Noncommutative Geometry
Chapter 5:
Connes' Quantized Calculus

Sichuan University, Spring 2025

Additional References

- Lord, S.; Sukochev, F.; Zanin, D.: *Singular traces: theory and applications*. De Gruyter, 2012.
- Ponge, R.: *Connes' integration and Weyl's laws*. J. Noncomm. Geom. **17** (2023), 719–767.

Setup

Throughout this chapter $\mathcal{H}$ is a separable Hilbert space.

Quantized Calculus (Connes)

Classical	**Quantum** (Connes)
Complex variable	Operator on Hilbert space $\mathcal{H}$
Real variable	Selfadjoint operator on $\mathcal{H}$
Infinitesimal variable	Compact operator on $\mathcal{H}$
Infinitesimal of order α	Compact operator s.t. $\mu_j(T) = \mathrm{O}(j^{-\alpha})$
Integral $\int f(x)dx$	NC integral $⨍ T$

Here the $\mu_j(A)$ are the singular values of A.

Infinitesimal Operators

Intuitive Definition

An infinitesimal is an object that it is smaller than any positive number.

Remark

For an operator $T \in \mathcal{L}(\mathcal{H})$ the condition

$$\|T\| < \epsilon \quad \text{for all } \epsilon > 0$$

gives the solution $T = 0$!

Definition (Infinitesimal Operator)

An operator $T \in \mathcal{L}(\mathcal{H})$ is infinitesimal if, for all $\epsilon > 0$, there is a subspace $E \subset \mathcal{H}$, $\dim E < \infty$, such that

$$\left\| T_{|E^\perp} \right\| < \epsilon.$$

Infinitesimal Operators

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$. *Then TFAE*

1. T *is a compact operator.*
2. T *is the norm-limit of finite rank operators.*
3. $\mu_j(T) \to 0$ *as* $j \to \infty$.
4. *For all* $\epsilon > 0$, *there is* $E \subset \mathcal{H}$, $\dim E < \infty$, *s.t.* $\|T_{|E^\perp}\| < \epsilon$.

Consequence

An operator T is an infinitesimal if and only if it is compact.

Infinitesimal Operators

Definition

A (compact) operator T is an infinitesimal of order α, $\alpha > 0$, if

$$\mu_j(T) = \mathrm{O}(j^{-\alpha}) \quad \text{as } j \to \infty.$$

Remark

In other words, T is an infinitesimal of order α if and only if it belongs to the weak Schatten class $\mathcal{L}_{p,\infty}$ with $p = \alpha^{-1}$.

From the properties of weak Schatten classes we get:

Proposition

For $j = 1, 2$, let T_j be infinitesimal of order α_j. Then

1. $T_1 + T_2$ *is infinitesimal of order* $\min(\alpha_1, \alpha_2)$.
2. $T_1 T_2$ *is infinitesimal of order* $\alpha_1 + \alpha_2$.

NC Integral - Ansatz

Ansatz for a NC Integral (Connes)

The NC integral should have the following properties:

1. It is defined on infinitesimals of order 1, i.e., on the weak trace class $\mathcal{L}^{1,\infty}$.
2. It should take non-negative values on positive operators.
3. It vanishes on infinitesimals of order > 1.
4. It vanishes on the commutator space,

$$\mathrm{Com}(\mathcal{L}^{1,\infty}) = \mathrm{Span}\left\{[A, T];\ A \in \mathcal{L}(\mathcal{H}),\ T \in \mathcal{L}^{1,\infty}\right\}.$$

That is, it should be a positive trace on $\mathcal{L}^{1,\infty}$.

Eigenvalue Sequences

Setup

$T =$ compact operator on $\mathcal{H}$.

Definition

Let $\lambda \in \mathrm{Sp}(T) \setminus 0$.

1. The root space relatively to λ is
$$E_\lambda(T) = \bigcup_{\ell \geq 1} \ker(T - \lambda)^\ell.$$
2. $\dim E_\lambda(T)$ is called the algebraic multiplicity of λ.

