

Noncommutative Geometry
Chapter 4:
Singular Values and Schatten Classes

Sichuan University, Spring 2024

Additional References

- Gohberg, I.C.; Krein, M.G.: *Introduction to the theory of linear nonselfadjoint operators*. American Mathematical Society, 1969.
- Simon, B.: *Trace ideals and their applications*. American Mathematical Society, 2005.

Setup

Throughout this chapter \mathcal{H} is a separable Hilbert space.

Singular Values

Definition (Singular Values)

Let $T \in \mathcal{L}(\mathcal{H})$. Given any $n \geq 0$, the $(n + 1)$ -th singular value (a.k.a. characteristic value) of T is

$$\mu_n(T) := \inf \{ \|T|_{E^\perp}\|; \dim E = n \}.$$

Remark

It follows from the above definition that

$$\mu_n(\lambda T) = |\lambda| \mu_n(T) \quad \forall \lambda \in \mathbb{C}.$$

Singular Values

Notation

\mathcal{R}_n = the space of operators $T \in \mathcal{L}(\mathcal{H})$ of rank $\leq n$.

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$. Then:

$$\begin{aligned}\mu_n(T) &= \text{dist}(T, \mathcal{R}_n), \\ \mu_m(T) &\leq \mu_n(T) \quad \forall m \geq n.\end{aligned}$$

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$. Then:

$$\begin{aligned}\mu_n(T) &= \mu_n(T^*) = \mu_n(|T|), \\ \mu_n(ATB) &\leq \|A\| \mu_n(T) \|B\| \quad \forall A, B \in \mathcal{L}(\mathcal{H}), \\ \mu_n(U^* T U) &= \mu_n(T) \quad \forall U \in \mathcal{L}(\mathcal{H}), \text{ } U \text{ unitary.}\end{aligned}$$

Remark

Let \mathcal{H}' be another (separable) Hilbert space.

- If $A \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ and let $B \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$, then

$$\mu_n(ATB) \leq \|A\|_{\mathcal{L}(\mathcal{H}', \mathcal{H})} \mu_n(T) \|B\|_{\mathcal{L}(\mathcal{H}, \mathcal{H}')}.$$

- If $U \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ is unitary, then

$$\mu_n(U^* T U) = \mu_n(T).$$

- Thus, the singular values are invariant under unitary isomorphisms.

Proposition

Let $S, T \in \mathcal{L}(\mathcal{H})$ and let $m, n \in \mathbb{N}_0$. Then:

$$\mu_{m+n}(S + T) \leq \mu_m(S) + \mu_n(T),$$

$$|\mu_m(S) - \mu_m(T)| \leq \|S - T\|,$$

$$\mu_{m+n}(ST) \leq \mu_m(S)\mu_n(T).$$

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$. TFAE:

- ① T is a compact operator.
- ② T is the norm-limit of finite-rank operators.
- ③ $\mu_n(T) \rightarrow 0$ as $n \rightarrow \infty$.
- ④ For any $\epsilon > 0$ there exists a finite-dimensional subspace E of \mathcal{H} such that $\|T|_{E^\perp}\| < \epsilon$.

Singular Values

Theorem (Min-Max Principle)

If T is compact, then

$\mu_n(T) = (n + 1)$ -th eigenvalue of $|T|$ counted with multiplicity.

Remark

This implies that eigenvalues of *positive* compact operators are continuous, since the singular values are continuous.

Corollary

Let $T = U|T|$ be the polar decomposition of T and $(\xi_n)_{n \geq 0}$ an orthonormal family such that $|T|\xi_n = \mu_n(T)\xi_n$ for all n . Then

$$T = \sum_{n \geq 0} \mu_n(T) |U\xi_n\rangle\langle\xi_n|,$$

where the series converges in norm.

Singular Values

Corollary

Assume that $f(t)$, $t \geq 0$, is non-negative, non-decreasing and converges to 0 as $t \rightarrow 0^+$. Then

$$\begin{aligned}\mu_n(f(|T|)) &= f(\mu_n(T)) \quad \forall n \geq 0, \\ f(|T|) &= \sum_{n \geq 0} f(\mu_n(T)) |\xi_n\rangle\langle\xi_n|,\end{aligned}$$

where the series converges in norm.

Example

Let $p > 0$. Applying the above result to $f(t) = t^p$ gives

$$\mu_n(|T|^p) = \mu_n(T)^p \quad \forall n \geq 0.$$

The Trace-Class \mathcal{L}^1

Definition

- For $T \in \mathcal{L}(\mathcal{H})$ we set

$$\|T\|_1 := \sum_{n \geq 0} \mu_n(T).$$

- The trace-class \mathcal{L}^1 is the set of operators T such that $\|T\|_1 < \infty$.

Remarks

- We always have $\|T\| = \mu_0(T) \leq \|T\|_1$.
- As $\|T\|_1 = \sum \mu_n(T) < \infty \Rightarrow \lim \mu_n(T) = 0$, any trace-class operator is compact.

