Differentiable Forms in Algebraic Topology
The Tangent Bundle

Sichuan University, Spring 2025

The Tangent Bundle as a Manifold

Objective

Let M be smooth manifold of dimension n. We would like to bundle together all the tangent spaces T_pM so as to get a smooth manifold, called the *tangent bundle*.

Definition

As a set, the *tangent bundle* of M is the disjoint union,

$$TM := \bigsqcup_{p\in M} T_pM = \{(p,v);\, p\in M,\ \ v\in T_pM\}\,.$$

Remarks

1. For $p\in M$ we identify the subset $\{p\}\times T_pM$ with the tangent space T_pM. This allows us to see T_pM as a subset of TM.
2. In particular, we write an element of TM either as (p,v) with $p\in M$ and $v\in T_pM$, or simply as v.

The Tangent Bundle as a Manifold

Remark

Let U be an open set in M. If $p \in U$, then $T_pU = T_pM$. Thus,

$$TU = \bigsqcup_{p \in U} T_pU = \bigsqcup_{p \in U} T_pM.$$

Definition

The *canonical map* $\pi : TM \to M$ is defined by

$$\pi\left((p, v)\right) = p, \qquad p \in M, \quad v \in T_pM.$$

Remarks

1. The map $\pi : TM \to M$ is onto.
2. If $p \in M$, then $\pi^{-1}(p) = T_pM$.

Example

Let U be an open in $\mathbb{R}^n$. If $p \in U$, then $T_pU = T_p\mathbb{R}^n = \mathbb{R}^n$. Recall that, if $(r^1, \ldots, r^n)$ are the standard coordinates on $\mathbb{R}^n$, then we identify

$$T_p\mathbb{R}^n \ni v = \sum v^i \frac{\partial}{\partial r^i}\bigg|_p \longleftrightarrow \langle v^1, \ldots, v^n \rangle \in \mathbb{R}^n.$$

Thus, the pair (p, v) is naturally identified with $(p, v^1, \ldots, v^n)$. Therefore, we have

$$TU = \bigsqcup_{p \in U} T_pU = \bigsqcup_{p \in U} \mathbb{R}^n = U \times \mathbb{R}^n.$$

Aim

We wish to equip TM with a smooth structure. Therefore, we need to do the following:

1. Define a topology on TM.
2. Construct a C^∞-atlas for TM.

The Tangent Bundle as a Manifold

Facts

Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart for M.

- Here $\phi(U)$ is an open in $\mathbb{R}^n$, and so $T(\phi(U)) = \phi(U) \times \mathbb{R}^n$.
- For every $p \in U$, the differential $\phi_{*,p}$ is an isomorphism from $T_pM = T_pU$ onto $T_{\phi(p)}(\phi(U)) = \mathbb{R}^n$.
- Therefore, we define a map $\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n$ by

$$\tilde{\phi}(p, v) = \left(\phi(p), \phi_{*,p}v\right), \qquad p \in U, \ v \in T_pU.$$

- This is a bijection with inverse $(x, v) \to (\phi^{-1}(x), \phi^{-1}_{*,\phi^{-1}(x)}v)$.
- This allows us to define a topology on TU by pulling back the topology of $\phi(U) \times \mathbb{R}^n$:

$$W \subseteq TU \text{ is open} \iff \tilde{\phi}(W) \text{ is open in } \phi(U) \times \mathbb{R}^n.$$

- With respect to this topology $\tilde{\phi}$ is a homeomorphism.

The Tangent Bundle as a Manifold

Remark

- If $p \in U$, then $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^1}\Big|_p \right\}$ is a basis of T_pM.
- The differential $\phi_{*,p} : T_pU \to T_{\phi(p)}V = \mathbb{R}^n$ maps $\frac{\partial}{\partial x^i}\Big|_p$ to $\frac{\partial}{\partial r^i}\Big|_{\phi(p)}$. Thus,

$$\sum v^i \frac{\partial}{\partial x^i}\bigg|_p \xrightarrow{\phi_*} \sum v^i \frac{\partial}{\partial r^i}\bigg|_{\phi(p)} \longleftrightarrow \langle v^1, \ldots, v^n \rangle \in \mathbb{R}^n.$$

- If $\phi(p) = (x^1(p), \ldots, x^n(p))$ and $v = \sum v^i \frac{\partial}{\partial x^i}\Big|_p \in T_pM$, then

$$\tilde{\phi}(p, v) = \big(\phi(p), \phi_{*,p} v\big) = \big(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n\big).$$

  In particular, this defines coordinates on TU.