Facts (see Gohberg-Krein)

1. The algebraic multiplicity is always finite (if $\lambda \neq 0$).
2. On $E_\lambda(T)$ the operator T takes the form $T = \lambda + N_\lambda$, where N_λ is nilpotent.
3. If T is normal, then $E_\lambda(T) = \ker(T - \lambda)$.

Eigenvalue Sequences

Definition

An eigenvalue sequence $\lambda(T) = \{\lambda_j(T)\}_{j\geq 0}$ is any sequence s.t.:

1. $\lambda_j(T)$ is an eigenvalue of T and is repeated according to algebraic multiplicity.
2. $|\lambda_0(T) \geq |\lambda_1(T)| \geq \cdots$.

Remarks

1. An eigenvalue sequence need not be unique.
2. If $T \geq 0$, then $\lambda_j(T) = \mu_j(T)$.
3. We shall denote by $\lambda(T)$ *any* eigenvalue sequence for T.

Eigenvalue Sequences

Proposition (Weyl)

For all $N \geq 1$*, we have*

$$\sum_{j<N} |\lambda_j(T)| \leq \sum_{j<N} \mu_j(T).$$

Remark

In general we don't have $|\lambda_j(T)| \leq \mu_j(T)$.

Theorem (Lidskii)

If $T \in \mathcal{L}^1$*, then*

$$\mathrm{Tr}(T) = \sum_{j\geq 0} \lambda_j(T).$$

Eigenvalue Sequences

Definition

A (compact) operator Q is called quasi-nilpotent if $\mathrm{Sp}(Q) = \{0\}$.

Fact

Q is quasi-nilpotent if and only if $\lim_{n\to\infty} \|Q^n\|^{\frac{1}{n}} = 0$.

Proof.

By Gel'fand-Mazur theorem,

$$\lim_{n\to\infty} \|Q^n\|^{\frac{1}{n}} = \sup\{|\lambda|;\ \lambda \in \mathrm{Sp}(Q)\}.$$

□

Eigenvalue Sequences

Theorem (Ringrose)

Any compact operator T can be put in the form,

$$T = A + Q,$$

where A and Q are compact operators such that

- *A is normal and $\lambda(A) = \lambda(T)$.*
- *Q is quasi-nilpotent.*

Lemma (see Reed-Simon)

Every $A \in \mathcal{L}(\mathcal{H})$ is linear combination of 4 unitaries.

Proof.

- If $A = A^*$ and $\|A\| = 1$, then

$$A = \frac{1}{2}(U + U^*) \quad \text{where } U = A + i\sqrt{1 - A^2}.$$

- In general $A = c_1 A_1 + i c_2 A_2$ with $c_i \geq 0$ and A_i as above. □

Traces on $\mathcal{L}^{1,\infty}$

Lemma

Let $\varphi : \mathcal{L}^{1,\infty} \to \mathbb{C}$ *be a linear functional. TFAE:*

(i) φ *is a trace, i.e.,*

$$\varphi(AT) = \varphi(TA) \qquad \forall T \in \mathcal{L}^{1,\infty}\ \forall A \in \mathcal{L}(\mathcal{H}).$$

(ii) φ *is unitarily invariant, i.e.,*

$$\varphi(U^*TU) = \varphi(T)\ \forall T \in \mathcal{L}^{1,\infty}\ \forall U \in \mathcal{L}(\mathcal{H}) \text{ unitary.}$$

Proof.

- If U is unitary, then

$$U^*TU - T = (U^*T)U - U(U^*T) = [U^*T, U]$$

- Here $U^*T \in \mathcal{L}^{1,\infty}$. Thus, if φ is a trace, then

$$\varphi(U^*TU) - \varphi(T) = \varphi([U^*T, U]) = 0.$$

- If φ is unitarily invariant, then

$$\varphi(UT) = \varphi(U^*(UT)U) = \varphi(TU).$$

- Thanks to the previous lemma the unitaries span $\mathcal{L}(\mathcal{H})$.
- Thus, by linearity $\varphi(TA) = \varphi(AT)$ for all $A \in \mathcal{L}(\mathcal{H})$, i.e., φ is a trace.

