Noncommutative Geometry
Chapter 3:
Operators on Hilbert Space

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Operators on a Hilbert Space

Overview

This chapter is a review of basic results regarding operators on Hilbert space.

References for this Chapter

- Reed, M.; Simon, B.: *Methods of modern mathematical physics. I. Functional analysis*. Second edition. Academic Press, Inc., New York, 1980.

Notation

- $\mathcal{H}$ is a separable Hilbert space.
- $\mathcal{L}(\mathcal{H})$ is the C^*-algebra of bounded operators on $\mathcal{H}$.

Positive Operators

Definition

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be positive if can be written in the form $T = S^*S$ with $S \in \mathcal{L}(\mathcal{H})$.

Remark

We denote by $\mathcal{L}(\mathcal{H})_+$ the set of positive elements of $\mathcal{L}(\mathcal{H})$.

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$. TFAE:

(i) T *is positive.*

(ii) *There exists $S \in \mathcal{L}(\mathcal{H})$ selfadjoint such that $T = S^2$.*

(iii) T *is selfadjoint and $\operatorname{Sp} T \subset [0, \infty)$.*

(iv) $\langle T\xi|\xi\rangle \geq 0$ *for all $\xi \in \mathcal{H}$.*

Corollary

The set of positive operators $\mathcal{L}(\mathcal{H})_+$ *is a positive cone of* $\mathcal{L}(\mathcal{H})$*, i.e.,*

$$\lambda_1 T_1 + \lambda_2 T_2 \in \mathcal{L}(\mathcal{H})_+ \qquad \forall T_j \in \mathcal{L}(\mathcal{H})_+ \ \forall \lambda_j \geq 0.$$

Corollary

Let $T \in \mathcal{L}(\mathcal{H})$ *be normal. If* $f \in C(\mathrm{Sp}\, T)$ *is non-negative, then* $f(T)$ *is a positive operator.*

Absolute Value

Facts

- If $T \in \mathcal{L}(\mathcal{H})$, then T^*T is positive.
- Thus, it is selfadjoint and $\mathrm{Sp}(T^*T) \subset [0, \infty)$.
- By continuous functional calculus we then can define its square root $\sqrt{T^*T}$.
- We get a positive operator.

Definition

The operator $\sqrt{T^*T}$ is denoted $|T|$ and is called the absolute value of T.

Absolute Value

Lemma

Let $T \in \mathcal{L}(\mathcal{H})$*. The following hold:*

(i) $|T|$ *is the unique element of* $\mathcal{L}(\mathcal{H})_+$ *whose square is* T^*T.

(ii) *We have*

$$\| |T|\xi \| = \| T\xi \| \qquad \forall \xi \in \mathcal{H},$$

and hence $\ker |T| = \ker T$.

Polar Decomposition

Proposition (Polar Decomposition)

Let $T \in \mathcal{L}(\mathcal{H})$. Then there exists a unique $U \in \mathcal{L}(\mathcal{H})$, called the phase of T, such that

(i) $T = U|T|$.

(ii) $\ker U = \ker |T|$.

Definition

1. The operator U is called the phase of U.
2. The decomposition $T = U|T|$ is called the polar decomposition of T.

Polar Decomposition

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$ *have polar decomposition* $T = U|T|$ *and denote by* $\Pi_0(T)$ *(resp.* $\Pi_0(T^*)$*) the orthogonal projection onto* $\ker T$ *(resp.* $\ker T^*$*).*

1. *The range of* U *is* $\overline{\operatorname{ran} T}$.
2. *We have*
$$U^*U = 1 - \Pi_0(T) \qquad \textit{and} \qquad UU^* = 1 - \Pi_0(T^*)$$
Thus, U *is a partial isometry and has norm* 1 *unless* $T = 0$.
3. *If* T *is injective and has dense range, then* U *is unitary.*
4. *We have*
$$T| = U^*T, \qquad T^* = UTU^*, \qquad |T^*| = TU^* = U|T|U^*.$$
5. *The phase of* T^* *is* U^*.

The Spectral Theorem

Setup

- T = normal operator in $\mathcal{L}(\mathcal{H})$ (i.e., $T^*T = TT^*$).
- We set $S = \mathrm{Sp}(T)$.

