Differentiable Forms in Algebraic Topology
Constant Rank Theorem. Immersions and Submersions

Sichuan University, Spring 2025

Constant Rank Theorem

Reminder

Let N be a manifold of dimension n and M a manifold of dimension m.

- The rank at $p \in N$ of a smooth map $f : N \to M$ is the rank of its differential $f_{*,p} : T_pN \to T_{f(p)}M$.
- The rank is always $\leq \min(m, n)$.

Constant Rank Theorem

Theorem (Constant Rank Theorem; Theorem B.4)

Let $f : U \to \mathbb{R}^m$ be a C^∞ map, where $U \subseteq \mathbb{R}^n$ is open. Assume that f has constant rank k near $p \in U$. Then there are:

- *A diffeomorphism F from a neighborhood of p onto a neighborhood of $0 \in \mathbb{R}^n$ with $F(p) = 0$,*
- *A diffeomorphism G from a neighborhood of $f(p)$ onto a neighborhood of $0 \in \mathbb{R}^m$ with $G(f(p)) = 0$,*

in such a way that

$$G \circ f \circ F^{-1}(r^1, \ldots, r^n) = (r^1, \ldots, r^k, 0, \ldots, 0).$$

Remark

If $k = m$, then

$$\left(\psi \circ f \circ \phi^{-1}\right)(r^1, \ldots, r^n) = (r^1, \ldots, r^m).$$

Constant Rank Theorem

Theorem (Constant Rank Theorem for Manifolds; Theorem 11.1)

Suppose that M is a manifold of dimension m and N is a manifold of dimension n. Let $f : N \to M$ be a smooth map that has constant rank k near a point $p \in N$. Then, there are a chart (U, ϕ) centered at p in N and a chart (V, ψ) centered at $f(p)$ in M such that, for all $(r^1, \dots, r^n) \in \phi(U)$, we have

$$\left(\psi \circ f \circ \phi^{-1}\right)(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

Constant Rank Theorem

A consequence of the constant rank theorem is the following extension of the regular level set theorem (Theorem 9.9).

Theorem (Constant-Rank Level Set Theorem; Theorem 11.2)

Let $f : N \to M$ be a smooth map and $c \in M$. If f has constant rank k in a neighborhood of the level set $f^{-1}(c)$ in N, then $f^{-1}(c)$ is a regular submanifold of codimension k.

Remark

A neighborhood of a subset $A \subseteq N$ is an open set containing A.

Constant Rank Theorem

Example (Orthogonal group $O(n)$; Example 11.3)

The *orthogonal group* $O(n)$ is the subgroup of $GL(n, \mathbb{R})$ of matrices A such that $A^T A = I_n$ (identity matrix),

- This is the level set $f^{-1}(I_n)$, where $f : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$, $A \to A^T A$.
- It can be shown that f has constant rank (in fact it has rank $k = \frac{1}{2}n(n+1)$).
- Therefore, by the constant-rank level set theorem $O(n)$ is a regular submanifold of $GL(n, \mathbb{R})$ (of codimension $\frac{1}{2}n(n+1)$).

The Immersion and Submersion Theorems

Reminder

Suppose that M is a manifold of dimension m and N is a manifold of dimension n, and let $f : N \to M$ be a smooth map.

- f is an *immersion at* p if $f_{*,p} : T_pN \to T_{f(p)}M$ is injective.
- f is a *submersion at* p if $f_{*,p} : T_pN \to T_{f(p)}M$ is surjective.

Remark

Equivalently,

$$f \text{ is an immersion at } p \Longleftrightarrow \big(n \leq m \text{ and } \operatorname{rk} f_{*,p} = n\big),$$
$$f \text{ is a submersion at } p \Longleftrightarrow \big(n \geq m \text{ and } \operatorname{rk} f_{*,p} = m\big).$$

As we always have $\operatorname{rk} f_{*,p} \leq \min(m,n)$, we see that

$$f \text{ is an immersion/submersion at } p \Longleftrightarrow f_{*,p} \text{ has maximal rank.}$$

Fact (See Tu2011)

Maximal rank is an open property, i.e., if f_* has maximal rank at p, then it has maximal rank near p.

As a consequence we obtain:

Proposition (Proposition 11.4)

If a smooth map $f : N \to M$ is a immersion (resp., a submersion) at a point $p \in N$, then it is an immersion (resp., submersion) near p. In particular, it has constant rank near p.