□

Traces on $\mathcal{L}^{1,\infty}$

Proposition

Any positive trace on $\mathcal{L}^{1,\infty}$ is continuous.

Remark

This is a folk result.

- It can be shown that any positive linear form on a C^*-algebra is continuous (see, e.g., Murphy's book).
- The same arguments show that any positive linear form on $\mathcal{L}^{1,\infty}$ is continuous.

Proposition (Connes-McDonald-Sukochev-Zanin '19)

Every continuous trace on $\mathcal{L}^{1,\infty}$ is the linear combination of 4 positive traces.

Spectral Characterization of Commutators

Lemma (see Lord-Sukochev-Zanin's book)

If $S, T \in \mathcal{L}^{1,\infty}$, *then*

$$\sum_{j<N} \lambda_j(S+T) = \sum_{j<N} \lambda_j(S) + \sum_{j<N} \lambda_j(T) + O(1).$$

Corollary

If $(\lambda_j(T))_{j\geq 0}$ *and* $(\lambda'_j(T))_{j\geq 0}$ *are two eigenvalue sequences, then*

$$\sum_{j<N} \lambda'_j(T) = \sum_{j<N} \lambda_j(T) + O(1).$$

Proof.

Apply the lemma to $S=0$ with $\lambda_j(S+T) = \lambda'_j(T)$. □

Spectral Characterization of Commutators

Corollary

If $T \in \mathrm{Com}(\mathcal{L}^{1,\infty})$, *then*

$$\sum_{j<N} \lambda_j(T) = \mathrm{O}(1).$$

Proof.

- As the unitaries span $\mathcal{L}(\mathcal{H})$, the space $\mathrm{Com}(\mathcal{L}^{1,\infty})$ is span by operators of the form,

$$[T, U] = TU - UT = U^*(UT)U - UT.$$

with U unitary and $T \in \mathcal{L}^{1,\infty}$.

- Substituting U^*T for T shows that $\mathrm{Com}(\mathcal{L}^{1,\infty})$ is span by operators of the form $U^*TU - T$.
- As $U^*TU = U^{-1}TU$ has same spectrum as T, we may take $\lambda_j(U^*TU) = \lambda_j(T)$ to get

$$\sum_{j<N} \lambda_j\,(U^*TU - T) = \sum_{j<N} \lambda_j\,(U^*TU) - \sum_{j<N} \lambda_j(T) + \mathrm{O}(1) = \mathrm{O}(1).$$

□

Spectral Characterization of Commutators

Theorem (Dykema-Figiel-Weiss-Wodzicki)

If $S, T \in \mathcal{L}^{1,\infty}$, *then*

$$S - T \in \mathrm{Com}(\mathcal{L}^{1,\infty}) \Longleftrightarrow \sum_{j<N} \lambda_j(S) = \sum_{j<N} \lambda_j(T) + \mathrm{O}(1).$$

In particular,

$$T \in \mathrm{Com}(\mathcal{L}^{1,\infty}) \Longleftrightarrow \sum_{j<N} \lambda_j(T) = \mathrm{O}(1).$$

Remarks

- This is a special case of a more general result for operator ideals.
- The proof for $\mathcal{L}^{1,\infty}$ is much simpler (see LSZ book).

Spectral Characterization of Commutators

Corollary

$\mathcal{L}^1 \subset \mathrm{Com}(\mathcal{L}^{1,\infty})$.

Proof.

- If $T \in \mathcal{L}^1$, then by Weyl's inequality,

$$\sum_{j<N} |\lambda_j(T)| \leq \sum_{j<N} \mu_j(T) \leq \sum_{j\geq 0} \mu_j(T) < \infty.$$

- Thus, $\sum \lambda_j(T) = \mathrm{O}(1)$, and hence $T \in \mathrm{Com}(\mathcal{L}^{1,\infty})$. □

Corollary

Every trace on $\mathcal{L}^{1,\infty}$ vanishes on trace-class operators, including infinitesimal operators of order > 1.

