Sichuan University

Differential Forms in Algebraic Topology – Spring 2024

Homeworks 1–4

Due date: **June 22, 2024**

Homework 1

Problem 1 (Grassmannian manifolds)**.** For $n \geq 1$ and $k = 1, \ldots, n$, the Grassmannian $G(n,k)$ is the set of all linear subspaces of $\mathbb{R}^n$ that have dimension k. Problem 7.8 in Tu2011 is about showing that $G(n,k)$ is a smooth manifold. Parts (a)–(d) of this problem show that $G(n,k)$ naturally inherits a quotient topology which is Hausdorff and 2nd countable. Assuming this result do the remaining parts (e)–(h) on the construction of a C^∞-atlas for $G(n,k)$.

Homework 2

Problem 2 (Cohomology of multi-punctured planes)**.**

(1) Let p and q be distinct points of $\mathbb{R}^n$. Compute the de Rham cohomology of $\mathbb{R}^n \setminus \{p, q\}$.
(2) Let $p_1, \ldots, p_m$ be distinct points in $\mathbb{R}^n$ compute the the de Rham cohomology of $\mathbb{R}^n \setminus \{p_1, \ldots, p_m\}$.

Problem 3. Recall that any open interval $I \subseteq \mathbb{R}$ is diffeomorphic to $\mathbb{R}$. Therefore, any product of n open intervals in $\mathbb{R}$ is diffeomorphic to $\mathbb{R}^n$.

(1) Find a finite good cover for $\mathbb{R}^2 \setminus \{0\}$. Deduce from this that $\mathbb{R}^2 \setminus 0$ satisfies Poincaré duality.
(2) Compute the compactly supported de Rham cohomology of $H_c^2(\mathbb{R}^2 \setminus 0)$.
(3) Let $p_1, \ldots, p_m$ be distinct points in $\mathbb{R}^2$ compute the compactly supported de Rham cohomology $\mathbb{R}^2 \setminus \{p_1, \ldots, p_m\}$.

Homework 3

Problem 4. Let M and N be smooth manifolds. The goal of this problem is to show that smooth homotopy is an equivalence relation on smooth maps from M to N. We let $\psi : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\psi(t) = \frac{\varphi(t)}{\varphi(t) + \varphi(1-t)}, \qquad \text{where } \varphi(t) = \begin{cases} e^{-1/t} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

We saw in the section on bump functions that φ and ψ are C^∞-functions, and we have

$$\psi^{-1}(0) = (-\infty, 0], \qquad 0 \leq \psi \leq 1, \qquad \psi^{-1}(1) = [1, \infty).$$

(1) Show that smooth homotopy relation is reflexive and symmetric.

(2) Let $f, g : M \to N$ be smooth maps and $F : M \times \mathbb{R} \to N$ a smooth homotopy such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in M$. Define $\tilde{F} : M \times \mathbb{R} \to N$ by

$$\tilde{F}(x,t) = F\left(x, \psi(t)\right), \qquad (x,t) \in M \times \mathbb{R}.$$

Check that $\tilde{F}$ is a smooth homotopy between f and g such that

$$\tilde{F}(x,t) = f(x) \quad \text{for } t \leq 0, \qquad \tilde{F}(x,t) = g(x) \quad \text{for } t \geq 1.$$

(3) Suppose that $h : M \to N$ is a smooth map which is smoothly homotopic to g. Construct a smooth map $H : M \times \mathbb{R} \to N$ such that

$$H(x,t) = f(x) \quad \text{for } t \leq 0, \qquad H(x,t) = h(x) \quad \text{for } t \geq 4,$$
$$H(x,t) = g(x) \quad \text{for } 1 \leq t \leq 3.$$

(4) Use (3) to show that f and h are smoothly homotopic.

(5) Deduce from this that smooth homotopy is an equivalence relation on smooth maps from M to N.

Problem 5. The aim of this problem is to show that homotopy type is an equivalence relation on smooth manifolds.

(1) Show that the homotopy type relation is reflexive and symmetric.

(2) Use the previous problem to show that if M and N have the same homotopy type and N and P have the same homotopy type, then M and P have the same homotopy type.

(3) Deduce from this that homotopy type is an equivalence relation.

Problem 6. Show that if M is a smooth manifold, then M is a deformation retract of $M \times \mathbb{R}^n$, and hence $H^k(M \times \mathbb{R}^n) \simeq H^k(M)$ for all $k \geq 0$.

Homework 4

Problem 7. Suppose that M is an oriented manifold of dimension n.

(1) Construct a top form $\eta \in \Omega_c^n(M)$ such that $\int \eta = 1$.

(2) Assume that M is connected and has a finite good cover. Show that η generates $H_c^n(M)$.

(3) What does the above result means for compact connected oriented manifolds of dimension n?

Problem 8. We know from Problem 3 that $\mathbb{R}^2 \setminus 0$ satisfies Poincaré duality. We let $I_+ = \{((x,0); x > 0\}$ be the positive real axis in $\mathbb{R}^2 \setminus \{0\}$. We also regard the unit circle $\mathbb{S}^1$ as a submanifold of $\mathbb{R}^2 \setminus 0$. Note that polar coordinates (r, θ) provide us with a diffeomorphism of $\mathbb{R}^2 \setminus 0$ with $I_+ \times \mathbb{S}^1$. With $r = \sqrt{x^2 + y^2}$ we then have $dr = r^{-1}(xdx + ydy)$ and $d\theta = r^{-2}(xdy - ydx)$.

(1) Show that $(2\pi)^{-1} d\theta$ is a generator of $H^1(\mathbb{R}^2 \setminus 0)$ and is the Poincaré dual of I_+.

(2) Let $\rho \in C_c^\infty(0, \infty)$ be such that $\int_0^\infty \rho(t) dt = 1$. Show that $\rho(r) dr$ is a generator of $H_c^1(\mathbb{R}^2 \setminus 0)$ and is the compact Poincaré dual of $\mathbb{S}^1$.

(3) What is the closed Poincaré dual of $\mathbb{S}^1$?