Theorem (Spectral Theorem)

There exist a finite measure space (X, μ), a unitary operator $U : \mathcal{H} \to L^2_\mu(X)$, and a function $f \in L^\infty_\mu(X)$ such that

$$UTU^*\xi = f\xi \qquad \forall \xi \in L^2_\mu(X).$$

Definition

For $F \in L^\infty_\mu(X)$ the multiplication operator $M_F : L^2_\mu(X) \to L^2_\mu(X)$ is given by

$$M_F\xi = F\xi, \qquad \xi \in L^2_\mu(X).$$

Remark

The Spectral Theorem asserts that $UTU^* = M_f$.

Definition

The essential range of $F \in L^\infty_\mu(X)$, denoted $\text{ess-ran}(F)$, consists of all $\lambda \in \mathbb{C}$ such that $\mu(|F - \lambda| < \epsilon) > 0$ for all $\epsilon > 0$.

Fact

$\text{Sp}(M_F) = \text{ess-ran}(F)$.

Borel Functional Calculus

Fact

$L^\infty_\mu(X)$ is a unital commutative C^*-algebra with respect to the involution $F \to \overline{F}$ and the norm,

$$\|F\|_{L^\infty} = \{|\lambda|;\ \lambda \in \text{ess-ran}(|F|)\}, \qquad F \in L^\infty_\mu(X).$$

Proposition

The map $L^\infty_\mu(X) \ni F \to M_F \in \mathcal{L}(L^2_\mu(X))$ *is an isometric* $*$*-representation.*

Reminder

By the Spectral Theorem $UTU^* = M_f$, i.e, $T = U^* M_f U$.

Borel Functional Calculus

Lemma

(i) $\text{ess-ran}(f) = S$. *(Here $S = \text{Sp}(T)$.)*

(ii) $g(T) = U^* T_{g\circ f} U$ *for all* $g \in C(S)$.

Proof.

- $\text{ess-ran}(f) = \text{Sp}(M_f) = \text{Sp}(U^* T U) = \text{Sp}(T) = S$.
- As $T = U^* M_f U$, if $g = z^m \bar{z}^n$, then
$$g(T) = T^m (T^*)^n = (U^* M_f U)^m (U^* M_f^* U)^m = U^* M_f^m (M_f^*)^n U.$$
- Here $M_f^m (M_f^*)^n = M_{f^m} M_{\bar{f}^n} = M_{f^m \bar{f}^n} = M_{g\circ f}$. Thus,
$$g(T) = U^* M_{g\circ f} U.$$
- By linearity this holds for any polynomial $g(z) = \sum c_{m,n} z^m \bar{z}^n$.
- By density of those polynomials and continuity of the continuous functional calculus, this holds for all $g \in C(S)$. □

Borel Functonial Calculus

Definition

If $g \in L^\infty(S)$, then the operator $g(T) \in \mathcal{L}(\mathcal{H})$ is given by

$$g(T) := U^* M_{g \circ f} U.$$

Theorem (Borel Functional Calculus)

1. *The map $g \to g(T)$ is a $*$-homomorphism from $L^\infty(S)$ to $\mathcal{L}(\mathcal{H})$ such that*

$$\|g(T)\| \le \|g\|_{L^\infty(S)} \qquad \forall g \in L^\infty(S).$$

2. *If $(g_n)_{n \ge 0}$ is a bounded sequence in $L^\infty(S)$ such that $g_n \to g$ a.e., then $g_n(T) \to g(T)$ strongly (i.e., $g_n(T)\xi \to g(T)\xi$ for all $\xi \in \mathcal{H}$).*
3. *If $g \in L^\infty(S)$ is real-valued (resp., non-negative), then $g(T)$ is selfadjoint (resp., positive).*

Unbounded Operators

Definition

- An (unbounded) operator on $\mathcal{H}$ is a linear operator $T : D(T) \to \mathcal{H}$, where the domain $D(T)$ is a subspace of $\mathcal{H}$.
- It is called densily defined if $D(T)$ is a dense subspace of $\mathcal{H}$.

Definition

- The *graph* of an operator T is
$$G(T) = \{(\xi, \eta) \in \mathcal{H} \oplus \mathcal{H};\ \eta = T\xi\}.$$
- We say that T is closed if $G(T)$ is closed.
- We say that T is closable if there is a (closed) operator $\overline{T}$ such that $\overline{G(T)} = G(\overline{T})$.
- The operator $\overline{T}$ is called the closure of T.

Remark

- An operator S is called an extension of T, and we write $T \subset S$ if $G(T) \subset G(S)$.
- Equivalently, $D(T) \subset D(S)$ and $S = T$ on $D(T)$.