Dixmier Traces

Notation

- $\ell^\infty = C^*$-algebra of bounded complex-valued sequences.
- $\mathfrak{c}_0 =$ closed ideal of sequences converging to 0.

Definition

For $T \in \mathcal{L}^{1,\infty}$ set

$$\Lambda_N(T) = \frac{1}{\log N} \sum_{j<N} \lambda_j(T), \qquad N \geq 1.$$

Lemma

1. *The sequence* $(\Lambda_N(T))_{N\geq 1}$ *is bounded.*
2. *If* $(\lambda'_j(T))_{j\geq 1}$ *is another eigenvalue sequence for* T, *then*

$$\frac{1}{\log N} \sum_{j<N} \lambda'_j(T) - \frac{1}{\log N} \sum_{j<N} \lambda_j(T) \in \mathfrak{c}_0.$$

Proof.

- By Weyl's inequality $\sum_{j<N} |\lambda_j(T)| \leq \sum_{j<N} \mu_j(T)$.
- As $\mu_j(T) = \mathrm{O}(j^{-1})$, we have $\sum_{j<N} \mu_j(T) = \mathrm{O}(\log N)$.
- Thus,

$$|\Lambda_N(T)| \leq \frac{1}{\log N} \sum_{j<N} |\lambda_j(T)| \leq \frac{1}{\log N} \sum_{j<N} \mu_j(T) = \mathrm{O}(1).$$

  That is, $(\Lambda_N(T))_{N\geq 1}$ is a bounded sequence.
- If $(\lambda'_j(T))_{j\geq 1}$ is another eigenvalue sequence for T, then

$$\sum_{j<N} \lambda'_j(T) = \sum_{j<N} \lambda_j(T) + \mathrm{O}(1).$$

- Thus,

$$\frac{1}{\log N} \sum_{j<N} \lambda'_j(T) - \frac{1}{\log N} \sum_{j<N} \lambda_j(T) = \mathrm{O}\left((\log N)^{-1}\right) = \mathrm{o}(1).$$

  In particular, the above sequence is in $\mathfrak{c}_0$. □

Consequence

The class of $(\Lambda_N(T))$ in $\ell^\infty/\mathfrak{c}_0$ does not depend on the choice of the eigenvalue sequence.

Definition

The map $\tau : \mathcal{L}^{1,\infty} \to \ell^\infty/\mathfrak{c}_0$ given by

$$\tau(T) = \text{class of } \left\{ \frac{1}{\log N} \sum_{k<N} \lambda_k(T) \right\}_{N\geq 1} \text{ in } \ell^\infty/\mathfrak{c}_0.$$

Lemma

τ is a positive linear map that vanishes on $\mathrm{Com}(\mathcal{L}^{1,\infty})$ and $\mathcal{L}_0^{1,\infty}$.

Proof.

- If $S, T \in \mathcal{L}^{1,\infty}$, then
$$\sum_{j<N} \lambda_j(S+T) = \sum_{j<N} \lambda_j(S) + \sum_{j<N} \lambda_j(T) + \mathrm{O}(1).$$
- Thus,
$$\Lambda_N(S+T) - \Lambda_N(S) - \Lambda_N(T) = \mathrm{O}((\log N)^{-1}) = \mathrm{o}(1).$$
That is, $\Lambda(S+T) - \Lambda(S) - \Lambda(T) \in \mathfrak{c}_0$, and hence $\tau(S+T) = \tau(S) + \tau(T)$.
- If $T \in \mathrm{Com}(\mathcal{L}^{1,\infty})$, then $\sum_{j<N} \lambda_j(T) = \mathrm{O}(1)$, and so
$$\Lambda_N(T) = \mathrm{O}((\log N)^{-1}) = \mathrm{o}(1).$$
That is, $\Lambda(T) \in \mathfrak{c}_0$, and hence $\tau(T) = 0$. □

Proof (Continued).

