Noncommutative Geometry
Chapter 2:
Examples of Noncommutative Quotients

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Examples of Noncommutative Quotients

Overview

In what follows we're going to look at various examples of noncommutative spaces:

1. Crossed-product algebras.
2. Dual of a locally compact group.
3. Group actions on manifolds.
4. Noncommutative tori.

Crossed-Product Algebras

Setup

- $A = C^*$-algebra.
- $G =$ locally compact group.
- There is a continuous left-action $G \times A \ni (g, x) \to \alpha_g(x) \in A$.

Definition

A covariant representation of (A, G, α) in a Hilbert space $\mathcal{H}$ is a pair (π_A, π_G) such that:

- $\pi_A : A \to \mathcal{L}(\mathcal{H})$ is a $*$-representation of A in $\mathcal{H}$.
- $\pi_G : G \to \mathcal{L}(\mathcal{H})$ is a unitary representation of G in $\mathcal{H}$, i.e., $\pi(g)$ is unitary for all $g \in G$.
- For all $x \in A$ and $g \in G$,

$$\pi_G(g)\pi_A(x)\pi_G(g)^{-1} = \pi_A(\alpha_g(x)).$$

The covariant representation is called isometric if π_A is isometric.

Remark

Isometric covariant representations of (A, G, α) always exist.

Proof.

- By Gel'fand-Naimark Theorem there always exists an isometric $*$-representation $\pi : A \to \mathcal{L}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$.
- We then have an isometric covariant representation in the Hilbert space $L^2(G, \mathcal{H})$ with $\pi_A : A \to \mathcal{L}(L^2(G, \mathcal{H}))$ and $\pi_G : G \to \mathcal{L}(L^2(G, \mathcal{H}))$ given by

$$[\pi_A(x)\xi](h) := [\pi(\alpha_{h^{-1}}x)\xi](h), \quad x \in A,\ \xi \in L^2(G, \mathcal{H}),\ h \in G,$$
$$[\pi_G(g)\xi](h) := \xi(g^{-1}h), \quad g, h \in G,\ \xi \in L^2(G, \mathcal{H}).$$

- Here $L^2(G, \mathcal{H})$ is defined as the completion of $C_c(G, \mathcal{H})$ with respect to the inner product,

$$\langle \xi | \eta \rangle = \int_G \langle \xi(h) | \eta(h) \rangle_{\mathcal{H}} \, d\lambda(h), \qquad \xi, \eta \in C_c(G, \mathcal{H}),$$

where $\lambda(h)$ is the left-invariant Haar measure of G. □

Crossed-Product Algebras

Setup

$C_c(A, G)$ = algebra of continuous maps $f : G \to A$ with compact support.

Definition

- The convolution product of $C_c(A, G)$ is given by

$$(f_1 * f_2)(g) := \int_G f_1(h)\alpha_h\left[f_2(h^{-1}g)\right] d\lambda(g), \quad f_j \in C_c(G, A),\ g \in G.$$

- Its antilinear involution is given by

$$f^*(g) := \Delta(g)^{-1} f(g^{-1})^*, \qquad f \in C_c(G, A),\ g \in G,$$

where $\Delta(g)$ is the modular function of G so that

$$d\lambda(g^{-1}) = \Delta(g)^{-1} d\lambda(g).$$

Crossed-Product Algebras

Setup

(π_A, π_G) = covariant $*$-representation of (A, G, α) in some Hilbert space $\mathcal{H}$.

Fact

- *We define a* $*$*-representation* $\pi : C_c(G, A) \to \mathcal{L}(\mathcal{H})$ *by*

$$\pi(f) = \int_G \pi_A(f(g))\pi_G(g)d\lambda(g), \quad f \in C_c(G, A).$$

- *In particular, its range* $\pi(C_c(G, A))$ *is a* $*$*-subalgebra of* $\mathcal{L}(\mathcal{H})$.

Definition

The (reduced) crossed-product algebra of A by G, denoted $A \rtimes_r G$, is the closure of $\pi(C_c(G, A))$ in $\mathcal{L}(\mathcal{H})$.

Remark

$A \rtimes_r G$ is a C^*-algebra. Up to $*$-isomorphism it does not depend on the covariant representation (π_A, π_G).

