Differential Forms in Algebraic Topology: Connections and Curvature on Vector Bundles

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References

Main References

- Sections 7, 10, 11 & 22 of Tu2017.
- Section 5.1 of Chern-Chen.

Connections on a Vector Bundle

Setup

- E is a vector bundle over a smooth manifold M.
- $\Gamma(E)$ is the $C^\infty(M)$-module of smooth sections of E.

Definition

A connection on E is an $\mathbb{R}$-bilinear map,

$$\nabla : \mathscr{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E) \qquad (X, s) \longrightarrow \nabla_X s.$$

satisfying the following two properties:

(i) C^∞-linearity in X:

$$\nabla_{fX} s = f \nabla_X s \qquad \forall f \in C^\infty(M).$$

(ii) Leibniz Rule in s:

$$\nabla_X(fs) = (Xf)s + f\nabla_X s \quad \forall f \in C^\infty(M).$$

Here X ranges over $\mathscr{X}(M)$ and s ranges over $\Gamma(E)$.

Connections on a Vector Bundle

Example

An affine connection is just a connection on the tangent bundle TM.

In the same way as with affine connections, by using a partition of unity argument we get:

Theorem

Every smooth vector bundle admits a connection.

Curvature of a Connection

Setup

- $\nabla : \mathscr{X}(M) \times \Gamma(E) \to \Gamma(E)$ is a connection on E.

Definition

The curvature of ∇ is the bilinear map,

$$R : \mathscr{X}(M) \times \mathscr{X}(M) \longrightarrow \mathrm{End}(\Gamma(E)),$$
$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Curvature of a Connection

Remark

- If $X, Y \in \mathscr{X}(M)$ and $s \in \Gamma(E)$, then $R(X,Y) \in \operatorname{End}(\Gamma(E))$, and so $R(X,Y)s \in \Gamma(E)$.
- Therefore, we may regard the curvature as a map

$$\begin{aligned} R : \mathscr{X}(M) \times \mathscr{X}(M) \times \Gamma(E) &\longrightarrow \Gamma(E), \\ (X, Y, s) &\longrightarrow R(X,Y). \end{aligned}$$

Proposition

The curvature $(X, Y, s) \to R(X,Y)s$ is $C^\infty(M)$-linear in X, Y, and s.

Metric Connections

Definition

A Riemannian metric (or Euclidean metric) on E is the datum for each $p \in M$ of an inner-product product $\langle \cdot, \cdot \rangle_p$ on E_p that depends smoothly on p in the following sense: for all sections $s_1, s_2 \in \Gamma(E)$ the function $M \ni p \to \langle s_1(p), s_2(p) \rangle_p \in \mathbb{R}$ is smooth.

Example

A Riemmanian manifold is manifold together with a Riemannian metric on its tangent bundle.

Definition

A Riemannian bundle is a smooth vector equipped with a Riemannian metric.

Theorem

Every smooth vector bundle admits a Riemannian metric.

Definition

A metric connection on a Riemannian bundle $(E, \langle\cdot,\cdot\rangle)$ is a connection ∇ on E that is compatible with the metric, i.e.,

$$\langle \nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle = X(\langle s_1, s_2\rangle), \quad X \in \mathscr{X}(M),\ s_i \in \Gamma(E).$$

Theorem

Every Riemannian bundle admits a metric connection.

Local Operators

Setup

- E and F are smooth vector bundles over M.
- $\Gamma(E)$ and $\Gamma(F)$ are the $C^\infty(M)$-modules of smooth sections.
- If $U \subseteq M$ is open, then $\Gamma(U, E)$ (resp., $\Gamma(U, F)$) is the $C^\infty(U)$-module of smooth sections of E (resp., F) over U.

Definition

A linear operator $\alpha : \Gamma(E) \to \Gamma(F)$ is a local operator if, given any section $s \in \Gamma(E)$ and any open $U \subseteq M$, we have

$$s = 0 \text{ on } U \implies \alpha(s) = 0 \text{ on } U.$$

Remark

Equivalently, α is a local operator if

$$\operatorname{supp}(\alpha(s)) \subseteq \operatorname{supp}(s) \quad \forall s \in \Gamma(E).$$

Local Operators

Example

If E and F are the trivial line bundle $M \times \mathbb{R}$, then

$$\Gamma(E) = \Gamma(F) \simeq C^\infty(M).$$

Any (smooth) vector field $X : C^\infty(M) \to C^\infty(M)$ then is a local operator.