- If $T \in \mathcal{L}_0^{1,\infty}$, then $\mu_j(T) = \mathrm{o}(j^{-1})$, and so we have

$$\sum_{j<N} \mu_j(T) = \mathrm{o}(\log N).$$

- By Weyl's inequality,

$$\sum_{j<N} |\lambda_j(T)| \leq \sum_{j<N} \mu_j(T).$$

- Thus,

$$\left|\Lambda_N(T)\right| \leq \frac{1}{\log N} \sum_{j<N} \mu_j(T) = \mathrm{o}(1).$$

That is, $\Lambda(T) \in \mathfrak{c}_0$, and so $\tau(T) = 0$. □

Definition

A state on a unital C^*-algebra $\mathcal{A}$ is a positive linear form $\omega : \mathcal{A} \to \mathbb{C}$ such that $\omega(1) = 1$.

Remark

Every state is continuous.

Definition

An extended limit is any positive linear map $\lim_\omega : \ell^\infty \to \mathbb{C}$ s.t.:

(i) $\lim_\omega 1 = 1$.

(ii) $\lim_\omega a_j = 1$ if $(a_j) \in \mathfrak{c}_0$.

Remark

- If $a_j \to L$, then $(a_j) - L \in \mathfrak{c}_0$.
- Thus, for every extended limit $\lim_\omega$, we have

$$\lim_\omega a_j = \lim_\omega L = L \lim_\omega 1 = L.$$

Remark

- Any state ω on $\ell^\infty/\mathfrak{c}_0$ defines an extended limit by

$$\lim_\omega a_j = \omega([a]), \qquad a = (a_j) \in \ell^\infty$$

  where $[a]$ is the class of a in $\ell^\infty/\mathfrak{c}_0$.
- Conversely, any extended limit descends to a state on $\ell^\infty/\mathfrak{c}_0$.
- We thus have a one-to-one correspondence,

$$\{\text{extended limits}\} \longleftrightarrow \{\text{states on } \ell^\infty/\mathfrak{c}_0\}.$$

Remark

If $(a_j) \in \ell^\infty$ is real-valued, for every extended limit $\lim_\omega$ we have

$$\liminf a_j \leq \lim_\omega a_j \leq \limsup a_j.$$

Lemma

Given any $(a_j) \in \ell^\infty$, TFAE:

(i) $a_j \to L$.

(ii) $\lim_\omega a_j = L$ *for every extended limit* $\lim_\omega$.

Dixmier Traces

Definition

If $\lim_\omega$ is an extended limit, then $\mathrm{Tr}_\omega : \mathcal{L}^{1,\infty} \to \mathbb{C}$ is given by

$$\mathrm{Tr}_\omega(T) := \lim_\omega \left\{ \frac{1}{\log N} \sum_{j<N} \lambda_j(T) \right\}, \qquad T \in \mathcal{L}^{1,\infty}.$$

Proposition (Dixmier)

1. Tr_ω *is a positive linear trace on* $\mathcal{L}^{1,\infty}$.
2. *It is annihilated by* $\mathcal{L}_0^{1,\infty}$*, and hence it vanishes on infinitesimals of order* > 1.

Proof.

- If ω is the state on $\ell^\infty/\mathfrak{c}_0$ defined by $\lim_\omega$, then

$$\mathrm{Tr}_\omega(T) = \lim_\omega \Lambda_N(T) = \omega\left([\Lambda(T)]\right) = \omega \circ \tau(T).$$

- It then follows from properties of τ and states that Tr_ω that:
  - It is a positive linear form on $\mathcal{L}^{1,\infty}$.
  - It vanishes on $\mathrm{Com}(\mathcal{L}^{1,\infty})$ and $\mathcal{L}_0^{1,\infty}$.
  - In particular, this is a trace.

□

Definition

Tr_ω is called the Dixmier trace associated with the extended limit $\lim_\omega$.