Dual of a Locally Compact Group

Setup

- G = locally compact group.
- α = trivial action $G \times \mathbb{C} \to \mathbb{C}$, i.e., $\alpha_g(x) = x$.
- $\pi_G : G \to \mathcal{L}(L^2(G))$ is the left-regular representation,

$$[\pi_G(g)\xi]\,(h) = \xi(g^{-1}h), \qquad g, h \in G,\ \xi \in L^2(G).$$

Fact

If $\pi_0 : \mathbb{C} \to \mathcal{L}(L^2(G))$ is the trivial representation, then (π_0, π_G, α) is a covariant representation.

Dual of a Locally Compact Group

Fact

The corresponding $*$-representation $\pi : C_c(G) \to \mathcal{L}(L^2(G))$ is given by

$$\begin{aligned}[\pi(f)\xi](h) &= \int_G f(g)\,[\pi_G(g)\xi](h)d\lambda(g) \\ &= \int_G f(g)\xi(g^{-1}h)d\lambda(g) = f * \xi(h).\end{aligned}$$

Here $*$ is the convolution for functions on G.

Definition

The reduced C^*-algebra of G, denoted $C_r(G)$, is the closure of $\pi(C_c(G))$ in $\mathcal{L}(L^2(G))$.

Remark

$C_r(G) = \mathbb{C} \rtimes_r G$.

Dual of a Locally Compact Group

Assumption

G is Abelian.

Definition

- A character is any continuous group morphism $\chi : G \to \mathbb{S}^1$.
- The set of all characters is denoted $\hat{G}$ and is called the Pontryagin dual of G.

Remarks

- $\hat{G}$ is a group with respect to the pointwise product.
- It is locally compact with respect to the topology of convergence on compact sets.

Dual of a Locally Compact Group

Definition

The Fourier transform $F : L^1(G) \to C(\hat{G})$, $f \to \hat{f}$ is given by

$$\hat{f}(\chi) := \int_G f(g)\overline{\chi(g)}d\lambda(g), \quad f \in L^1(G),\ \chi \in \hat{G}.$$

Example

Let $G = \mathbb{R}$.

- The characters of $\mathbb{R}$ are of the form $\chi_x(t) = e^{ixt}$, $x \in \mathbb{R}$.
- If $f \in L^1(\mathbb{R})$, then

$$\hat{f}(\chi_x) = \int_{\mathbb{R}} f(t)\overline{\chi_x(t)}dt = \int_{\mathbb{R}} f(t)e^{-ixt}dt.$$

Therefore, we recover the usual Fourier transform on $\mathbb{R}$.

Dual of a Locally Compact Group

Proposition

1. *If* $f \in L^1(G)$, *then* $\hat{f} \in C_0(\hat{G})$ *and* $(f^*)^\wedge = \overline{\hat{f}}$
2. *If* $f_1, f_2 \in L^1(G)$, *then* $(f_1 * f_2)^\wedge = \hat{f}_1 \cdot \hat{f}_2$.
3. F *extends to an isometric isomorphism* $F : L^2(G) \to L^2(\hat{G})$.

Reminder

- The $*$-representation $\pi : C_c(G) \to \mathcal{L}(L^2(G))$ is given by $\pi(f)\xi = f * \xi$, $f \in C_c(G)$, $\xi \in L^2(G)$.
- By definition $C_r(G) = \overline{\pi(C_c(G))}$.

Proposition

Let $f \in C_c(G)$.

1. $[\pi(f)\xi]^\wedge = \hat{f}\hat{\xi}$ *for all* $\xi \in L^2(G)$.
2. $\|\pi(f)\| = \|\hat{f}\|_{C_0(\hat{G})}$. *In particular,* π *is one-to-one.*

Dual of a Locally Compact Group

Proof.

- $[\pi(f)\xi]^\wedge = [f * \xi]^\wedge = \hat{f}\hat{\xi}$.
- As the Fourier transform is an isometric isomorphism from $L^2(G)$ onto $L^2(\hat{G})$, we have

$$\|[\pi(f)\xi\|_{L^2(G)} = \|[\pi(f)\xi]^\wedge\|_{L^2(\hat{G})} = \|\hat{f}\hat{\xi}\|_{L^2(\hat{G})}.$$

- Thus,

$$\|\pi(f)\| = \sup_{\|\xi\|_{L^2(G)}=1} \|\pi(f)\xi\|_{L^2(G)} = \sup_{\|\xi\|_{L^2(G)}=1} \|\hat{f}\hat{\xi}\|_{L^2(\hat{G})}.$$

- Using once again the fact that the Fourier transform is an isometric isomorphism, we get

$$\|\pi(f)\| = \sup_{\|\eta\|_{L^2(\hat{G})}=1} \|\hat{f}\eta\|_{L^2(\hat{G})} = \|\hat{f}\|_{C_0(\hat{G})}.$$