The Tangent Bundle as a Manifold

Facts

- Let $(V,\psi) = (V, y^1, \ldots, y^n)$ be another chart of M such that $U \cap V \neq \emptyset$. Define $\tilde{\psi} : TV \to \psi(V) \times \mathbb{R}^n$ by
$$\tilde{\psi}(p,v) = \big(\psi(p), \psi_{*,p} v\big), \qquad p \in V,\ v \in T_pM.$$
- On $T(U \cap V) = (TU) \cap (TV)$ we have two topologies induced by the topologies of TU and TV (defined by the charts $(U \cap V, \phi_{|U\cap V})$ and $(U \cap V, \psi_{|U\cap V})$).
- On $\tilde{\phi}(TU \cap TV) = \phi(U \cap V) \times \mathbb{R}^n$ we have

$$\tilde{\psi}\circ\tilde{\phi}^{-1}(r,v) = \big(\psi\circ\phi^{-1}(r), \psi_*\circ\phi_*^{-1} v\big) = \big(\psi\circ\phi^{-1}(r), (\psi\circ\phi^{-1})_{*,r} v\big).$$

- Here $(\psi \circ \phi^{-1})_{*,r}$ is the multiplication by the Jacobian matrix $J_{\psi\circ\phi^{-1}}(r) = \left[\partial(y^j \circ \phi^{-1})/\partial r^i(r)\right]$ whose entries depends smoothly on r.
- Therefore, $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$ is smooth map. Its inverse map $\tilde{\phi} \circ \tilde{\psi}^{-1}$ is smooth as well, and so $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is a diffeomorphism.

The Tangent Bundle as a Manifold

Facts (Continued)

- $T(U \cap V)$ is open in TU and in TV.
- As $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$ is a diffeomorphism, this is a homeomorphism.
- If $W \subseteq T(U \cap V)$, then
$$\begin{aligned} W \text{ open in } TU &\Longleftrightarrow \tilde{\phi}(W) \text{ open in } \phi(U) \times \mathbb{R}^n, \\ &\Longleftrightarrow \tilde{\psi} \circ \tilde{\phi}^{-1}\left[\tilde{\phi}(W)\right] \text{ open in } \psi(U \cap V) \times \mathbb{R}^n, \\ &\Longleftrightarrow \tilde{\psi}(W) \text{ open in } \psi(V) \times \mathbb{R}^n, \\ &\Longleftrightarrow W \text{ open in } TV. \end{aligned}$$
  Thus, TU and TV induce the same topology on $T(U \cap V)$.
- It follows that, if W is open in TU and X is open in TV, then $W \cap X = (W \cap T(U \cap V)) \cap (X \cap T(U \cap V))$ is open in $T(U \cap V)$ (by definition of the subspace topology).

Summary

(a) If (U, ϕ) is a chart for M, then $\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n$ allows us to define a topology on TU by pulling back the topology of $\phi(U) \times \mathbb{R}^n$. This map then becomes a homeomorphism.

(b) If (V, ψ) is another chart for M, then the transition map $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$ is a diffeomorphism.

(c) TU and TV induce the same topology on $T(U \cap V)$. In particular, if W is open in TU and X is open in TV, then $W \cap X = (W \cap T(U \cap V)) \cap (X \cap T(U \cap V))$ is open in $T(U \cap V)$.

In particular, (b) would allow us to get a C^∞ atlas for TM provided we can define a topology on TM by patching together the TU-topologies.

The Tangent Bundle as a Manifold

Reminder (Topological Bases; see Appendix A)

Let X be a topological space. A *basis for the topology of X* is a collection $\mathscr{B}$ of open sets such that, for every open $U \subseteq X$ and every $p \in U$, there is an open set $V \in \mathscr{B}$ such that $p \in V \subseteq U$.

Remark

If $\mathscr{B}$ is a basis for the topology of X, then every open set is the union of open sets in $\mathscr{B}$. We then say that $\mathscr{B}$ generates the topology of X.

Proposition (Proposition A.8)

Let X be a set and $\mathscr{B}$ a collection of subsets such that:

(i) $X = \cup_{V \in \mathscr{B}} V$.