Measurable Operators

Definition (Connes)

1. An operator $T \in \mathcal{L}^{1,\infty}$ is called measurable if the value of $\mathrm{Tr}_\omega(T)$ does not depend on the extended limit.
2. We denote by $\mathcal{M}$ the class of measurable operators.
3. If $T \in \mathcal{M}$, we define its NC integral by
$$⨍ T := \mathrm{Tr}_\omega(A),$$
where Tr_ω is any Dixmier trace.

Proposition (Connes, Lord-Sukochev-Zanin)

Given $T \in \mathcal{L}^{1,\infty}$, TFAE:

1. *T is measurable and $⨍ T = L$.*
2. *We have*
$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \lambda_j(T) = L.$$

Measurable Operators

Proof.

We have

$$T \text{ meas. } \& \ ⨍ T = L \iff \mathrm{Tr}_\omega(T) = L \quad \forall \lim_\omega,$$
$$\iff \lim_\omega \left\{ \frac{1}{\log N} \sum_{j<N} \lambda_j(T) \right\} = L \quad \forall \lim_\omega,$$
$$\iff \lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \lambda_j(T) = L.$$

□

Consequence

If T is measurable, then

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \lambda_j(T) = ⨍ T.$$

Measurable Operators

Proposition

1. $\mathcal{M}$ *is a closed subspace of* $\mathcal{L}^{1,\infty}$ *that contains* $\mathrm{Com}(\mathcal{L}^{1,\infty})$ *and* $\mathcal{L}_0^{1,\infty}$.
2. $⨍ : \mathcal{M} \to \mathbb{C}$ *is a positive linear functional that vanishes on* $\mathrm{Com}(\mathcal{L}^{1,\infty})$ *and* $\mathcal{L}_0^{1,\infty}$.
3. *In particular, this is a positive trace that annihilates infinitesimals of order* > 1.

Remarks

- The C^*-algebra $\ell^\infty/\mathfrak{c}_0$ is not separable.
- The existence of states follows from Hahn-Banach theorem.
- In the non-separable case the proof relies on the Axiom of Choice.

Question (Connes, Fudan U. '17)

- Show the existence of a limit for measurable operators without using extended limits.
- Produce a purely spectral theoretic construction of the NC integral.

Tauberian Approach

Reminder

If $(\lambda_j(T))$ and $(\lambda'_j(T))$ are two eigenvalue sequences for $T \in \mathcal{L}^{1,\infty}$, then

$$\frac{1}{\log N}\sum_{j<N}\lambda'_j(T) = \frac{1}{\log N}\sum_{j<N}\lambda_j(T) + \mathrm{o}(1).$$

Lemma

Let $T \in \mathcal{L}^{1,\infty}$. TFAE:

(i) $\lim_{N\to\infty}(\log N)^{-1}\sum_{j<N}\lambda_j(T)$ *exists for some eigenvalue sequence.*

(ii) $\lim_{N\to\infty}(\log N)^{-1}\sum_{j<N}\lambda_j(T)$ *exists for every eigenvalue sequence.*

Tauberian Approach

Definition (Lord-Sukochev-Zanin)

1. An operator $T \in \mathcal{L}^{1,\infty}$ is called Tauberian if
$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \lambda_j(T) \text{ exists.}$$
2. The class of Tauberian operators is denoted $\mathcal{T}$.

Definition

For $T \in \mathcal{T}$ set
$$\int' T := \lim_{N\to\infty} (\log N)^{-1} \sum_{j<N} \lambda_j(T).$$

Tauberian Approach

Proposition

1. *$\mathcal{T}$ is a closed subspace of $\mathcal{L}^{1,\infty}$ that contains $\mathrm{Com}(\mathcal{L}^{1,\infty})$ and $\mathcal{L}_0^{1,\infty}$.*
2. *$f' : \mathcal{T} \to \mathbb{C}$ is a positive linear functional that vanishes on $\mathrm{Com}(\mathcal{L}^{1,\infty})$ and $\mathcal{L}_0^{1,\infty}$.*
3. *In particular, this is a positive trace that annihilates infinitesimals of order > 1.*

Proof.

