Sichuan University

Differential Forms in Algebraic Topology – Spring 2024

Homeworks 1–4

Due date: **July 7, 2024**

Homework 5

Problem 1. Suppose that $E \to M$ is a smooth vector bundle and $\nabla : \mathscr{X}(M) \times \Gamma(E) \to \Gamma(E)$ is a connection on E. Show that its curvature

$$R : \mathscr{X}(M) \times \mathscr{X}(M) \times \Gamma(E) \ni (X, Y, s) \longrightarrow R(X, Y)s \in \Gamma(E)$$

is $C^\infty(M)$-linear in X, Y, and s.

Problem 2. Suppose that M is a smooth manifold and $\nabla : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ is a connection on M.

(1) Show that the torsion

$$\mathscr{X}(M) \times \mathscr{X}(M) \ni (X, Y) \longrightarrow T(X, Y) \in \mathscr{X}(M)$$

is $C^\infty(M)$-linear in X and Y.

(2) Assume that M has a Riemannian metric $\langle \cdot, \cdot \rangle$ and ∇ is the Levi-Civita connection. Let $(X, Y, Z) \to R(X, Y)Z$ be it curvature. It can be shown (see Tu2017) that the curvature has the following symmetries:

$$R(X, Y) = -R(Y, X),$$
$$\langle R(X, Y)Z, W \rangle = -\langle R(X, Y)W, Z \rangle,$$
$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0.$$

The last equalities is the vector form of the 1st Bianchi identity (see Tu2017). Use these properties to establish the 4th symmetry,

$$\langle R(X, Y)Z, W \rangle = \langle R(Z, W)X, Y \rangle.$$