- In particular $\|\pi(f)\| \Rightarrow \hat{f} = 0 \Rightarrow f = 0$, i.e., π is one-to-one. □

Consequence

- As $\pi : C_c(G) \to \mathcal{L}(L^2(G))$ is one-to-one there is a unique linear map $\hat{\phi} : \pi(C_c(G)) \to C_0(\hat{G})$ such that

$$\hat{\phi}(\pi(f)) = \hat{f} \qquad \forall f \in C_c(G).$$

- As π and $f \to \hat{f}$ are $*$-homomorphisms, $\hat{\phi}$ is $*$-homomorphism as well.

Reminder

By definition $C_r(G)$ is the closure of $\pi(C_c(G))$ in $\mathcal{L}(L^2(G))$.

Proposition

$\hat{\phi}$ *uniquely extends to an isometric* $*$*-isomorphism,*

$$\hat{\phi} : C_r(G) \xrightarrow{\sim} C_0(\hat{G})$$

Dual of a Locally Compact Group

Proof.

- $\hat{\phi}$ is isometric, since, for all $f \in C_c(G)$,

$$\left\|\hat{\phi}\left(\pi(f)\right)\right\| = \left\|\hat{f}\right\| = \left\|\pi(f)\right\|.$$

- Thus, $\hat{\phi}$ uniquely extends to an isometric $*$-homomorphism,

$$\hat{\phi} : C_r(G) \xrightarrow{\sim} C_0(\hat{G}).$$

- Its range is closed and contains $\hat{\phi}(\pi(C_c(G)))$ as a dense subspace.
- $\hat{\phi}(\pi(C_c(G)))$ is a $*$-subalgebra of $C_0(\hat{G})$ separating the points of $\hat{G}$, and so it's dense by Stone-Weierstrass Theorem.
- Thus, the range of $\hat{\phi}$ is all $C_0(\hat{G})$, and hence we have an isomorphism. □

Dual of a Locally Compact Group

Remarks

- If G is not Abelian, then the Pontryagin dual $\hat{G}$ is defined in terms of irreducible unitary representations.
- Its topology need not be Hausdorff, and point set topology cannot be used to get information on $\hat{G}$.
- However, the C^*-algebra $C_r^*(G)$ always makes sense and its representations are closely related to the unitary representations of G.
- This the main impetus for studying other C^*-algebra $C_r^*(G)$ to gain information on G and its unitary representations.

Group Actions on Manifolds

Setup

- $M =$ smooth manifold equipped with a smooth measure $\rho(x)$.
- $G =$ Lie group acting smoothly on M, i.e., we have a smooth map $G \times M \ni (g, x) \to g \cdot x \in M$.
- We then get a continuous action $\alpha : G \times C_0(M) \to C_0(M)$,

$$\alpha_g(f) = f(g^{-1} \cdot x), \quad f \in C_0(M),\ g \in G.$$

Definition

- The regular representation $\pi_1 : C_0(M) \to \mathcal{L}(L^2(M))$ is

$$\pi_1(f)\xi = f\xi, \qquad f \in C_0(M)\ \forall \xi \in L^2(M).$$

- The unitary representation $\pi_2 : G \to \mathcal{L}(L^2(M))$ is given by

$$\left[\pi_2(g)\xi\right](x) := \kappa_g(x)^{\frac{1}{2}} \xi(g^{-1}.x), \quad g \in G,\ \xi \in L^2(M),\ x \in M,$$

where $\kappa_g(x) = \frac{d\rho(g.x)}{d\rho(x)}$.

Proposition

The pair (π_1, π_2) is an isometric covariant representation of $(G, C_0(M), \alpha)$.

Facts

- We get an isometric $*$-representation π of $C_c(G, C_0(M))$ in $\mathcal{L}(L^2(M))$.
- If $f \in C_c(G \times M) \subset C_c(G, C_0(M))$, then

$$[\pi(f)\xi](x) = \int_G f(g,x)\xi(g^{-1}.x)\kappa_g(x)^{\frac{1}{2}} d\lambda(g), \quad \xi \in L^2(M),\ x \in M.$$

Proposition

The crossed-product algebra $G \rtimes_r C_0(M)$ is the closure of $\pi[C_c(G \times M)]$ in $\mathcal{L}(L^2(M))$.