Proposition

Any $C^\infty(M)$-linear operator $\alpha : \Gamma(E) \to \Gamma(F)$ is a local operator.

Local Operators

Lemma

Let $U \subseteq M$ *be an open set. Given any* $s \in \Gamma(U, E)$ *and* $p \in U$*, there is a global section* $\tilde{s} \in \Gamma(E)$ *such that*

$$\tilde{s} = s \qquad \text{near } p.$$

Theorem

Let $\alpha : \Gamma(E) \to \Gamma(F)$ *be a local operator. Given any open* $U \subseteq M$*, there is a unique linear operator* $\alpha_U : \Gamma(U, E) \to \Gamma(U, F)$*, called the restriction to* U*, such that*

$$\alpha(s)_{|U} = \alpha_U(s_{|U}) \qquad \forall s \in \Gamma(E).$$

Local Operators

Remark

The operator $\alpha_U : \Gamma(U, E) \to \Gamma(U, F)$ is defined as follows:

- If $s \in \Gamma(U, E)$ and $p \in U$, then

  $$\alpha_U(s)(p) := \alpha(\tilde{s})(p),$$

  where $\tilde{s} \in \Gamma(E)$ is such that $\tilde{s} = s$ near p.
- If $\tilde{s}, \bar{s} \in \Gamma(E)$ are such that $\tilde{s} = \bar{s} = s$ near p, then $\tilde{s} - \bar{s} = 0$ near p.
- As α is a local operator, we then have

  $$\alpha(\tilde{s}) - \alpha(\bar{s}) = \alpha(\tilde{s} - \bar{s}) = 0 \quad \text{near } p.$$
- In particular, $\alpha(\tilde{s})(p) = \alpha(\bar{s})(p)$.
- This shows that $\alpha(\tilde{s})(p)$ does not depend on the choice of $\tilde{s}$, and so $\alpha_U(s)(p)$ is well defined.

Remark

If $\alpha : \Gamma(E) \to \Gamma(F)$ is $C^\infty(M)$-linear, then $\alpha_U : \Gamma(U, E) \to \Gamma(U, F)$ is $C^\infty(U)$-linear.

Local Operators

Setup

- E_1, E_2, and F are smooth vector bundles over M.

Definition

A bilinear map $\beta : \Gamma(E_1) \times \Gamma(E_2) \to \Gamma(F)$ is a local operator if, given any $s_i \in \Gamma(E_i)$ and any open $U \subseteq M$, we have

$$(s_1 = 0 \text{ or } s_2 = 0 \text{ on } U) \implies \beta(s_1, s_2) = 0 \text{ on } U.$$

Example

Any $C^\infty(M)$-bilinear map $\beta : \Gamma(E_1) \times \Gamma(E_2) \to \Gamma(F)$ is a local operator.

Proposition

If $\beta : \Gamma(E_1) \times \Gamma(E_2) \to \Gamma(F)$ *is a local bilinear operator, then, for any open* $U \subseteq M$*, there is a unique bilinear operator* $\beta_U : \Gamma(U, E_1) \times \Gamma(U, E_2) \to \Gamma(U, F)$ *such that*

$$\beta(s_1, s_2)_{|U} = \beta_U(s_{1|U}, s_{2|U}) \qquad \forall s_i \in \Gamma(U, E_i).$$

Remark

If $s_i \in \Gamma(U, E_i)$ and $p \in U$, then

$$\beta_U(s_{1|U}, s_{2|U})(p) = \beta(\tilde{s}_1, \tilde{s}_2)(p),$$

where $\tilde{s}_i \in \Gamma(E_i)$ is such that $\tilde{s}_i = s_i$ near p.

Restricting a Connection to an Open Set

Setup

- E is a smooth vector bundle over M.
- $\nabla : \mathscr{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ is a connection on E.

Proposition

The connection ∇ is a local operator.