(ii) *If $V_1, V_2 \in \mathscr{B}$ and $p \in V_1 \cap V_2$, then there is $W \in \mathscr{B}$ such that $p \in W \subseteq V_1 \cap V_2$.*

Then:

1. *$\mathscr{B}$ is a basis for a unique topology on X.*
2. *The open sets for this topology consists of unions of sets in $\mathscr{B}$.*

Remark

The condition (ii) holds automatically if $\mathscr{B}$ is closed under finite intersection.

The Tangent Bundle as a Manifold

Facts

Let $\mathscr{A} = \{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas of M (which defines its smooth structure). Define

$$\mathscr{B} = \bigcup_\alpha \{W;\ W \text{ is an open in } TU_\alpha\}.$$

Note that $TU_\alpha \in \mathscr{B}$.

- As $\cup U_\alpha = M$, we have
$$\bigcup_\alpha TU_\alpha = \bigcup_\alpha \bigsqcup_{p \in U_\alpha} T_pM = \bigsqcup_{p \in \cup U_\alpha} T_pM = \bigsqcup_{p \in M} T_pM = TM.$$
- If W_α is an open in TU_α and W_β is an open in TU_β, then $W_1 \cap W_2$ is open in $T(U_\alpha \cap U_\beta)$, and hence is contained in $\mathscr{B}$.

It follows that $\mathscr{B}$ satisfies the conditions (i) and (ii) of Proposition A.8, and so it's a basis for a unique topology on TM.

The Tangent Bundle as a Manifold

Definition

The topology of TM is the topology generated by $\mathscr{B}$. The open sets are unions of sets in $\mathscr{B}$.

Remark

Each TU_α is open in TM, since it is contained in $\mathscr{B}$.

Proposition (Proposition 12.4)

As a topological space TM is Hausdorff.

Remark

Each open TU_α is Hausdorff since it is homeomorphic to the open set $\phi_\alpha(U) \times \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n$. This can be used to show that TM is Hausdorff.

The Tangent Bundle as a Manifold

Proposition (Proposition 12.3)

The topology of TM is second countable.

Remarks

- It can be shown that the topology of M admits a countable basis $\{U_i\}_{i\in I}$ consisting of domains of charts (*cf.* Lemma 12.2 of Tu's book).
- Each TU_i is second countable since it is homeomorphic to an open in $\mathbb{R}^n \times \mathbb{R}^n$.
- If for each $i \in I$, we let $\{W_{i,j}\}_{j\in\mathbb{N}}$ be a countable basis for the topology of TU_i, then $\{W_{i,j};\, i \in I,\ j \in \mathbb{N}\}$ is a countable basis for the topology of TM (see Tu's book).

The Tangent Bundle as a Manifold

Facts

- Each TU_α is an open in TM.
- We knows that $TM = \cup_\alpha TU_\alpha$.
- The local trivializations $\tilde{\phi}_\alpha : TU_\alpha \to \phi_\alpha(U_\alpha) \times \mathbb{R}^n$ are homeomorphisms onto open sets in $\mathbb{R}^n \times \mathbb{R}^n$.
- All the transition maps $\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}$ are smooth.

Proposition

The collection $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$ is a C^∞ atlas for TM, and hence TM is a smooth manifold of dimension $2n$.

Remark

If $\{(V_\beta, \psi_\beta)\}$ is any C^∞-atlas for M, then we also get a C^∞ atlas $\{(TV_\beta, \tilde{\psi}_\beta)\}$ for TM. It is compatible with the atlas $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$, and so it defines the same smooth structure.

The Tangent Bundle as a Manifold

Facts

- The canonical map $\pi : TM \to M$ is such that $\pi(v) = p$ if $v \in T_pM$. It is onto.
- Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart for M. Then

$$\begin{aligned}
&\phi \circ \pi \circ \tilde{\phi}^{-1}(r^1, \ldots, r^n, v^1, \ldots, v^n) \\
&\qquad = \phi \circ \pi \left[\phi^{-1}(r^1, \ldots, r^n), \sum v^i \partial / \partial x^i \right] \\
&\qquad\qquad = \phi \circ \phi^{-1}(r^1, \ldots, r^n) = (r^1, \ldots, r^n).
\end{aligned}$$

- As (U, ϕ) and $(TU, \tilde{\phi})$ are charts this shows that π is C^∞.
- By the converse of the submersion theorem (exercise!) this also shows that π is a submersion.

Proposition

The canonical projection $\pi : TM \to M$ is a surjective submersion.