Definition

- The action of G on M is called free if no $g \in G \setminus \{1\}$ has fixed points.
- It is called proper if $G \times M \ni (g, x) \to (x, g \cdot x) \in M \times M$ is a proper map.

Proposition

If the action is free and proper, then M/G is a smooth manifold and the canonical map $\pi : M \to M/G$ is a submersion.

Proposition

If the action of G on M is free and proper, then we have a strong Morita equivalence,

$$C_0(M) \rtimes_r G \underset{M.E.}{\simeq} C_0(M/G).$$

Remark

- Two algebras A and B are said to be Morita equivalent if there is an (A, B)-bimodule $\mathcal{M}_1$ and a (B, A)-bimodule $\mathcal{M}_2$ such that $\mathcal{M}_1 \otimes_B \mathcal{M}_2 \simeq A$ and $\mathcal{M}_2 \otimes_A \mathcal{M}_1 \simeq B$.
- Strong Morita equivalence is an analogous notion in the setting of C^*-algebras.
- Many key properties of C^*-algebras are preserved by Morita equivalence.

Remark

- If the action is not free or proper, then M/G need not be Hausdorff.
- However, the crossed-product algebra $C_0(M) \rtimes_r G$ always make sense.
- This is the impetus for using this C^*-algebra to extract info on the action of G on M.

Assumption

G is discrete and its action preserves the measure $\rho(x)$, that is, $\rho(g \cdot x) = \rho(x)$ (i.e., $\kappa_g(x) = 1$).

Notation

- For $f \in C_0(M)$ denote $\pi_1(f)$ by f and set $g \cdot f = \alpha_g(f)$, i.e., $(g \cdot f)(x) = f(g^{-1} \cdot x)$.
- For $g \in G$, set $U_g = \pi_2(g)$. This a unitary operator of $\mathcal{L}(L^2(M))$ such that $U_g^* = U_g^{-1} = U_{g^{-1}}$.

Proposition

$C_0(M) \rtimes_r G$ is the C^*-subalgebra of $\mathcal{L}(L^2(M))$ generated by the operators,

$$fU_g, \qquad f \in C_0(M),\ g \in G,$$

with relations,

$$U_g f = (g \cdot f) U_g.$$

Proof.

- For $g \in G$, let $\delta_g : G \to \mathbb{C}$ be such that
$$\delta_g(g) = 1, \qquad \delta_g(h) = 0, \ h \neq g.$$
- $C_c(G, C_0(M))$ is spanned by $f\delta_g$, $f \in C_0(M)$, $g \in G$.
- If $f \in C_0(M)$ and $g \in G$, then, for all $\xi \in L^2(M)$,
$$\begin{aligned}[\pi(f\delta_g)\xi](x) &= \int_G f(x)\delta_g(h)\xi(h^{-1}.x)d\lambda(h) \\ &= f(x)\xi(g^{-1}.x) = f(x)(U_g\xi)(x).\end{aligned}$$
That is, $\pi[f\delta_g] = fU_g$.
- Thus, $\pi(C_0(G, C_0(M))$ is spanned by the fU_g, and so they generate $C_0(M) \rtimes G$.
- Moreover,
$$(U_g f)\xi(x) = U_g(f\xi)(x) = f(g^{-1}{\cdot}x)\xi(g^{-1}{\cdot}x) = (g{\cdot}f)(x)(U_g\xi)(x).$$
That is, $U_g f = (g \cdot f)U_g$.

□

Noncommutative Tori

Setup

Given $\theta \in \mathbb{R}$, we let $\mathbb{Z}$ act on $\mathbb{S}^1$ by

$$k \cdot z := e^{-2ik\pi\theta} z, \qquad (k, z) \in \mathbb{Z} \times \mathbb{S}^1.$$

Proposition

If $\theta \notin \mathbb{Q}$, then the orbits are dense. In particular, the orbit space $\mathbb{S}^1/\mathbb{Z}$ is not Hausdorff.

Facts

- We represent $C(\mathbb{S}^1)$ by multiplication operators on $L^2(\mathbb{S}^1)$,

$$\pi_1(f)\xi = f\xi, \qquad f \in C(\mathbb{S}^1),\ \xi \in L^2(M).$$

- The action of $\mathbb{Z}$ on $\mathbb{S}^1$ yields an action of $\mathbb{Z}$ on $C(\mathbb{S}^1)$ by

$$\alpha_k(f) = f(e^{2ik\pi\theta}z), \qquad f \in C(\mathbb{S}^1),\ k \in \mathbb{Z}.$$

- We also have the unitary representation $\pi_{\mathbb{Z}} : \mathbb{Z} \to \mathcal{L}(L^2(\mathbb{T}))$,

$$\pi_2(k)\xi = \xi(e^{2ik\pi\theta}z), \qquad k \in \mathbb{Z},\ \xi \in L^2(\mathbb{T}).$$

Proposition

(π_1, π_2, α) *is an isometric covariant representation in* $L^2(\mathbb{S}^1)$.