Proposition

The properties of singular values imply the following:

$$\begin{aligned}\|T\|_1 &= \|T^*\|_1 = \| |T| \|_1, \\ \|\lambda T\|_1 &= |\lambda| \|T\|_1, \quad \lambda \in \mathbb{C}, \\ \|S + T\|_1 &\leq \|S\|_1 + \|T\|_1, \\ \|ATB\|_1 &\leq \|A\| \|T\|_1 \|B\|, \quad A, B \in \mathcal{L}(\mathcal{H}).\end{aligned}$$

Remark

The top equalities imply that

$$T \in \mathcal{L}^1 \iff T^* \in \mathcal{L}^1 \iff |T| \in \mathcal{L}^1.$$

The Trace-Class \mathcal{L}^1

Proposition

- ① \mathcal{L}^1 is a two-sided ideal of $\mathcal{L}(\mathcal{H})$.
- ② $\|\cdot\|_1$ is a norm which respect to which \mathcal{L}^1 is a Banach space.

Proposition

Assume $T \in \mathcal{L}^1$. Let $T = U|T|$ be its polar decomposition and $(\xi_n)_{n \geq 0}$ an orthonormal family such that $|T|\xi_n = \mu_n(T)\xi_n$ for all n . Then

$$T = \sum_{n \geq 0} \mu_n(T) |U\xi_n\rangle\langle\xi_n|,$$

where the series converges in \mathcal{L}^1 .

Corollary

The space of finite-rank operators is dense in \mathcal{L}^1 .

The Trace-Class \mathcal{L}^1

Lemma

Let $T \in \mathcal{L}(\mathcal{H})$ be positive. Then, for every orthonormal basis $(\xi_n)_{n \geq 0}$ of \mathcal{H} , we have

$$\|T\|_1 = \sum_{n \geq 0} \langle \xi_n | T \xi_n \rangle.$$

Lemma

Let $T \in \mathcal{L}^1$.

- ① For every orthonormal basis $(\xi_n)_{n \geq 0}$ of \mathcal{H} , the series

$$\sum_{n \geq 0} \langle \xi_n | T \xi_n \rangle$$

converges absolutely.

- ② The value of its sum does not depend on the basis $(\xi_n)_{n \geq 0}$.

The Trace-Class \mathcal{L}^1

Definition

Let $T \in \mathcal{L}(\mathcal{H})$. Then the trace of T is

$$\mathrm{Tr}(T) := \sum_{n \geq 0} \langle \xi_n | T \xi_n \rangle,$$

where $(\xi_n)_{n \geq 0}$ is any orthonormal basis.

Example

We have

$$\mathrm{Tr}(|T|) = \sum \langle \xi_n | |T| \xi_n \rangle = |||T|||_1 = \|T\|_1 = \sum \mu_n(T).$$

Proposition

- ① $T \rightarrow \text{Tr}(T)$ is a continuous linear form on \mathcal{L}^1 such that

$$|\text{Tr}(T)| \leq \text{Tr}(|T|) = \|T\|_1,$$
$$\text{Tr}(T^*) = \overline{\text{Tr}(T)}.$$

- ② This is a trace on the ideal \mathcal{L}^1 , i.e.,

$$\text{Tr}(AT) = \text{Tr}(TA) \quad \forall T \in \mathcal{L}^1 \quad \forall A \in \mathcal{L}(\mathcal{H}).$$

Example

Let $\xi, \eta \in \mathcal{H}$. Assume that $(\xi_n)_{n \geq 0}$ is an orthonormal basis of \mathcal{H} such that $\xi_0 = \|\xi\|^{-1} \xi$.

- For all $n \geq 0$, we have

$$\langle \xi_n | (|\xi\rangle\langle\eta|) \xi_n \rangle = \langle \xi_n | \xi \rangle \langle \eta | \xi_n \rangle = \delta_{n,0} \|\xi\| \langle \eta | \xi_0 \rangle = \delta_{n,0} \langle \eta | \xi \rangle .$$

- Thus,

$$\text{Tr}(|\xi\rangle\langle\eta|) = \sum_{n \geq 0} \langle \xi_n | (|\xi\rangle\langle\eta|) \xi_n \rangle = \langle \eta | \xi \rangle .$$

Example

Assume $T \in \mathcal{L}^1$ is normal, and let $(\xi_n)_{n \geq 0}$ be an orthonormal eigenbasis of \mathcal{H} , i.e., $T\xi_n = \lambda_n \xi_n$ for all $n \geq 0$.

- We have

$$\mathrm{Tr}(T) = \sum_{n \geq 0} \langle \xi_n | T \xi_n \rangle = \sum_{n \geq 0} \lambda_n \langle \xi_n | \xi_n \rangle = \sum_{n \geq 0} \lambda_n.$$

- That is, the trace of T is the sum of its eigenvalues.