Definition

If T is densily defined, then its *adjoint* is the operator T^* given by the graph

$$G(T^*) = \{(\xi, \eta) \in \mathcal{H} \oplus \mathcal{H};\ \langle T\zeta|\xi\rangle = \langle \zeta|\eta\rangle\ \ \forall \zeta \in D(T)\}\,.$$

Remark

As $G(T^*)$ is a closed subspace, we see that T^* is always a closed operator.

Definition

Let T be densily defined. We say that:

- T is selfadjoint if $T^* = T$.
- T is symmetric if $T \subset T^*$.
- T is essentially selfadjoint if T is symmetric and closable.

Remarks

- T is symmetric if and only if $\langle T\xi|\eta\rangle = \langle\xi|T\eta\rangle \;\; \forall\xi,\eta \in D(T)$.
- T is selfadjoint iff T is symmetric and $D(T) = D(T^*)$.
- If T is selfadjoint, then it is closed.
- If T is essentially selfadjoint, then T^* is the closure of T.
- A symmetric operator may have several different selfadjoint extensions.

Proposition

Let T be symmetric. TFAE:

1. T *is selfadjoint.*
2. T *is closed and* $\ker(T^* \pm i) = \{0\}$.
3. $\operatorname{ran}(T \pm i) = \mathcal{H}$.

Corollary

Let T be symmetric. TFAE:

1. T *is essentially selfadjoint.*
2. $\ker(T^* \pm i) = \{0\}$.
3. $\operatorname{ran}(T \pm i)$ *is dense.*

Unbounded Operators

Definition

Suppose that T is closed.

- The *resolvent set* of T consists of all $\lambda \in \mathbb{C}$ such that
  (i) $T - \lambda : D(T) \to \mathcal{H}$ is a bijection.
  (ii) The inverse $(T - \lambda)^{-1} : \mathcal{H} \to D(T)$ is bounded.
- The *spectrum* of T, denoted $\mathrm{Sp}(T)$, is the complement of the resolvent set.

Proposition

1. $\mathrm{Sp}(T)$ *is (possibly empty) closed subset of* $\mathbb{C}$.
2. *The resolvent map* $\mathbb{C} \setminus \mathrm{Sp}(T) \to (T - \lambda)^{-1} \in \mathcal{L}(\mathcal{H})$ *is analytic.*

Remark

If T is selfadjoint, then $\mathrm{Sp}(T) \subset \mathbb{R}$.

Setup

T = selfadjoint operator on $\mathcal{H}$.

Theorem (Spectral Theorem;)

There exist a measured space (X, μ) *with* $\mu(X) < \infty$, *a unitary operator* $U : \mathcal{H} \to L^2_\mu(X)$, *and a measurable real-valued function* f *on* X *such that*

$$U(D(T)) = \left\{\xi \in L^2_\mu(X);\ f\xi \in L^2_\mu(X)\right\},$$

$$UTU^*\xi = f\xi \qquad \forall \xi \in U(D(T)).$$

Definition

If g is any bounded Borel function on $\mathbb{R}$, then $g(T)$ as the bounded operator on $\mathcal{H}$ defined by

$$g(T) := U^* M_{g \circ f} U.$$

Remark

If g is Borel and bounded, then $g \circ f \in L^\infty_\mu(X)$, and so $M_{g \circ f}$ is a bounded operator.

Theorem (Borel Functional Calculus)

1. *The map* $g \to g(T)$ *is a* $*$*-homomorphism from* $L^\infty(\mathbb{R})$ *to* $\mathcal{L}(\mathcal{H})$ *such that*

$$\|g(T)\| \leq \|g\|_{L^\infty(\mathbb{R})} \qquad \forall g \in L^\infty(\mathbb{R}).$$

2. *If* $(g_n)_{n\geq 0}$ *is a bounded sequence in* $L^\infty(\mathbb{R})$ *such that* $g_n \to g$ *a.e., then* $g_n(T) \to g(T)$ *strongly.*
3. *If* $g \in L^\infty(\mathbb{R})$ *is real-valued (resp., non-negative), then* $g(T)$ *is selfadjoint (resp., positive).*

Unbounded Operators

Remark

If g is a possibly unbounded Borel function on $\mathbb{R}$, then $g(T)$ makes sense as an unbounded operator as follows.