Restricting a Connection to an Open Set

Corollary

For every open set $U \subseteq M$, the connection restricts to a bilinear operator,

$$\nabla^U : \mathscr{X}(U) \times \Gamma(U, E) \longrightarrow \Gamma(U, E).$$

Remarks

- $\nabla^U : \mathscr{X}(U) \times \Gamma(U, E) \longrightarrow \Gamma(U, E)$ is the unique bilinear operator such that
$$\left(\nabla^U_X s\right)_{|U} = \nabla^U_{X_{|U}}(s_{|U}) \qquad \forall (X, s) \in \mathscr{X}(M) \times \Gamma(E).$$
- If $X \in \mathscr{X}(U)$ and $s \in \Gamma(U, E)$, then
$$\left(\nabla^U_X s\right)(p) = \left(\nabla_{\tilde{X}} \tilde{s}\right)(p),$$
where $\tilde{X} \in \mathscr{X}(M)$ and $\tilde{s} \in \Gamma(E)$ are such that $\tilde{X} = X$ and $\tilde{s} = s$ near p.

Restricting a Connection to an Open Set

Proposition

If $U \subseteq M$ is an open set, then ∇^U is a connection on $E_{|U}$.

$C^\infty(M)$-Linear Operators and Bundle Maps

Setup

- $E \xrightarrow{\pi_E} M$ and $F \xrightarrow{\pi_F} M$ are smooth vector bundles over M.

Definition

A smooth bundle map $\varphi : E \to F$ is a smooth map satisfying the following two conditions:

(i) $\pi_F \circ \varphi = \pi_E$, i.e., $\varphi(E_p) \subseteq F_p$ for all $p \in M$.

(ii) For every $p \in M$, the induced map $\varphi_p : E_p \to F_p$ is linear.

$C^\infty(M)$-Linear Operators and Bundle Maps

Facts

Let $\varphi : E \to F$ be a smooth bundle map.

- If $s \in \Gamma(E)$, then $p \to \varphi(s(p))$ is a section of F.
- It can be shown this is a smooth section.
- If $f \in C^\infty(M)$, then $\varphi(f(p)s(p)) = f(p)\varphi(s(p))$.

Therefore, we obtain the following result:

Proposition

Any smooth bundle map $\varphi : E \to F$ defines a $C^\infty(M)$-linear map,

$$\varphi_\sharp : \Gamma(E) \longrightarrow \Gamma(F),$$
$$\varphi_\sharp(s)(p) := \varphi(s(p)), \quad s \in \Gamma(E),\ p \in M.$$

$C^\infty(M)$-Linear Operators and Bundle Maps

Lemma

Let $\alpha : \Gamma(E) \to \Gamma(F)$ *be a* $C^\infty(M)$*-linear map. Given any* $s \in \Gamma(E)$ *and* $p \in M$*, we have*

$$s(p) = 0 \implies \alpha(s)(p) = 0.$$

$C^\infty(M)$-Linear Operators and Bundle Maps

Proposition

If $\alpha : \Gamma(E) \to \Gamma(F)$ is a $C^\infty(M)$-linear map, then there is a unique smooth bundle map $\varphi : E \to F$ such that $\varphi_\sharp = \alpha$.

Remark

The bundle map $\varphi : E \to F$ is defined as follows:

- If $p \in M$ and $\xi \in E_p$, then
$$\varphi(\xi) = \alpha(s)(p),$$
where $s \in \Gamma(E)$ is such that $s(p) = \xi$.
- Thanks to the previous lemma the r.h.s. does not depend on the choice of s.

$C^\infty(M)$-Linear Operators and Bundle Maps

Consequence

We have a one-to-one consequence,

$$\left\{\begin{array}{c}\text{smooth bundle maps}\\ \varphi : E \longrightarrow F\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}C^\infty(M)-\text{linear maps}\\ \alpha : \Gamma(E) \longrightarrow \Gamma(F)\end{array}\right\}.$$

Corollary

Any $C^\infty(M)$-linear map $\omega : \mathscr{X}(M) \to C^\infty(M)$ is a smooth 1-form.