Notation

If $S \in \mathcal{L}^1$ and $T \in \mathcal{L}(\mathcal{H})$, then $ST \in \mathcal{L}^1$, and so we may set

$$(S, T) := \text{Tr}(ST).$$

Remark

We have

$$|(S, T)| = |\text{Tr}(ST)| \leq \|ST\|_1 \leq \|S\|_1 \|T\|.$$

The Trace-Class \mathcal{L}^1

Lemma

(i) For all $S \in \mathcal{L}^1$,

$$\|S\|_1 = \sup_{\|T\|=1} |\mathrm{Tr}(ST)|.$$

(ii) For all $T \in \mathcal{L}(\mathcal{H})$,

$$\|T\| = \sup_{\|S\|_1=1} |\mathrm{Tr}(ST)|.$$

Consequence

We isometric linear maps,

$$\begin{aligned}\mathcal{L}^1 \ni S &\longrightarrow (S, \cdot) \in \mathcal{L}(\mathcal{H})', \\ \mathcal{L}(\mathcal{H}) \ni T &\longrightarrow (\cdot, T) \in (\mathcal{L}^1)'. \end{aligned}$$

The Trace-Class \mathcal{L}^1

Proposition

The map $T \rightarrow (\cdot, T)$ is an isometric isomorphism from $\mathcal{L}(\mathcal{H})$ onto $(\mathcal{L}^1)'$.

Remark

This means that \mathcal{L}^1 is the pre-dual of $\mathcal{L}(\mathcal{H})$.

Proposition

The map $S \rightarrow (S, \cdot)$ is an isometric isomorphism from \mathcal{L}^1 onto \mathcal{K}' .

Hilbert-Schmidt Operators

Definition

- For any $T \in \mathcal{L}(\mathcal{H})$, we set

$$\|T\|_2 = \left(\sum \mu_n(T)^2 \right)^{\frac{1}{2}}.$$

- We say that T is a Hilbert-Schmidt operator if $\|T\|_2 < \infty$.
- The class of Hilbert-Schmidt operators is denoted \mathcal{L}^2 .

Remarks

- We have $\mathcal{L}^1 \subset \mathcal{L}^2 \subset \mathcal{K}$.
- If T is compact, then $\mu_n(T)^2 = \mu_n(|T|^2)$, and so

$$\|T\|_2^2 = \sum \mu_n(T)^2 = \sum \mu_n(|T|^2) = \text{Tr}(|T|^2) = \text{Tr}(T^*T).$$

- Thus,

$$T \in \mathcal{L}^2 \iff \sum \mu_n(|T|^2) < \infty \iff |T|^2 \in \mathcal{L}^1.$$

Example

Let $\xi, \eta \in \mathcal{H}$. Assume that $(\xi_n)_{n \geq 0}$ is an orthonormal basis of \mathcal{H} such that $\xi_0 = \|\xi\|^{-1}\xi$.

- We have $(|\xi \rangle \langle \eta|)^* = |\eta \rangle \langle \xi|$, and so

$$(|\xi \rangle \langle \eta|)^* (|\xi \rangle \langle \eta|) = (|\eta \rangle \langle \xi|)(|\xi \rangle \langle \eta|) = \langle \xi | \xi \rangle |\eta \rangle \langle \eta| = \|\xi\|^2 |\eta \rangle \langle \eta|.$$

- Thus,

$$\| |\xi \rangle \langle \eta| \|_2^2 = \text{Tr} [(|\xi \rangle \langle \eta|)^* (|\xi \rangle \langle \eta|)] = \|\xi\|^2 \text{Tr} [|\eta \rangle \langle \eta|] = \|\xi\|^2 \langle \eta | \eta \rangle.$$

- That is,

$$\| |\xi \rangle \langle \eta| \|_2 = \|\xi\| \|\eta\|.$$

Proposition

The properties of singular values imply the following:

$$\begin{aligned}\|T\|_2 &= \|T^*\|_2 = \| |T| \|_2, \\ \|\lambda T\|_2 &= |\lambda| \|T\|_2, \quad \lambda \in \mathbb{C}, \\ \|ATB\|_2 &\leq \|A\| \|T\|_2 \|B\|, \quad A, B \in \mathcal{L}(\mathcal{H}).\end{aligned}$$

Remark

The top equalities imply that

$$T \in \mathcal{L}^2 \iff T^* \in \mathcal{L}^2 \iff |T| \in \mathcal{L}^2.$$

Hilbert-Schmidt Operators

Lemma

For every orthonormal basis $(\xi_n)_{n \geq 0}$ of \mathcal{H} , we have

$$\|T\|_2^2 = \sum_{n \geq 0} \|T\xi_n\|^2.$$

Proof.

- We have

$$\|T\|_2^2 = \text{Tr}(|T|^2) = \sum \langle \xi_n | |T|^2 \xi_n \rangle.$$

- Here

$$\langle \xi_n | |T|^2 \xi_n \rangle = \langle \xi_n | T^* T \xi_n \rangle = \langle T \xi_n | T \xi_n \rangle = \|T \xi_n\|^2.$$

- Thus,

$$\|T\|_2^2 = \sum \|T \xi_n\|^2.$$



Corollary

We have

$$\|S + T\|_2 \leq \|S\|_2 + \|T\|_2, \quad S, T \in \mathcal{L}(\mathcal{H}).$$

Proposition

- ① \mathcal{L}^2 is a two-sided ideal of \mathcal{L}^2 .
- ② $\|\cdot\|_2$ is a norm which respect to which \mathcal{L}^2 is a Banach space.

