Differentiable Forms in Algebraic Topology
Submanifolds

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Submanifolds

Definition (Regular Submanifold)

Given a manifold N of dimension n, a subset $S \subset N$ is called a *regular submanifold* of dimension k if, for every $p \in S$, there is a chart $(U, x^1, \ldots, x^n)$ about p in N such that

$$U \cap S = \left\{ q \in U; x^{k+1}(q) = \cdots = x^n(q) = 0 \right\}.$$

Remarks

1. A chart $(U, x^1, \ldots, x^n)$ as above is called an *adapted chart* relative to S.
2. We call $n - k$ the *codimension* of S.
3. We always assume that S is equipped with the induced topology.
4. There are other types of submanifolds. By a submanifold we shall always mean a regular submanifold.

Remark

Let $S \subset N$ be a regular submanifold of dimension k, and $(U, \phi) = (U, x^1, \ldots, x^n)$ be an adapted chart relative to S.

- We have $\phi = (x^1, \ldots, x^k, 0, \ldots, 0)$ on $U \cap S$.
- Define $\phi_S : U \cap S \to \mathbb{R}^k$ by

$$\phi_S(q) = \big(x^1(q), \ldots, x^k(q)\big), \qquad q \in U \cap S.$$

  Then ϕ_S is a homeomorphism from $U \cap S$ onto its image
- Let $(r^1, \ldots, r^n)$ be the coordinates in $\mathbb{R}^n$. We have

$$\phi_S(U \cap S) \times \{0\}^{n-k} = \phi(U \cap S) = \phi(U) \cap \{r^{k+1} = \cdots = r^n = 0\}.$$

  Thus, $\phi_S(U \cap S) \times \{0\}^{n-k}$ is open in $\mathbb{R}^k \times \{0\}^{n-k}$, and so $\phi_S(U \cap S)$ is an open in $\mathbb{R}^k$.
- It then follows that $(U \cap S, \phi_S)$ is a (continuous) chart for S.

Submanifolds

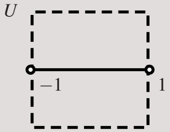
Example

Any open set $U \subset N$ is a regular submanifold of codimension 0.

Example

- The open interval $S = (-1, 1)$ on the x-axis is a regular submanifold of dimension 1 of the xy-plane.
- An adapted chart is (U, x, y), with $U = (-1, 1) \times (-1, 1)$, since

$$U \cap \{y = 0\} = (-1, 1) \times \{0\} = S.$$



Submanifolds

Facts

Let $(U, \phi) = (U, x^1, \ldots, x^n)$ and $(V, \psi) = (V, y^1, \ldots, y^n)$ be adapted charts relative to S about a point $p \in S$. Denote by $(r^1, \ldots, r^n)$ the coordinates in $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$.

- On $U \cap V \cap S$ we have

$$\phi = (x^1, \ldots, x^k, 0, \ldots, 0) = (\phi_S, 0, \ldots, 0),$$
$$\psi = (y^1, \ldots, y^k, 0, \ldots, 0) = (\psi_S, 0, \ldots, 0).$$

- Thus, on $\phi(U \cap V \cap S) = \phi_S(U \cap V \cap S) \times \{0\}^{n-k}$ we have $\psi \circ \phi^{-1}(r^1, \ldots, r^k, 0, \ldots, 0) = (\psi_S \circ \phi_S^{-1}(r^1, \ldots, r^k), 0, \ldots, 0).$

- As $\psi \circ \phi^{-1} = (y^1 \circ \phi^{-1}, \ldots, y^n \circ \phi^{-1})$, we get

$$\psi_S \circ \phi_S^{-1} = (z^1, \ldots, z^k), \quad \text{where } z^i = y^i \circ \phi^{-1}(r^1, \ldots, r^k, 0, \ldots, 0).$$

In particular, the transition map $\psi_S \circ \phi_S^{-1}$ is smooth.

Submanifolds

Proposition (Proposition 9.4)

Let S be a regular submanifold of dimension k in a manifold N of dimension n. Let $\{(U, \phi)\}$ be a collection of adapted charts relative to S that covers S. Then:

1. *The collection $\{(U \cap S, \phi_S)\}$ is a C^∞ atlas for S.*
2. *S is a manifold of dimension k.*

Remark

It can be shown that the differentiable structure on S defined above is unique, i.e., it does not depend on the choice of the collection $\{(U, \phi)\}$.

Level Sets of a Function

Definition

- Given $F : N \to M$ and $c \in M$, the preimage $F^{-1}(c)$ is called a *level set* of level c.
- When $M = \mathbb{R}^m$ we call $F^{-1}(0)$ the *zero set* of F and denote it by $Z(F)$.

Reminder

If $F : N \to M$ is a smooth map, then we say that c is a regular value when, either $c \notin F(M)$, or for every point $p \in F^{-1}(c)$ the differential $F_{*,p} : T_pM \to T_cN$ is onto.

Level Sets of a Function

Definition

Let $F : N \to M$ be a smooth map, and let $c \in M$.

- If c is a regular value, then $F^{-1}(c)$ is called a *regular level set*.
- If $M = \mathbb{R}^m$ and 0 is a regular value, then we say that $Z(F)$ is a *regular zero set*.

Remark

Let $f : N \to \mathbb{R}$ be a smooth function.