- The domain of $g(T)$ is

$$D(g(T)) = \left\{\xi\mathcal{H};\ (g\circ f)(U\xi) \in L^2_\mu(X)\right\},$$

- On $D(T)$ we have

$$g(T)\xi := U^*\left((g\circ f)U\xi\right), \qquad \xi \in D(g(T)).$$

- For instance, if $g(t) = t$, then $g(T) = T$.
- If (g_n) is a sequence of Borel functions on $\mathbb{R}$ such that $g_n \to g$ and $|g_n| \leq |g|$, then

$$g_n(T)\xi \longrightarrow g(T)\xi \qquad \forall \xi \in D(g(T)).$$

Example

Let $\Delta = -(\partial_{x_1}^2 + \ldots + \partial_{x_n}^2)$ be the (positive) Laplacian on $\mathbb{R}^n$.

- This is a selfadjoint operator on $\mathbb{R}^n$ with domain
$$D(\Delta) = \left\{u \in L^2(\mathbb{R}^n); (1 + |\xi|^2)|\hat{u}(\xi) \in L^2(\mathbb{R}^n)\right\}.$$
Here $\hat{u}$ is the Fourier transform.
- The Fourier transform $U : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a unitary operator (with the suitable normalization).
- As $(\Delta u)^\wedge = |\xi|^2 \hat{u}$ we see that
$$\Delta = U^* M_{|\xi|^2} U.$$
- If $g \in L^\infty(\mathbb{R})$, then $g(T) = U^* M_{g(|\xi|^2)} U$. That is,
$$g(T)u = \left(g(|\xi|^2)\hat{u}\right)^\vee \qquad \forall u \in L^2(\mathbb{R}^n).$$
Here $v \to \check{v}$ is the inverse Fourier transform.

Compact Operators

Definition

- An operator $T \in \mathcal{L}(\mathcal{H})$ is compact if the image by T of the unit ball $B(0,1)$ is precompact.
- The set of compact operators is denoted $\mathcal{K}$.

Example

Every finite rank operator is compact.

Proof.

Let $T \in \mathcal{L}(\mathcal{H})$ have finite rank, i.e., $\dim \operatorname{ran}(T) < \infty$.

- By Riesz Theorem any bounded subset of $\operatorname{ran}(T)$ is precompact.
- Here $T(B(0,1))$ is a bounded subset of $\operatorname{ran}(T)$.
- Therefore, it is precompact, and hence T is compact. □

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$. *TFAE:*

1. T *is a compact operator.*
2. *For any bounded sequence* $(\xi_n)_{n\geq 0} \subset \mathcal{H}$ *there is a subsequence* $(\xi_{n_k})_{k\geq 0}$ *such that the sequence* $(T\xi_{n_k})_{k\geq 0}$ *converges in norm.*
3. *For any sequence* $(\xi_n)_{n\geq 0} \subset \mathcal{H}$ *converging weakly to* 0 *the sequence* $(T\xi_n)_{n\geq 0}$ *converges to* 0 *in norm.*
4. *There is an orthonormal basis* $(\xi_n)_{n\geq 0}$ *of* $\mathcal{H}$ *such that*
$$\lim_{N\to\infty} \| T_{|E_N^{\perp}} \| = 0, \quad E_N := \operatorname{Span}\{\xi_0, \ldots, \xi_{N-1}\}.$$
5. *For any orthonormal basis* $(\xi_n)_{n\geq 0}$ *of* $\mathcal{H}$, *we have*
$$\lim_{N\to\infty} \| T_{|E_N^{\perp}} \| = 0, \quad E_N := \operatorname{Span}\{\xi_0, \ldots, \xi_{N-1}\}.$$

Compact Operators

Proposition

$\mathcal{K}$ *is a closed two-sided ideal of* $\mathcal{L}(\mathcal{H})$.

Corollary

Let $T \in \mathcal{L}(\mathcal{H})$. *TFAE:*

1. T *is compact.*
2. T *is the norm limit of finite-rank operators.*

Proof.