Proposition

Assume $T \in \mathcal{L}^2$. Let $T = U|T|$ be its polar decomposition and $(\xi_n)_{n \geq 0}$ an orthonormal family such that $|T|\xi_n = \mu_n(T)\xi_n$ for all n . Then

$$T = \sum_{n \geq 0} \mu_n(T) |U\xi_n\rangle\langle\xi_n|,$$

where the series converges in \mathcal{L}^2 .

Corollary

The space of finite-rank operators is dense in \mathcal{L}^2 .

Hilbert-Schmidt Operators

Proposition

If $S, T \in \mathcal{L}^2$, then ST is trace-class, and we have

$$\begin{aligned} |\operatorname{Tr}(ST)| &\leq \|ST\|_1 \leq \|S\|_2 \|T\|_2, \\ \operatorname{Tr}(ST) &= \operatorname{Tr}(TS). \end{aligned}$$

Corollary

\mathcal{L}^2 is a Hilbert space with respect to the inner product,

$$\langle S|T \rangle_{\mathcal{L}^2} := \operatorname{Tr}(S^* T), \quad S, T \in \mathcal{L}^2.$$

Remark

We have

$$\langle T|T \rangle_{\mathcal{L}^2} = \operatorname{Tr}(T^* T) = \operatorname{Tr}(|T|^2) = \|T\|_2^2.$$

Integral Operators

Setup

(X, μ) is a σ -finite measure space s.t. $L^2_\mu(X)$ is separable.

Example

X is a smooth manifold and μ is a Radon measure.

Definition (Integral Operators)

If $K(x, y) \in L^2_{\mu \otimes \mu}(X \times X)$, then $T_K : L^2_\mu(X) \rightarrow L^2_\mu(X)$ is given by

$$T_K f(x) := \int_X K(x, y) f(y) d\mu(y), \quad f \in L^2_\mu(X).$$

Remark

- By Cauchy-Schwartz's inequality,

$$|T_K f(x)|^2 = \left(\int_X K(x, y) f(y) d\mu(y) \right)^2 \leq \int_X |K(x, y)|^2 d\mu(y) \int_X |f(y)|^2 d\mu(y).$$

- Thus,

$$\int_X |T_K f(x)|^2 d\mu(x) \leq \|f\|_{L^2}^2 \iint_{X \times X} |K(x, y)|^2 d\mu(y) d\mu(x) < \infty.$$

- That is,

$$T_K f \in L^2_\mu(X) \quad \text{and} \quad \|T_K\|_{L^2} \leq \|K\|_{L^2} \|f\|_{L^2}.$$

- It follows that

$$T_K \in \mathcal{L}(L^2_{\mu \otimes \mu}(X \times X)) \quad \text{and} \quad \|T_K\| \leq \|K\|_{L^2}.$$

Example

Let $\varphi, \psi \in L^2_\mu(X)$, and set $K(x, y) = \varphi(x)\overline{\varphi(y)} \in L^2_{\mu \otimes \mu}(X \times X)$.

- If $f \in L^2_\mu(X)$, then

$$T_K f(x) = \int_X \varphi(x)\overline{\varphi(y)}f(y)d\mu(y) = f(x)\langle \psi|f \rangle.$$

- Thus,

$$T_K = |\varphi\rangle\langle \psi|$$

- It follows that every rank 1 operator on $L^2_\mu(X)$ is an integral operator.
- By linearity every finite-rank operator is an integral operator

Proposition

For $K(x, y)$ and $K'(x, y)$ in $L^2_{\mu \otimes \mu}(X \times X)$, we have

$$T_K^* = T_{K^*} \quad \text{and} \quad T_K T_{K'} = T_{K * K'},$$

where we have set

$$K^*(x, y) = \overline{K(y, x)},$$
$$K * K'(x, y) = \int_X K(x, z) K'(z, y) d\mu(z).$$

Proposition

- ① If $K(x, y) \in L^2_{\mu \otimes \mu}(X \times X)$, then T_K is Hilbert-Schmidt.
- ② The linear map $K \rightarrow T_K$ is an isometric isomorphism from $L^2_{\mu \otimes \mu}(X \times X)$ onto \mathcal{L}^2 .

Proof.

Let $(\varphi_n)_{n \geq 0}$ be an orthonormal basis of $L^2_\mu(X)$.