- Reminder (LSZ Lemma): if $S, T \in \mathcal{L}^{1,\infty}$, then

$$\sum_{j<N} \lambda_j(S+T) = \sum_{j<N} \lambda_j(S) + \sum_{j<N} \lambda_j(T) + O(1).$$

- Thus,

$$\frac{1}{\log N}\sum_{j<N} \lambda_j(S+T) = \frac{1}{\log N}\sum_{j<N} \lambda_j(S) + \frac{1}{\log N}\sum_{j<N} \lambda_j(T) + o(1).$$

- Therefore, if $S, T \in \mathcal{T}$, then

$$\lim_{N\to\infty} \frac{1}{\log N}\sum_{j<N} \lambda_j(S+T) =$$

$$\lim_{N\to\infty} \frac{1}{\log N}\sum_{j<N} \lambda_j(S) + \lim_{N\to\infty} \frac{1}{\log N}\sum_{j<N} \lambda_j(T) = \int' S + \int' T.$$

- $S+T \in \mathcal{T}$ and $\int'(S+T) = \int' S + \int' T$. □

Proof (Continued).

- Reminder: if $T \in \mathrm{Com}(\mathcal{L}^{1,\infty})$, then $\sum_{j<N} \lambda_j(T) = \mathrm{O}(1)$, and hence
$$\frac{1}{\log N} \sum_{j<N} \lambda_j(T) = \mathrm{o}(1).$$
- Thus,
$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \lambda_j(T) = 0.$$
- That is,
$$T \in \mathcal{T} \quad \text{and} \quad ⨍' T = 0.$$

□

Tauberian Approach

Proof (Continued).

- Reminder: if $T \in \mathcal{L}_0^{1,\infty}$, then $\mu_j(T) = \mathrm{o}(j^{-1})$, and hence

$$\sum_{j<N} \mu_j(T) = \mathrm{o}(\log N).$$

- By Weyl's inequality,

$$\Big|\sum_{j<N} \lambda_j(T)\Big| \leq \sum_{j<N} |\lambda_j(T)| \leq \sum_{j<N} \mu_j(T)$$

- Thus,

$$\frac{1}{\log N}\Big|\sum_{j<N} \lambda_j(T)\Big| \leq \frac{1}{\log N}\sum_{j<N} \mu_j(T) = \mathrm{o}(1).$$

- It follows that $(\log N)^{-1} \sum \lambda_j(T) \to 0$.
- As before, this implies that

$$T \in \mathcal{T} \quad \text{and} \quad \int' T = 0.$$

□

Tauberian Approach

The two approaches agree.

Proposition

$\mathcal{T} = \mathcal{M}$ *and* $⨍' = ⨍$.

Proof.

- We know that

$$T \in \mathcal{M} \iff \lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \lambda_j(T) \text{ exists.}$$

- Thus, $\mathcal{T} = \mathcal{M}$.
- Moreover, if $T \in \mathcal{M} = \mathcal{T}$, then

$$⨍' T = \lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \lambda_j(T) = ⨍ T.$$

□

Tauberian Approach

Corollary (Spectral Invariance)

Let $S, T \in \mathcal{L}^{1,\infty}$ have the same non-zero eigenvalues with same multiplicity. Then:

1. *S is measurable if and only if T is measurable.*
2. *In this case $⨍ S = ⨍ T$.*

Proof.

- The assumptions imply that $\lambda_j(S) = \lambda_j(T)$.
- Thus,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \lambda_j(S) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \lambda_j(T),$$

  provided any of the above limit exists.
- Therefore, $S \in \mathcal{M}$ iff $T \in \mathcal{M}$, and in this case $⨍ S = ⨍ T$. □

Definition

A trace $\varphi : \mathcal{L}^{1,\infty} \to \mathbb{C}$ is called normalized if

$$\left(T \geq 0 \text{ and } \lambda_j(T) = (j+1)^{-1}\right) \Longrightarrow \varphi(T) = 1.$$

Remark

Every Dixmier trace Tr_ω is a normalized trace.