The Immersion and Submersion Theorems

Combining the previous proposition with the Constant Rank Theorem gives the following result.

Theorem (Theorem 11.5)

Let $f : N \to M$ be a smooth map.

1. **Immersion Theorem**. *If f is an immersion at p, then there are a chart (U, ϕ) centered at p in N and a chart (V, ψ) centered at $f(p)$ in M such that near $\phi(p)$ we have*
$$\left(\psi \circ f \circ \phi^{-1}\right)(r^1, \ldots, r^n) = (r^1, \ldots, r^n, 0, \ldots, 0).$$
2. **Submersion Theorem**. *If f is a submersion at p, then there are a chart (U, ϕ) centered at p in N and a chart (V, ψ) centered at $f(p)$ in M such that near $\phi(p)$ we have*
$$\left(\psi \circ f \circ \phi^{-1}\right)(r^1, \ldots, r^m, r^{m+1}, \ldots, r^n) = (r^1, \ldots, r^m).$$

The Immersion and Submersion Theorems

Remark

- The submersion theorem implies that if $f : N \to M$ is a submersion then, for every $p \in N$, there are a chart $(U, x^1, \ldots, x^n)$ centered at p in N and a chart $(V, y^1, \ldots, y^m)$ centered at $f(p)$ in M relative to which f is such that
$$(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) \longrightarrow (x^1, \ldots, x^m).$$
- The projection $(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) \to (x^1, \ldots, x^m)$ is an open map (see Problem A.7). This implies that f maps any neighborhood of p onto a neighborhood of $f(p)$.
- As this is true for every $p \in N$, we see that f is an open map. Therefore, we obtain:

Corollary (Corollary 11.6)

Every submersion $f : N \to M$ is an open map.

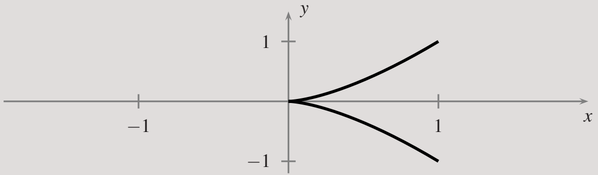
Images of Smooth Maps

Let us look at some examples of smooth maps $f : \mathbb{R} \to \mathbb{R}^2$.

Example (Example 11.7)

Let $f(t) = (t^2, t^3)$.

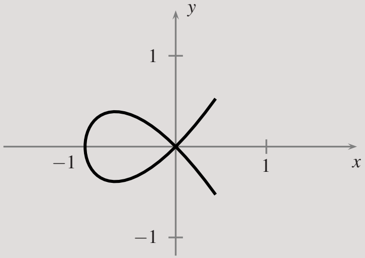
- This is a one-to-one map, since $t \to t^3$ is one-to-one.
- As $f'(0) = (0, 0)$ the differential $f_{*,0}$ is zero, and so f is not an immersion at 0.
- The image of f is the cuspidal cubic $y^2 = x^3$.



Example (Example 11.8)

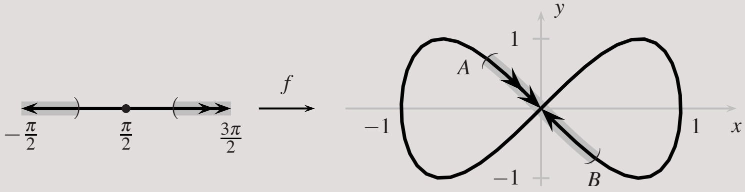
Let $f(t) = (t^2 - 1, t^3 - t)$.

- As $f'(t) = (2t, 3t^2 - 1) \neq (0,0)$ the differential f_* is one-to-one everywhere, and hence f is an immersion.
- However, f is not one-one since $f(1) = f(-1) = (0,0)$.
- The image of f is the nodal cubic $y^2 = x^2(x+1)$ (see Tu's book).



Example (The Figure-eight; Example 11.12)

Set $I = (-\pi/2, 3\pi/2)$, and let $f : I \to \mathbb{R}^2$, $t \to (\cos t, \sin 2t)$.



- $f'(t) = (-\sin t, 2\cos 2t) \neq (0,0)$, and so f is an immersion.
- f is one-to-one, and so f is a bijection onto its image $f(I)$.
- The inverse map $f^{-1} : f(I) \to I$ is not continuous: if $t \to (3\pi/2)^-$, then $f(t) \to (0,0) = f(\pi/2)$, but
$$f^{-1}(f(t)) = t \to 3\pi/2 \notin I.$$

In particular, $f : I \to f(I)$ is not a homeomorphism.