- If $K(x, y) \in L^2_{\mu \otimes \mu}(X \times X)$, then

$$\|T_K\|_2^2 = \sum_{n \geq 0} \|T_K \varphi_n\|^2 = \sum_{n, m} |\langle \varphi_m | T_K \varphi_n \rangle|^2.$$

- Here

$$\langle \varphi_m | T_K \varphi_n \rangle = \iint \overline{\varphi_m(x)} K(x, y) \varphi_n(y) d\mu(y) = \langle \varphi_m \otimes \overline{\varphi_n} | K \rangle.$$

- As $(\varphi_m \otimes \overline{\varphi_n})_{m, n \geq 0}$ is an orthonormal basis of $L^2_\mu(X \times X)$, we get

$$\|T_K\|_2^2 = \sum_{n, m} |\langle \varphi_m \otimes \overline{\varphi_n} | K \rangle|^2 = \|K\|_{L^2}^2 < \infty.$$

- Thus, $T_K \in \mathcal{L}^2$ and we have isometric linear map $K \rightarrow T_K$ from $L^2_{\mu \otimes \mu}(X \times X)$ to \mathcal{L}^2 . In particular, it has closed range.
- As the range contains finite-rank operators, which are dense in \mathcal{L}^2 , the map is onto, and hence is an isometric isomorphism. \square

Assumptions

- X = separable metrizable locally compact Hausdorff space (e.g., X is a smooth manifold).
- μ = Radon measure.

Remark

Under those assumptions $L^2_\mu(X)$ is a separable Hilbert space.

Theorem (Duflo)

Assume that

- $K(x, y) \in L^2_{\mu \otimes \mu}(X \times X) \cap C(X \times X)$.
- T_K is trace-class.

Then $K(x, x)$ is in $L^1_\mu(X)$, and we have

$$\mathrm{Tr}(T_K) = \int_X K(x, x) d\mu(x).$$

Remark

In the above generality we need to assume T_K to be trace-class, since otherwise the trace formula may fail.

Trace Theorems

If $T_K \geq 0$, then we don't need to assume T_K is trace-class.

Theorem

Assume that

- $K(x, y) \in L^2_{\mu \otimes \mu}(X \times X) \cap C(X \times X)$.
- T_K is positive.

Then:

- ① $K(x, x) \geq 0$ for all $x \in X$.
- ② We have

$$\mathrm{Tr}(T_K) = \int_X K(x, x) d\mu(x).$$

- ③ In particular,

$$T_K \in \mathcal{L}^1 \iff K(x, x) \in L^1_\mu(X).$$

Definition

A Banach ideal is a two-sided ideal \mathcal{I} of $\mathcal{L}(\mathcal{H})$ together with a norm $\|\cdot\|_{\mathcal{I}}$ such that

- (i) $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a Banach space.
- (ii) We have

$$\|ATB\|_{\mathcal{I}} \leq \|A\| \|T\|_{\mathcal{I}} \|B\| \quad \forall T \in \mathcal{I} \quad \forall A, B \in \mathcal{L}(\mathcal{H}).$$

Examples

The following are Banach ideals:

- The ideal of compact operators \mathcal{K} .
- The trace-class \mathcal{L}^1 .
- The ideal of Hilbert-Schmidt operators \mathcal{L}^2 .

Remark

If \mathcal{I} is a two-sided ideal, then

$$T \in \mathcal{I} \iff |T| \in \mathcal{I} \iff T^* \in \mathcal{I}.$$

Proof.

If $T = U|T|$, then $|T| = U^*T$ and $T^* = U^*TU^*$ and $T = UT^*U$. □

Proposition

Assume $(\mathcal{I}, \|\cdot\|)$ is a Banach ideal. If $T \in \mathcal{I}$, then:

$$\begin{aligned} \|T\|_{\mathcal{I}} &= \||T|\|_{\mathcal{I}} = \|T^*\|_{\mathcal{I}}, \\ \|U^*TU\|_{\mathcal{I}} &= \|T\|_{\mathcal{I}} \quad \forall U \in \mathcal{L}(\mathcal{H}), \ U \text{ unitary.} \end{aligned}$$

Corollary

If $T \in \mathcal{I}$ and $\mu_n(S) = \mu_n(T)$ for all $n \geq 0$, then

$$S \in \mathcal{I} \quad \text{and} \quad \|S\|_{\mathcal{I}} = \|T\|_{\mathcal{I}}.$$

Thus, $\|T\|_{\mathcal{I}}$ only depends on the singular values $\mu_n(T)$.

Proof.

- Let $(\xi_n)_{n \geq 0}$ and $(\eta_n)_{n \geq 0}$ be orthonormal bases such that

$$|T|\xi_n = \mu_n(T)\xi_n \quad \text{and} \quad |S|\eta_n = \mu_n(T)\eta_n.$$

- Let U be the unitary operator s.t. $U\xi_n = \eta_n$. Then

$$U|T|\xi_n = \mu_n(T)U\xi_n = \mu_n(T)\eta_n = |S|\eta_n = |S|U\xi_n.$$

- That is, $U|T| = |S|U$, i.e., $U|T|U^* = |S|$. Thus,

$$T \in \mathcal{I} \iff |T| \in \mathcal{I} \iff U|T|U^* \in \mathcal{I} \iff |S| \in \mathcal{I} \iff S \in \mathcal{I}.$$

Moreover, $\|S\|_{\mathcal{I}} = \||S|\|_{\mathcal{I}} = \|U|T|U^*\|_{\mathcal{I}} = \||T|\|_{\mathcal{I}} = \|T\|_{\mathcal{I}}. \square$

Remark

If \mathcal{I} is a two-sided ideal, then, we always have

$$\{\text{finite-rank operators}\} \subset \mathcal{I} \subset \mathcal{K}$$

Definition

\mathcal{I}^0 is the closure in \mathcal{I} of the ideal of finite-rank operators.