$C^\infty(M)$-Linear Operators and Bundle Maps

Remark

Any 2-form $\omega \in \Omega^2(M)$ gives rise to an alternating $C^\infty(M)$-bilinear map,

$$\mathscr{X}(M) \times \mathscr{X}(M) \ni (X, Y) \longrightarrow \omega(X, Y) \in C^\infty(M)$$

Lemma

If $\beta : \mathscr{X}(M) \times \mathscr{X}(M) \to C^\infty(M)$ is $C^\infty(M)$-linear, then, given any $X, Y \in \mathscr{X}(M)$ and $p \in M$, we have

$$(X(p) = 0 \text{ or } Y(p) = 0) \implies \beta(X, Y)(p) = 0.$$

$C^\infty(M)$-Linear Operators and Bundle Maps

Proposition

Let $\omega : \mathscr{X}(M) \times \mathscr{X}(M) \to C^\infty(M)$ be $C^\infty(M)$-bilinear and alternating. Then ω is a smooth 2-form on M.

Setup

- ∇ is a connection on a vector bundle E of rank r over M.
- $(e_1, \ldots, e_r)$ is a frame of E over an open subset $U \subseteq M$.
- Any (smooth) section s of E over U can then be uniquely written as $s = \sum a^j e_j$ with $a^i \in C^\infty(U)$.
- We denote by ∇ the restriction to U of ∇.

Connection and Curvature Forms

Facts

- If $X \in \mathscr{X}(U)$, then $\nabla_X e_j$ is a smooth section of E over U.
- We thus have a unique decomposition,

$$\nabla_X e_j = \sum_i \omega_j^i(X) e_i, \qquad \omega_j^i(X) \in C^\infty(U).$$

Lemma

Each map $\mathscr{X}(U) \ni X \to \omega_j^i(X) \in C^\infty(U)$ is $C^\infty(U)$-linear, and hence is a smooth 1-form on U.

Definition

1. The 1-forms ω_j^i are called the connection 1-forms of ∇ relative to the frame $(e_1, \ldots, e_r)$.
2. The matrix $\omega = (\omega_j^i)$ is called the connection matrix of ∇ relative to the frame $(e_1, \ldots, e_r)$.

Connection and Curvature Forms

Facts

- If $X, Y \in \mathscr{X}(U)$, then $R(X,Y)e_j$ is a smooth section of E over U.
- We thus have a unique decomposition,

$$R(X,Y)e_j = \sum_i \Omega^i_j(X,Y)e_i, \qquad \Omega^i_j(X,Y) \in C^\infty(U).$$

Lemma

Each map $\mathscr{X}(U) \times \mathscr{X}(U) \ni (X,Y) \to \Omega^i_j(X,Y) \in C^\infty(U)$ *is* $C^\infty(U)$*-bilinear and alternating, and hence is a smooth 2-form.*

Definition

1. The 2-forms Ω^i_j are called the curvature forms of ∇ relative to the frame $(e_1, \ldots, e_r)$.
2. The matrix $\Omega = (\Omega^i_j)$ is called the curvature matrix of ∇ relative to the frame $(e_1, \ldots, e_r)$.

Connection and Curvature Forms

Theorem (Second Structural Equation)

For $i, j = 1, \ldots, r$, we have

$$\Omega^i_j = d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j.$$

Remark

- If $A = (\alpha^i_j)$ and $B = (\beta^i_j)$ are matrices whose entries are differential forms, then we set

$$dA = (d\alpha^i_j),$$

$$A \wedge B = \left((A \wedge B)^i_j\right), \quad \text{with } (A \wedge B)^i_j = \sum_k A^i_k \wedge B^k_j.$$

- Thus, with $\omega = (\omega^i_j)$ and $\Omega = (\Omega^i_j)$ the structural equation takes the form,

$$\Omega = d\omega + \omega \wedge \omega.$$

Connection and Curvature Forms

Proposition

Assume E is a Riemannian bundle and ∇ is a metric connection. If $(e_1, \ldots, e_r)$ is an orthonormal frame, the connection matrix $\omega = (\omega^i_j)$ and the curvature matrix $\Omega = (\Omega^i_j)$ are both skew-symmetric.

We have the following converse result:

Proposition

Assume E is a Riemannian bundle. If near any point $p \in M$ there is an orthonormal frame relative to which the connection matrix is skew-symmetric, then ∇ is a metric connection.

Bianchi Identity

Proposition (Second Bianchi identity)

We have

$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega.$$

Proof.