Proof.

- If $\lambda_j(T) = (j+1)^{-1}$, then

$$\frac{1}{\log N} \sum_{j<N} \lambda_j(T) = \frac{1}{\log N} \sum_{j<N} \frac{1}{j+1} \longrightarrow 1.$$

- Thus T is measurable and $⨍ T = 1$.
- In particular, $\mathrm{Tr}_\omega(T) = 1$. □

Strong Measurability

Remark

There are many normalized positive traces on $\mathcal{L}^{1,\infty}$ that are not Dixmier traces.

Definition

An operator $T \in \mathcal{L}^{1,\infty}$ is called strongly measurable (or positively measurable) if $\varphi(T)$ takes the same value as φ ranges over all normalized positive traces.

Remark

If T is strongly measurable, then: its is measurable, and, for every normalized positive trace $\varphi : \mathcal{L}^{1,\infty} \to \mathbb{C}$, we have

$$\varphi(T) = ⨍ T = \lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \lambda_j(T).$$

Strong Measurability

Reminder

1. Every positive linear form on $\mathcal{L}^{1,\infty}$ is continuous.
2. Every continuous trace on $\mathcal{L}^{1,\infty}$ is linear combinations of 4 positive traces (Connes *et al.*).

Remark

It can be shown that every non-zero positive trace is normalized up to scalar multiple.

Consequence

The space of continuous traces on $\mathcal{L}^{1,\infty}$ is spanned by normalized positive traces.

Strong Measurability

Notation

$T_0 =$ any positive operator in $\mathcal{L}^{1,\infty}$ such that $\lambda_j(T) = (j+1)^{-1}$.

Lemma

Given any $T \in \mathcal{L}^{1,\infty}$, TFAE:

(i) T *is strongly measurable and* $⨍ T = L$.

(ii) $\varphi(T) = \varphi(T_0)L$ *for every continuous trace on* $\mathcal{L}^{1,\infty}$.

Strong Measurability

Notation

$\mathcal{M}_s$ = class of strongly measurable operators.

Proposition

- *$\mathcal{M}_s$ is a closed subspace of $\mathcal{L}^{1,\infty}$.*
- *It contains $\mathrm{Com}(\mathcal{L}^{1,\infty})$ and $\mathcal{L}_0^{1,\infty}$. In particular, it contains all infinitesimals of order > 1.*
- *It does not depend on the inner product of $\mathcal{L}(\mathcal{H})$.*

Remark

- In fact, $\mathcal{M}_s$ contains the closure $\overline{\mathrm{Com}(\mathcal{L}^{1,\infty})}$.
- This closure contains $\mathrm{Com}(\mathcal{L}^{1,\infty}) \cup \mathcal{L}_0^{1,\infty}$.

Strong Measurability

Proposition

Let $T \in \mathcal{L}^{1,\infty}$ *be such that*

$$\sum_{j<N} \lambda_j(T) = L \cdot \log N + O(1).$$

Then T *is strongly measurable and* $⨍ T = L$.

Strong Measurability

Proof.

- The assumptions imply that T is measurable and $\fint T = L$.
- We have
$$\sum_{j<N} \lambda_j(T_0) = \sum_{j<N} (j+1)^{-1} = \log N + \mathrm{O}(1).$$
- Thus,
$$\sum_{j<N} \lambda_j(T) = L \cdot \log N + \mathrm{O}(1) = \sum_{j<N} \lambda_j(T_0) + \mathrm{O}(1).$$
- We know that by a theorem of Dykema *et al.* this implies that $T - LT_0 \in \mathrm{Com}(\mathcal{L}^{1,\infty})$.
- Here $T_0 \in \mathcal{M}_s$ and $\mathrm{Com}(\mathcal{L}^{1,\infty})$, and so $T \in \mathcal{M}_s$. □