Image of Smooth Maps

Summary

As the previous examples show:

- A one-to-one smooth map need not be an immersion.
- An immersion need not be one-to-one.
- A one-to-one immersion need not be a homeomorphism onto its image.

Definition

A smooth map $f : N \to M$ is called an *embedding* if f is an immersion and a homeomorphism onto its image $f(N)$ with respect to the subspace topology.

Remark

A one-to-one immersion $f : N \to M$ is an embedding if and only if it is an open map.

Image of Smooth Maps

The importance of embeddings stems from the following result.

Theorem (Theorem 11.13)

If $f : N \to M$ is an embedding, then its image $f(N)$ is a regular submanifold in M.

This result admits the following converse:

Theorem (Theorem 11.14)

If N is a regular submanifold in M, then the inclusion $i : N \to M$ is an embedding.

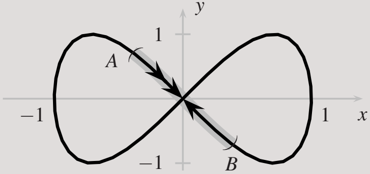
Image of Smooth Maps

Remarks

1. The images of smooth embeddings are called *embedded submanifolds*.
2. The previous two results show that the regular submanifolds and embedded submanifolds are the same objects.
3. The images of one-to-one immersions are called *immersed submanifolds*.

Example

The figure-eight is an immersed submanifold in $\mathbb{R}^2$ (but this is not a regular submanifold).



Smooth Maps into a Submanifold

Question

Suppose that $f : N \to M$ is smooth map such that $f(N)$ is contained in a given subset $S \subseteq M$. If S is manifold, then is the induced map $f : N \to S$ smooth as well?

Theorem (Theorem 11.15)

Suppose that $f : N \to M$ is a smooth map whose image is contained in a regular submanifold S in M. Then the induced map $f : N \to S$ is smooth.

Remarks

1. The above result does not hold if S is only an immersed submanifold (see Tu's book).
2. The converse holds. As S is a regular submanifold, the inclusion $i : S \to M$ is smooth. Thus, if $f : N \to S$ is a smooth map, then $i \circ f : N \to M$ is a C^∞ map that induces f.

Smooth Maps into a Submanifold

Example (Multiplication map of $\mathrm{SL}(n,\mathbb{R})$; Example 11.16)

$\mathrm{SL}(n,\mathbb{R})$ is the subgroup of $\mathrm{GL}(n,\mathbb{R})$ of matrices of determinant 1.

- This is a regular submanifold in $\mathrm{GL}(n,\mathbb{R})$ (Example 9.11), and so the inclusion $\iota : \mathrm{SL}(n,\mathbb{R}) \hookrightarrow \mathrm{GL}(n,\mathbb{R})$ is a smooth map.
- By Example 6.21 we have a smooth multiplication map,
$$\mu : \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) \longrightarrow \mathrm{GL}(n,\mathbb{R}).$$
- We thus get a smooth map,
$$\mu \circ (\iota \times \iota) : \mathrm{SL}(n,\mathbb{R}) \times \mathrm{SL}(n,\mathbb{R}) \longrightarrow \mathrm{GL}(n,\mathbb{R}).$$
- As it takes values in $\mathrm{SL}(n,\mathbb{R})$, and $\mathrm{SL}(n,\mathbb{R})$ is a regular submanifold in $\mathrm{GL}(n,\mathbb{R})$, we get a smooth multiplication map,
$$\mathrm{SL}(n,\mathbb{R}) \times \mathrm{SL}(n,\mathbb{R}) \longrightarrow \mathrm{SL}(n,\mathbb{R}).$$

Smooth Maps into a Submanifold

Theorem 11.5 and its converse are especially useful when $M = \mathbb{R}^m$. In this case we have:

Corollary

Let S be a regular submanifold in $\mathbb{R}^m$ and $f : N \to \mathbb{R}^m$ a map such that $f(N) \subseteq S$. Set $f = (f^1, \dots, f^m)$. Then TFAE:

(i) *f is smooth as a map from N to S.*

(ii) *f is smooth as a map from N to $\mathbb{R}^m$.*

(iii) *The components $f^1, \dots, f^m$ are smooth functions on N.*