

As $\lim_{n\to\infty} f_N(\lambda_n) = f(0) = f_N(0)$, we see that $f_N$ is a continuous function on $\operatorname{Sp} T$, and hence it satisfies (3.15). Moreover, for all $n \in \mathbb{N}_0$,

$$|f_N(\lambda_n)| \leq \|f\|_\infty \qquad \text{and} \qquad \lim_{N\to\infty} f_N(\lambda_n) = f(\lambda_n).$$

Thus $(f_N)_{N\geq 1}$ is a bounded sequence in $L^\infty(\operatorname{Sp} T)$ which converges pointwise to $f$. It then follows from Theorem 3.2.2 and (3.15) that, for all $n \in \mathbb{N}_0$,

$$f(T)\xi_n = \lim_{N\to\infty} f_N(T)\xi_n = \lim_{N\to\infty} f_N(\lambda_n)\xi_n = f(\lambda_n)\xi_n. \tag{3.16}$$

Let $\xi \in \mathcal{H}$. Since $\xi = \sum_{n\geq 0} \langle \xi_n, \xi\rangle \xi_n$ and $f(T)$ is continuous, using (3.16) we get

$$f(T)\xi = \sum_{n\geq 0} \langle \xi_n, \xi\rangle f(\xi_n) = \sum_{n\geq 0} \langle \xi_n, \xi\rangle f(\lambda_n)\xi_n = \sum_{n\geq 0} f(\lambda_n)(\xi_n \otimes \xi_n^*)\xi,$$

which proves (3.14).

Next, for $N \in \mathbb{N}$ let $E_N$ the span of $\xi_0, \ldots, \xi_{N_1}$ and denote by $\Pi_N$ be the orthogonal projection onto $E_N$. Then by Proposition 3.4.2 (iv) the operator $f(T)$ is compact if and only if

$$\|f(T)_{|E_N^\perp}\| = \|f(T)(1-\Pi_N)\| \longrightarrow 0 \qquad \text{as } N \longrightarrow 0. \tag{3.17}$$

Set $\nu_N = \sup_{n\geq N} |f(\lambda_n)|$. Since $f(T)\xi_n = f(\lambda_n)\xi_n$ for all $n \in \mathbb{N}_0$, we see that $\nu_N \leq \|f(T)_{|E_N^\perp}\| = \|f(T)(1-\Pi_N)\|$. Conversely, let $\xi \in \mathcal{H}$. Since $(\xi_n)_{n\geq 0}$ is an orthonormal basis, from (3.14) we get

$$\|f(T)(1-\Pi_N)\xi\|^2 = \sum_{n\geq N} |\langle \xi_n, \xi\rangle|^2 |f(\lambda_n)|^2 \leq \nu_N^2 \sum_{n\geq 0} |\langle \xi_n, \xi\rangle|^2 = \nu_N^2 \|\xi\|^2.$$

Thus $\|f(T)(1-\Pi_N)\| \leq \nu_N$. Thefore, we have

$$\|f(T)_{|E_N^\perp}\| = \|f(T)(1-\Pi_N)\| = \sup_{n\geq N} |f(\lambda_n)|. \tag{3.18}$$

Combining this with the condition (3.17) then shows that $f(T)$ is compact if and only if $\lim_{n\to\infty} f(\lambda_n) = 0$.

Finally, using (3.14) we get

$$f(T)(1 - \Pi_N) = \sum_{n \geq N} f(\lambda_n)(\xi_n \otimes \xi_n^*),$$

where the series converges strongly. Thus,

$$\left\| \sum_{n \geq N} f(\lambda_n)(\xi_n \otimes \xi_n^*) \right\| = \|f(T)(1 - \Pi_N)\| = \sup_{n \geq N} |f(\lambda_n)|.$$

Therefore, if $\lim_{n\to\infty} f(\lambda_n) = 0$, then the series (3.14) converges to $f(T)$ in norm. The proof is complete. □

EXAMPLE 3.4.9. Let $T \in \mathcal{K}$ be normal. Then, with the notation of Proposition 3.4.8, we have

$$T = \sum_{\lambda_n \neq 0} \lambda_n(\xi_n \otimes \xi_n^*), \tag{3.19}$$

where the series converges in norm. Moreover, if we let $T = U|T|$ be the polar decomposition, then

$$U = \sum_{\lambda_n \neq 0} |\lambda_n|^{-1} \lambda_n(\xi_n \otimes \xi_n^*) \qquad \text{and} \qquad |T| = \sum_{n \geq 0} |\lambda_n|(\xi_n \otimes \xi_n^*),$$

where the first series converges strongly (unless $T$ has finite rank, in which case it is a finite sum), and the second series converges in norm.

## Bibliography


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