- Any finite-rank operator is compact and $\mathcal{K}$ is closed.
- Thus, any norm-limit of finite-rank operators is compact.
- If T is compact and $(\xi_n)_{n\geq 0}$ is any orthonormal basis of $\mathcal{H}$, then

$$\lim_{N\to\infty} \| T_{|E_N^\perp} \| = 0, \quad E_N := \text{Span}\{\xi_0, \ldots, \xi_{N-1}\}.$$

- Let Π_N be the orthogonal projection onto E_N.
- Then $T_N := T\Pi_N$ has finite rank, and

$$\| T_N - T \| = \| T(1 - \Pi_N) \| = \| T_{|E_N^\perp} \| \to 0.$$

- Thus, T is the norm-limit of finite-rank operators. □

Corollary

Let $T \in \mathcal{L}(\mathcal{H})$. *Then*

$$T \in \mathcal{K} \Longleftrightarrow T^* \in \mathcal{K} \Longleftrightarrow |T| \in \mathcal{K}.$$

Proof.

- $\mathcal{K}$ is an ideal, and so $T \in \mathcal{K} \Rightarrow ATB \in \mathcal{K}\ \forall A, B \in \mathcal{L}(\mathcal{H})$.
- By polar decomposition $T = U|T|$, and so $|T| \in \mathcal{K} \Rightarrow T \in \mathcal{K}$.
- As $|T| = U^*T$, we have $|T| \in \mathcal{K} \Rightarrow T \in \mathcal{K}$.
- We have $T^* = |T|U^* = U^*TU^*$, and hence $T = UT^*U$.
- Thus, $T \in \mathcal{K} \Leftrightarrow T^* \in \mathcal{K}$.

□

Remark

The above result is true for any two-sided ideal of $\mathcal{L}(\mathcal{H})$.

Corollary

$\mathcal{K}$ is a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$, and hence is a C^*-algebra.

Proof.

- $\mathcal{K}$ is a closed two-sided ideal of $\mathcal{L}(\mathcal{H})$, and so this is a closed subalgebra.
- $\mathcal{K}$ is a $*$-subalgebra by the previous corollary.
- Thus, $\mathcal{K}$ is a C^*-subalgebra. □

Compact Operators. Spectrum

Theorem (Riesz-Schauder)

Let $T \in \mathcal{K}$. The following hold.

1. T *always contains* 0 *in its spectrum.*
2. *If* $\lambda \in \mathrm{Sp}\, T \setminus 0$, *then* λ *is an eigenvalue with finite multiplicity.*
3. $\mathrm{Sp}\, T$ *is either finite or consists of a sequence of complex numbers converging to* 0.

Notation

If $\xi, \eta \in \mathcal{H}$, then $|\xi\rangle\langle\eta| \in \mathcal{L}(\mathcal{H})$ is given by

$$(|\xi\rangle\langle\eta|)\zeta := \langle\eta|\zeta\rangle\,\xi, \qquad \zeta \in \mathcal{H}.$$

Remarks

- If $\xi \neq 0$ and $\eta \neq 0$, then $|\xi\rangle\langle\eta|$ has rank 1.
- If $\|\xi\| = 1$, then $|\xi\rangle\langle\xi|$ is the orthogonal projection onto $\mathbb{C}\xi$.

Theorem (Hilbert-Schmidt)

Let $T \in \mathcal{K}$ *be normal.*

1. T *diagonalizes in an orthonormal basis, i.e., there exists an orthonormal basis* $(\xi_n)_{n\geq 0}$ *of* $\mathcal{H}$ *and a sequence* $(\lambda_n)_{n\geq 0} \subset \mathbb{C}$ *converging to* 0 *such that*

$$T\lambda_n = \lambda_n \xi_n \qquad \forall n \geq 0.$$

2. T *is the sum of its Schmidt series,*

$$T = \sum \lambda_n |\xi_n\rangle\langle \xi_n|,$$

where the series converges in norm.

Compact Operators. Spectrum

Corollary (Borel Functional Calculus for Compact Operators)

Let $T \in \mathcal{K}$ be normal.

1. *For every bounded function f on $\mathrm{Sp}(T)$, we have*
$$f(T) = \sum_{n \geq 0} f(\lambda_n) |\xi_n \rangle\langle \xi_n|,$$
where the series converges strongly.
2. *$f(T)$ is a compact operator if and only*
$$\lim_{n \to \infty} f(\lambda_n) = 0.$$
Moreover, in this case the above series converges in norm.

Example

Suppose that $T \in \mathcal{K}$ is normal, and let $T = U|T|$ be the polar decomposition. Then

$$U = \sum_{\lambda_n \neq 0} |\lambda_n|^{-1}\lambda_n |\xi_n\rangle\langle\xi_n|, \qquad |T| = \sum_{n \geq 0} |\lambda_n||\xi_n\rangle\langle\xi_n|,$$

where the first series converges strongly and the second series converges in norm.