Remarks

- $(\mathcal{I}^0, \|\cdot\|_{\mathcal{I}})$ is a Banach ideal itself.
- As a Banach space \mathcal{I}_0 is separable (i.e., it contains a dense countable subset).

Lemma

Let $T \in \mathcal{L}(\mathcal{H})$. TFAE:

- ① $T \in \mathcal{I}$.
- ② Any Schmidt series $T = \sum \mu_n(T) |U\xi_n\rangle\langle\xi_n|$ converges in \mathcal{I} .

Proposition

TFAE:

- ① \mathcal{I} is separable.
- ② $\mathcal{I}^0 = \mathcal{I}$, i.e., finite-rank operators are dense in \mathcal{I} .

Proposition

\mathcal{I}^0 always contains the trace-class \mathcal{L}^1 and we can choose $\|\cdot\|_{\mathcal{I}}$ so that $\|\cdot\|_{\mathcal{I}} \leq \|\cdot\|_1$ on \mathcal{L}^1 .

Remark

In other words \mathcal{L}^1 is the smallest Banach ideal.

Definition

A quasi-norm on a vector space E is a function $\|\cdot\| : E \rightarrow [0, \infty)$ such that:

- ① $\|x\| = 0 \Leftrightarrow x = 0$.
- ② $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{C}$ and $x \in E$.
- ③ There is $C > 0$ such that

$$\|x + y\| \leq C (\|x\| + \|y\|) \quad \forall x, y \in E.$$

A vector space equipped with a quasi-norm is called quasi-normed.

Facts

- ① A quasi-norm generates a topological vector space topology.
- ② A basis of neighbourhoods of the origin is given by the “quasi-balls”,

$$B(0, 1/n) = \{x \in E; \|x\| < 1/n\}, \quad n \geq 1.$$

- ③ As we have a countable basis, the topology is metrizable.
- ④ In particular, $x_n \rightarrow x$ in E iff $\|x_n - x\| \rightarrow 0$.

Proposition

Let $T : E \rightarrow F$ be a linear map between quasi-normed spaces.
TFAE:

- ① T is continuous.
- ② T is continuous at 0.
- ③ There is $C > 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in E$.

Definition

A quasi-Banach space is a vector space E together with a complete quasi-norm (i.e., every Cauchy sequence is convergent).

Remark

- ❶ A quasi-Banach space need not be locally convex.
- ❷ In particular, Hahn-Banach Theorem need not hold.

Example

Let (X, μ) be a measure space.

- ❶ For $0 < p < 1$ the space $L^p_\mu(X)$ is a quasi-Banach space with quasi-norm,

$$\|f\|_{L^p} := \left(\int_X |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}.$$

- ❷ The triangular inequality is replaced by

$$\|f + g\|_{L^p}^p \leq \|f\|_{L^p}^p + \|g\|_{L^p}^p.$$

In particular $d(f, g) = \|f - g\|_{L^p}^p$ is a metric.

- ❸ Thanks to the convexity of $t \rightarrow t^{1/p}$ this gives the inequality,

$$\|f + g\|_{L^p} \leq 2^{\frac{1-p}{p}} (\|f\|_{L^p} + \|g\|_{L^p}).$$

Definition

A quasi-Banach ideal is a two-sided ideal \mathcal{I} of $\mathcal{L}(\mathcal{H})$ together with a quasi-norm $\|\cdot\|_{\mathcal{I}}$ such that

- (i) $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a quasi-Banach space.
- (ii) We have

$$\|ATB\|_{\mathcal{I}} \leq \|A\| \|T\|_{\mathcal{I}} \|B\| \quad \forall T \in \mathcal{I} \quad \forall A, B \in \mathcal{L}(\mathcal{H}).$$

Remark

We will see that the Schatten classes \mathcal{L}^p , $0 < p < 1$, and the weak Schatten classes $\mathcal{L}^{p,\infty}$, $0 < p \leq 1$, are instances of quasi-Banach spaces.

Proposition

Assume $(\mathcal{I}, \|\cdot\|)$ is a Banach ideal. If $T \in \mathcal{I}$, then:

$$\begin{aligned}\|T\|_{\mathcal{I}} &= \|\|T\|\|_{\mathcal{I}} = \|T^*\|_{\mathcal{I}}, \\ \|U^*TU\|_{\mathcal{I}} &= \|T\|_{\mathcal{I}} \quad \forall U \in \mathcal{L}(\mathcal{H}), U \text{ unitary}.\end{aligned}$$

Corollary

If $T \in \mathcal{I}$ and $\mu_n(S) = \mu_n(T)$ for all $n \geq 0$, then

$$S \in \mathcal{I} \quad \text{and} \quad \|S\|_{\mathcal{I}} = \|T\|_{\mathcal{I}}.$$

Thus, $\|T\|_{\mathcal{I}}$ only depends on the singular values $\mu_n(T)$.

Definition

\mathcal{I}^0 is the closure in \mathcal{I} of the ideal of finite-rank operators.