- If $p \in N$, then $f_{*,p} : T_pM \to T_{f(p)}R \simeq \mathbb{R}$ is onto if and only if it is non-zero.
- If $c \in f(M)$, then $f^{-1}(c)$ is a regular level set if and only if $f_{*,p} \neq 0$ for all $p \in f^{-1}(c)$.

Level Sets of a Function

Example (Example 9.6; the 2-sphere in $\mathbb{R}^3$)

- The unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ is the zero set of the function,

$$f(x, y, z) = x^2 + y^2 + z^2 - 1.$$

- For every $p = (x, y, z) \in \mathbb{S}^2$ we have

$$\left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p) \right) = (2x, 2y, 2z) \neq (0, 0, 0).$$

Therefore, $\mathbb{S}^2$ is a regular zero set.

Example (The 2-sphere in $\mathbb{R}^3$; continued)

- Suppose that $p = (x(p), y(p), z(p))$ is such that $x(p) \neq 0$.
- The Jacobian matrix of $F(x, y, z) = (f(x, y, z), y, z)$ is

$$J(F) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $J(F)(p)$ is invertible, since $x(p) \neq 0$.
- By the inverse function theorem, there is an open U about p such that F is a diffeomorphism from U onto its image.
- Thus, $(U, F_{|U}) = (U, f_{|U}, y_{|U}, z_{|U})$ is a chart about p in $\mathbb{R}^3$.

Level Sets of a Function

Example (The 2-sphere in $\mathbb{R}^3$; continued)

- Set $u^1 = y_{|U}$, $u^2 = z_{|U}$, and $u^3 = f_{|U}$. Then (U, u^1, u^2, u^3) is a chart about p in $\mathbb{R}^3$, and we have

$$\{u^3 = 0\} = \{f_{|U} = 0\} = U \cap \{f = 0\} = U \cap \mathbb{S}^2.$$

  Thus, (U, u^1, u^2, u^3) is an adapted chart relative to $\mathbb{S}^2$.
- Similarly, if $y(p) \neq 0$ or $z(p) \neq 0$, then there is an adapted chart about p.
- Therefore, $\mathbb{S}^2 \subset \mathbb{R}^3$ is a regular submanifold of codimension 1.

More generally, we have the following result:

Theorem (Theorem 9.8)

Let $g : N \to \mathbb{R}$ be a smooth function. Any non-empty regular level set $g^{-1}(c)$ is a regular submanifold of codimension 1.

Remark

A codimension 1 submanifold is called a *hypersurface*.

Example (Example 9.11)

Let S be the solution set of $x^3 + y^3 + z^3 = 1$ in $\mathbb{R}^3$.

- S is the zero set of $f(x, y, z) = x^3 + y^3 + z^3 - 1$.
- If $p = (x, y, z) \in S$, then

$$\left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p)\right) = (3x^2, 3y^2, 3z^2) \neq 0.$$

  Thus, every $p \in S$ is a regular point.
- Therefore, S is a regular zero set, and so this is a regular hypersurface.

Example (Example 9.13; Special Linear Group)

- Let $\mathbb{R}^{n\times n}$ be the vector space of $n \times n$ matrices with real entries. The general linear group is

$$\mathrm{GL}(n,\mathbb{R}) = \left\{A \in \mathbb{R}^{n\times n};\ \det A \neq 0\right\}.$$

This an open set in $\mathbb{R}^{n\times n}$, and so this is a manifold of dimension n^2.

- The *special linear group* is

$$\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R});\ \det A = 1\}.$$

This is the level set $f^{-1}(1)$ of the function $f(A) = \det A$.

Example (Special Linear Group, continued)

- If $A = [a_{ij}] \in \mathrm{GL}(n, \mathbb{R})$ and $m_{ij} = \det S_{ij}$ is the (i, j)-minor, then
$$\frac{\partial f}{\partial a_{ij}} = (-1)^{i+j} m_{ij}.$$
- If $A \in \mathrm{GL}(n, \mathbb{R})$, then at least one minor is non-zero, and so A is a regular point of f.
- In particular, every $A \in \mathrm{SL}(n, \mathbb{R})$ is a regular point.
- Therefore, $\mathrm{SL}(n, \mathbb{R})$ is a regular level set, and so this is a regular hypersurface in $\mathrm{GL}(n, \mathbb{R})$.

The Regular Level Set Theorem

Even more generally we have:

Theorem (Regular Level Set Theorem; Theorem 9.9)

Let $F : N \to M$ be a C^∞ map. Any non-empty regular level set $F^{-1}(c)$ is a regular submanifold of codimension equal to $\dim M$.

The Regular Level Set Theorem

Example (Example 9.12)

Let S be the solution set in $\mathbb{R}^3$ of the polynomial equations,

$$x^3 + y^3 + z^3 = 1, \quad x + y + z = 0.$$

- By definition S is the level set $F^{-1}(1,0)$, where $F : \mathbb{R}^3 \to \mathbb{R}^2$ is the smooth function given by

$$F(x,y,z) = \left(x^3 + y^3 + z^3, x + y + z\right).$$

- The Jacobian matrix of F is

$$J(F) = \begin{bmatrix} 3x^2 & 3y^2 & 3z^2 \\ 1 & 1 & 1 \end{bmatrix}.$$

  It has rank 2 unless $x^2 = y^2 = z^2$, i.e., $x = \pm y = \pm z$.
- For such a point $F(x,y,z) = \lambda(x^3, x) \neq (1,0)$, so all the points of S are regular points.
- Thus, S is a regular level set of F, and so this is a regular submanifold of codimension 2.