Lemma

Define

$$U := \pi_1(z) \qquad \textit{and} \qquad V := \pi_2(1).$$

1. *U and V both are unitary operators on $L^2(\mathbb{S}^1)$.*
2. *They satisfy the relation,*

$$VU = e^{2i\pi\theta} UV.$$

Proof.

- As $\bar{z} = z^{-1}$ we have

$$V^* = \pi_1(\bar{z}) = \pi_1(z^{-1}) = \pi_1(z)^{-1} = U^{-1}.$$

  That is, U is a unitary operator.

- As $V = \pi_2(1)$ is unitary, since π_2 is a unitary representation.
- Given any $\xi \in L^2(\mathbb{S}^1)$, we have

$$UV\xi = z(V\xi) = z\xi(e^{2i\pi\theta}z),$$
$$VU\xi = V(z\xi) = e^{2i\pi\theta}z\xi(e^{2i\pi\theta}z) = e^{2i\pi\theta}UV\xi.$$

  That is, $UV = e^{2i\pi\theta}UV$.

□

Proposition

$C(\mathbb{S}^1) \rtimes_{r,\theta} \mathbb{Z}$ is the C^*-subalgebra of $\mathcal{L}(L^2(\mathbb{S}^1))$ generated by the unitaries U and V

Proof.

- As $\mathbb{Z}$ is discrete $C(\mathbb{S}^1) \rtimes_{r,\theta} \mathbb{Z}$ is generated by the operators,

$$\pi_1(f)\pi_2(k), \qquad f \in C(\mathbb{S}^1),\ k \in \mathbb{Z}.$$

- $\pi_2(k) = \pi_2(1)^k = V^k$ with $V^{-1} = V^*$.
- As a C^*-algebra $C(S^1)$ is generated by z with $\bar{z} = z^{-1}$.
- Thus $C(\mathbb{S}^1) \rtimes_{r,\theta} \mathbb{Z}$ is generated by $\pi_1(z) = U$ and V. □

Noncommutative Tori

Definition

The noncommutative torus $C(\mathbb{T}^2_\theta)$ is the universal C^*-algebra generated by unitaries U and V subject to the relation,

$$VU = e^{2i\pi\theta} UV.$$

Remarks

- Universal here means that any other C^*-algebra with unitary generators U and V satisfying the above relations is isomorphic to $C(\mathbb{T}^2_\theta)$.
- If $\theta \notin \mathbb{Q}$, then $C(\mathbb{T}^2_\theta) \simeq C(\mathbb{S}^2) \rtimes_{r,\theta} \mathbb{Z}$.

Remark

- If $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, then as a C^*-algebra $C(\mathbb{T}^2)$ is generated by the multiplication operators,

$$U = e^{2i\pi x} \quad \text{and} \quad V = e^{2i\pi y}.$$

- They are unitaries such that $UV = VU$.
- Thus,

$$C(\mathbb{T}^2_\theta) \simeq C(\mathbb{T}^2) \qquad \text{for } \theta = 0.$$

Higher Dimensional Noncommutative Tori

Setup

- $\theta = (\theta_{jk})$ real anti-symmetric $n \times n$-matrix.
- $\theta_1, \ldots, \theta_n$ column vectors of θ.

Definition

The noncommutative torus $C(\mathbb{T}^n_\theta)$ is the C^*-algebra generated by the unitary operators,

$$U_j : L^2(\mathbb{T}^n) \longrightarrow L^2(\mathbb{T}^n), \quad (U_j\xi)(x) = e^{ix_j}\xi(x + \pi\theta_j),$$

subject to the relations,

$$U_k U_j = e^{2i\pi\theta_{jk}} U_j U_k.$$

Remark

For $\theta = 0$ we get the C^*-algebra $C(\mathbb{T}^n)$ represented by multiplication operators.

Remark

For $n = 2$ we recover the previous definition under the correspondence,

$$\mathbb{R} \ni \theta \longleftrightarrow \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \in M_2(\mathbb{R}).$$