Proposition

TFAE:

- 1 \mathcal{I} is separable.
- 2 $\mathcal{I}^0 = \mathcal{I}$, i.e., finite-rank operators are dense in \mathcal{I} .

Definition (Schatten Classes)

Let $p \in (0, \infty)$.

- For $T \in \mathcal{L}(\mathcal{H})$, set

$$\|T\|_p = \left(\sum \mu_n(T)^p \right)^{\frac{1}{p}}.$$

- The Schatten class \mathcal{L}^p consists of all operators T such that $\|T\|_p < \infty$.

Remarks

- We have $\mathcal{L}^p \subset \mathcal{L}^q \subset \mathcal{K}$ for $0 < p < q$.
- As $\mu_n(T)^p = \mu_n(|T|^p)$, we have

$$T \in \mathcal{L}^p \iff \sum \mu_n(|T|^p) < \infty \iff |T|^p \in \mathcal{L}^1.$$

Proposition

① We have

$$\begin{aligned}\|T\|_p &= \|T^*\|_p = \| |T| \|_p, \\ \|\lambda T\|_p &= |\lambda| \|T\|_p, \quad \lambda \in \mathbb{C}, \\ \|ATB\|_p &\leq \|A\| \|T\|_p \|B\|, \quad A, B \in \mathcal{L}(\mathcal{H}).\end{aligned}$$

② If $p \geq 1$, then

$$\|S + T\|_p \leq \|S\|_p + \|T\|_p.$$

③ If $0 < p < 1$, then

$$\|S + T\|_p^p \leq \|S\|_p^p + \|T\|_p^p.$$

Proposition

- ① \mathcal{L}^p is a two-sided ideal of $\mathcal{L}(\mathcal{H})$.
- ② If $p \geq 1$, then $\|\cdot\|_p$ is a norm which respect to which \mathcal{L}^p is a Banach ideal.
- ③ If $p < 1$, then $\|\cdot\|_p$ is a quasi-norm which respect to which \mathcal{L}^p is a quasi-Banach ideal.

Proposition

- ① If $T \in \mathcal{L}^p$, then all its Schmidt series converge in \mathcal{L}^p .
- ② Finite rank operators are dense in \mathcal{L}^p .

Proposition (Hölder Inequality)

Assume $p^{-1} + q^{-1} = r^{-1}$. If $S \in \mathcal{L}^p$ and $T \in \mathcal{L}^q$, then $ST \in \mathcal{L}^r$, and we have

$$\|ST\|_r \leq \|S\|_p \|T\|_q.$$

Remark

If $p_1^{-1} + \cdots + p_k^{-1} = r^{-1}$ and $T_i \in \mathcal{L}^{p_i}$, then $T_1 \cdots T_k \in \mathcal{L}^r$, and we have

$$\|T_1 \cdots T_k\|_r \leq \|T_1\|_{p_1} \cdots \|T_k\|_{p_k}.$$

Corollary

Assume $p^{-1} + (p')^{-1} = 1$. If $S \in \mathcal{L}^p$ and $T \in \mathcal{L}^{p'}$, then ST is trace-class, and we have

$$|\mathrm{Tr}(ST)| \leq \|ST\|_1 \leq \|S\|_p \|T\|_{p'}.$$

Notation

If $S \in \mathcal{L}^p$ and $T \in \mathcal{L}^{p'}$, then $(S, T) := \mathrm{Tr}(ST)$.

Remark

The above corollary implies that $T \rightarrow (\cdot, T)$ is a continuous linear map from $\mathcal{L}^{p'}$ to the (topological) dual $(\mathcal{L}^p)'$.

Proposition

The map $T \rightarrow (\cdot, T)$ is an isometric linear isomorphism from $\mathcal{L}^{p'}$ onto $(\mathcal{L}^p)'$.

Definition (Weak Schatten Classes $\mathcal{L}^{p,\infty}$)

Let $p \in (0, \infty)$.

- ① The weak Schatten class $\mathcal{L}^{p,\infty}$ consists of all $T \in \mathcal{L}(\mathcal{H})$ such that

$$\mu_n(T) = O\left(n^{-\frac{1}{p}}\right) \quad \text{as } n \rightarrow \infty.$$

- ② For $T \in \mathcal{L}(\mathcal{H})$, we set

$$\|T\|_{p,\infty} := \sup_{n \geq 0} (n+1)^{\frac{1}{p}} \mu_n(T).$$

Remark

For $0 < p < q$ we have strict inclusions,

$$\mathcal{L}^p \subsetneq \mathcal{L}^{p,\infty} \subsetneq \mathcal{L}^q \subsetneq \mathcal{K}.$$

Proposition

We have

$$\begin{aligned}\|T\|_{p,\infty} &= \|T^*\|_{p,\infty} = \| |T| \|_{p,\infty}, \\ \|\lambda T\|_{p,\infty} &= |\lambda| \|T\|_{p,\infty}, \quad \lambda \in \mathbb{C}, \\ \|ATB\|_{p,\infty} &\leq \|A\| \|T\|_{p,\infty} \|B\|, \quad A, B \in \mathcal{L}(\mathcal{H}), \\ \|S + T\|_{p,\infty} &\leq 2^{\frac{1}{p}} (\|S\|_{p,\infty} + \|T\|_{p,\infty}).\end{aligned}$$

Proposition

- ① $\mathcal{L}^{p,\infty}$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$.
- ② $\|\cdot\|_{p,\infty}$ is a quasi-norm which respect to which $\mathcal{L}^{p,\infty}$ is a quasi-Banach ideal.