- As $\Omega = d\omega + \omega \wedge \omega$, we get

$$d\Omega = d(\omega \wedge \omega) = d\omega \wedge \omega - \omega \wedge d\omega.$$

- As $d\omega = \Omega - \omega \wedge \omega$ we get

$$\begin{aligned} d\Omega &= (\Omega - \omega \wedge \omega) \wedge \omega - \omega \wedge (\Omega - \omega \wedge \omega) \\ &= \Omega \wedge \omega - \omega \wedge \Omega. \end{aligned}$$

The result is proved. □

Change of Frame

Setup

- $(\overline{e}_1, \ldots, \overline{e}_r)$ is another frame of E over U.
- We thus have

$$\overline{e}_j = \sum_i a^i_j e_i, \quad \text{with } a = (a^i_j) \in C^\infty(U, \mathrm{GL}_r(\mathbb{R})).$$

Theorem

Let $\overline{\omega}$ and $\overline{\Omega}$ be the connection matrix and curvature matrix relative to $(\overline{e}_1, \ldots, \overline{e}_r)$. Then, we have

$$\overline{\omega} = a^{-1}\omega a + a^{-1}da,$$

$$\overline{\Omega} = a^{-1}\Omega a.$$

Affine Connections

Setup

- ∇ is an affine connection, i.e., a connection on TM.
- $(e_1, \ldots, e_n)$ is a tangent frame over an open $U \subseteq M$.
- $(\theta^1, \ldots, \theta^n)$ is the dual coframe, i.e., the frame of T^*M such that $\theta^i(e_j) = \delta^i_j$.
- We let θ be the column vector with entries θ^i.

Remarks

- If $(U, x^1, \ldots, x^n)$ are local coordinates, then as tangent frame and dual coframe we may take $(\partial_{x^1}, \ldots, \partial_{x^n})$ and $(dx^1, \ldots, dx^n)$.
- For every vector field $X \in \mathscr{X}(U)$, we have

$$X = \sum_i \theta^i(X) e_i.$$

Affine Connections

Facts

- The torsion of ∇ is
$$T(X,Y) = \nabla_X Y - \nabla_Y X - \nabla_{[X,Y]}, \qquad X, Y \in \mathscr{X}(U).$$
- As $T(X,Y)$ is a smooth vector field on U, we may write
$$T(X,Y) = \sum_i \tau^i(X,Y) e_i, \qquad \tau^i(X,Y) \in C^\infty(U).$$

Lemma

Each $\mathscr{X}(U) \times \mathscr{X}(U) \ni (X,Y) \to T(X,Y) \in C^\infty(U)$ is $C^\infty(U)$-bilinear and alternating, and hence is a smooth 2-form.

Definition

The 2-forms τ^i are called the torsion forms of ∇ relative to the frame $(e_1, \ldots, e_r)$.

Remark

We denote by τ the r-column matrix whose entries are $\tau^1, \dots, \tau^r$.

Theorem (Structural Equations)

We have

$$\tau = d\theta + \omega \wedge \theta,$$
$$\Omega = d\omega + \omega \wedge \omega.$$

Corollary

Assume M is a Riemannian manifold and ∇ is the Levi-Civita connection. If $(e_1, \dots, e_n)$ is an orthonormal tangent frame, then the connection form ω is the unique skew-symmetric matrix of 1-forms such that

$$d\theta + \omega \wedge \theta = 0.$$

Bianchi Identities

Theorem

We have

$$d\tau = \Omega \wedge \theta - \omega \wedge \tau,$$
$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega.$$

Proof.

- The 2nd equality is known already. We only need to prove the 1st equality.
- As $\tau = d\theta + \omega \wedge \theta$, we get

$$d\tau = d(\omega \wedge \theta) = d\omega \wedge \theta - \omega \wedge d\theta.$$

- As $d\theta = \tau - \omega \wedge \theta$ and $d\omega = \Omega - \omega \wedge \omega$, we get

$$\begin{aligned} d\tau &= (\Omega - \omega \wedge \omega) \wedge \theta - \omega \wedge (\tau - \omega \wedge \theta) \\ &= \Omega \wedge \theta - \omega \wedge \tau. \end{aligned}$$

The proof is complete. □

Bianchi Identities

Remark

If ∇ is torsion-free (e.g., ∇ is a Levi-Civita connection), then the 1st Bianchi identity reduces to

$$\Omega \wedge \theta = 0.$$