Proposition

- ① If $p > 1$, then $\|\cdot\|_{p,\infty}$ is equivalent to the norm,

$$\|T\|'_{p,\infty} = \sup_{N \geq 1} \left\{ N^{-1+\frac{1}{p}} \sum_{j < N} \mu_j(T) \right\}, \quad T \in \mathcal{L}^{p,\infty}.$$

- ② In this case, $\mathcal{L}^{p,\infty}$ is a Banach ideal (w.r.t. that norm).

Weak Schatten Classes

Notation

$\mathcal{L}_0^{p,\infty}$ is the closure in $\mathcal{L}^{p,\infty}$ of the ideal of finite-rank operators.

Proposition

① We have

$$\mathcal{L}_0^{p,\infty} = \left\{ T \in \mathcal{L}(\mathcal{H}); \mu_n(T) = o\left(n^{-\frac{1}{p}}\right) \right\}.$$

② We have a strict inclusion $\mathcal{L}_0^{p,\infty} \subsetneq \mathcal{L}^{p,\infty}$.

③ In particular, $\mathcal{L}^{p,\infty}$ is not separable.

Remark

For $0 < p < q$ we have strict inclusions,

$$\mathcal{L}^p \subsetneq \mathcal{L}_0^{p,\infty} \subsetneq \mathcal{L}^{p,\infty} \subsetneq \mathcal{L}^q.$$

Proposition (Hölder's Inequality)

Assume that $p^{-1} + q^{-1} = r^{-1}$.

- ① There is $C_{pqr} > 0$ such that

$$\|ST\|_{r,\infty} \leq C_{pqr} \|S\|_{p,\infty} \|T\|_{q,\infty}.$$

- ② In particular,

$$(S \in \mathcal{L}^{p,\infty} \text{ and } T \in \mathcal{L}^{q,\infty}) \implies ST \in \mathcal{L}^{r,\infty}.$$

- ③ If $S \in \mathcal{L}_0^{p,\infty}$ or $T \in \mathcal{L}_0^{q,\infty}$, then $ST \in \mathcal{L}_0^{r,\infty}$.

Remark (Sukochev-Zanin '21)

The best constant C_{pqr} is equal to $p^{-\frac{1}{q}} q^{-\frac{1}{p}} (p+q)^{\frac{1}{r}}$.

Corollary

Suppose that $p_1^{-1} + \dots + p_k^{-1} = r^{-1}$.

- ① There is $C = C(p_1, \dots, p_k, r) > 0$ such that

$$\|T_1 \cdots T_k\|_{r, \infty} \leq C \|T_1\|_{p_1, \infty} \cdots \|T_k\|_{p_k, \infty}.$$

- ② If $T_i \in \mathcal{L}^{p_i, \infty}$ for $i = 1, \dots, k$, then $T_1 \cdots T_k \in \mathcal{L}^{r, \infty}$.
- ③ If in addition one of the T_i is in $\mathcal{L}_0^{p_i, \infty}$, then $T_1 \cdots T_k \in \mathcal{L}_0^{r, \infty}$.

The Dixmier-Macaev Ideal

Definition

The Dixmier-Macaev ideal is

$$\mathcal{M}^{1,\infty} := \left\{ T \in \mathcal{L}(\mathcal{H}); \sum_{j < N} \mu_j(T) = O(\log N) \right\}.$$

Proposition

- ① $\mathcal{M}^{1,\infty}$ is a Banach ideal with respect to the norm,

$$\|T\|'_{1,\infty} = \sup_{N \geq 1} \left\{ \frac{1}{\log(N+1)} \sum_{j < N} \mu_j(T) \right\}, \quad T \in \mathcal{M}^{1,\infty}.$$

- ② We have a strict inclusion $\mathcal{L}^{1,\infty} \subsetneq \mathcal{M}^{1,\infty}$.

Remark

In Connes' book (as well as elsewhere) the Dixmier-Macaev ideal is denoted $\mathcal{L}^{1,\infty}$.

Notation

$\mathcal{M}_0^{1,\infty}$ is the closure in $\mathcal{M}^{1,\infty}$ of the ideal of finite-rank operators.

Proposition

① *We have*

$$\mathcal{M}_0^{1,\infty} = \left\{ T \in \mathcal{L}(\mathcal{H}); \sum_{j < N} \mu_n(T) = o(\log N) \right\}.$$

② *There is a strict inclusion $\mathcal{M}_0^{1,\infty} \subsetneq \mathcal{M}^{1,\infty}$.*

③ *In particular, $\mathcal{M}^{1,\infty}$ is not separable.*